Analytic Integrals and Poincaré's Centre Problem

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Synopsis

As for Poincaré's centre-problem Poincaré [3] himself assumed that if the origin is a centre then there is an integral (constant of motion) being analytic in a neighborhood of the origin. We are going to prove this assumption and vice versa in a concise way by using techniques developed by C. L. Siegel [6].

1 Introduction

Consider a system of differential equations of the form

$$\left.\begin{array}{rcl} \dot{x} &=& y + q(x,y) \\ \dot{y} &=& -x - p(x,y) \end{array}\right\} \tag{1.1}$$

where p, q are real convergent power series whose terms of lowest order are of degree at least two. We want to present a new short proof that the origin is a centre of (1.1) if and only if there is an integral (constant of motion) being real analytic and non-constant in a neighborhood of the origin. According to [4, p. 6] Ljapunov [1] was the first to give a complete proof of this result; then however in [4] special attention is called to the proof in [2] which seems to be available only with great difficulties. It may be therefore worthwhile to present a proof which is easily accessible. Our method is based on Siegel's considerations on Poincaré's centre-problem in [6, §25].

2 Complex Systems

(1.1) is considered for complex valued x, y also whereas t remains real.

Definition 2.1: *Let us consider the complex (real) system*

$$\left.\begin{array}{rcl} \dot{x} &=& Q(x,y) \\ \dot{y} &=& P(x,y) \end{array}\right\} \tag{2.1}$$

where Q, P are convergent power series in the two complex variables x, y, whereas the curve parameter t is real. Let

$$Q(0,0) = P(0,0) = 0.$$

The equilibrium $(0,0)$ is called stable if for every sufficiently small polycylinder (square in $\mathbb{R}^2$) $U_\varepsilon(0,0) = \{(x,y)||x| < \varepsilon,\ |y| < \varepsilon\}$, $\varepsilon > 0$, there is a polycylinder (square in $\mathbb{R}^2$) $B_\delta(0,0) = \{|x| < \delta, |y| < \delta\}$, $\delta > 0$, such that the solution $(x(t,0,\xi), y(t,0,\eta))$ of (2.1) with initial values (ξ, η) for $t = 0$ exists for all times and satisfies

$$(x(t,0,\xi), y(t,0,\eta)) \in U_\varepsilon(0,0)$$

provided

$$(\xi, \eta) \in B_\delta(0,0).$$

The notion of stability we are going to use here thus means stability in the past and in the future. Instability is the logical negation of stability whereas in [6, p. 157] a stronger notion is used. Thus instability in the sense of [6, p. 157] implies instability in our sense. The linear part in (1.1) has eigenvalues i and $-i$. The eigenvectors are $\frac{1}{\sqrt{2}}(1,i)$, $\frac{1}{\sqrt{2}}(1,-i)$. Introducing new variables $(\widetilde{x},\widetilde{y})$ in (1.1) by the substitution

$$\begin{pmatrix} \widetilde{x} \\ \widetilde{y} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}i \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{2.2}$$

we arrive at the equivalent system

$$\left.\begin{aligned} \widetilde{x} &= i\widetilde{x} + f(\widetilde{x},\widetilde{y}), \\ \widetilde{y} &= -i\widetilde{y} + g(\widetilde{x},\widetilde{y}). \end{aligned}\right\} \tag{2.3}$$

f, g are convergent power series starting with quadratic terms and we have

$$f(\widetilde{x},\widetilde{y}) = \overline{g}(\widetilde{y},\widetilde{x})$$

$\overline{g}$ originates from g by replacing the coefficients of g by their complex conjugates ([6, p. 175]).

For a moment we use formal power series and apply a particular substitution which brings (2.2) into its normal form. There exist power series $\varphi(u,v), \psi(u,v)$ in the new variables u, v of the form

$$\begin{aligned} \widetilde{x} &= \varphi(u,v) &= u + \varphi_2 + \varphi_3 + \ldots, \\ \widetilde{y} &= \psi(u,v) &= v + \psi_2 + \psi_3 + \ldots \end{aligned}$$

with the homogeneous parts φ_i, ψ_i of degree i such that (2.3) becomes

$$\dot{u} = pu,\ \dot{v} = qv. \tag{2.4}$$

p, q are power series in $w = u \cdot v$. φ, ψ do not contain terms of the form uw^k, vw^k with $k \geq 1$. φ, ψ are determined uniquely by these requirements (cf [6, pp.175, 176]). Moreover

$$\varphi(u,v) = \overline{\psi}(v,u) \tag{2.5}$$

As for the stability of the origin we have
Theorem 2.1: *The origin is a stable point of equilibrium of* (2.3) *if and only if*

$$p + q = 0. \tag{2.6}$$

In this case the series for φ, ψ, p and q are convergent in a neighborhood of the origin.

Proof: [6, pp. 177, 178]. □

Stability can also be characterized by the existence of a holomorphic constant of motion.

Theorem 2.2: *The origin is a stable point of equilibrium of* (2.3) *if and only if there is a constant of motion $\widetilde{F}$ which is holomorphic in a neighborhood of the origin and whose power series around the origin contains the term $\widetilde{x} \cdot \widetilde{y}$.*
Proof: Let the origin be stable. (2.6) implies that $u \cdot v$ is a constant of motion for (2.4). Inverting the biholomorphic mapping

$$\left.\begin{aligned} \widetilde{x} &= \varphi(u,v) \\ \widetilde{y} &= \psi(u,v) \end{aligned}\right\} = \Phi(u,v)$$

we immediately see that with

$$\begin{aligned} \begin{pmatrix} u \\ v \end{pmatrix} &= \Phi^{-1}\begin{pmatrix} \widetilde{x} \\ \widetilde{y} \end{pmatrix}, \\ u &= \widetilde{x} + u_2(\widetilde{x},\widetilde{y}) + u_3(\widetilde{x},\widetilde{y}) + \ldots, \\ v &= \widetilde{y} + v_2(\widetilde{x},\widetilde{y}) + v_3(\widetilde{x},\widetilde{y}) + \ldots \end{aligned}$$

the function

$$\begin{aligned}\widetilde{F}(\widetilde{x},\widetilde{y}) &= u(\widetilde{x},\widetilde{y})\cdot v(\widetilde{x},\widetilde{y}),\\ &= \widetilde{x}\cdot\widetilde{y} + \text{ higher order terms}\end{aligned}$$

is a constant of motion for (2.3). As for the opposite direction we refer to [7, p. 197]. □

3 The Real System

We now apply Theorem 2.2 and in particular the proof of Theorem 2.1 in [6, p. 175] to the real system (1.1).

Theorem 3.1: *The origin is a stable point of equilibrium of the real system (1.1) if and only if it is a stable point of equilibrium of the complex system* (2.3)*.*

Proof: Let the origin be unstable for (2.3). Then it is so for the real system (1.1) as proved in [6, pp. 177, 178]. In this reference there is a misprint on p. 177, 4th line from below: On has to replace (7) by (1). See also [5, pp. 23 - 30] for a more detailed version. If the origin is unstable for the real system (1.1) then it is so for (2.3) since (2.3) originates from (1.1) by an invertible linear transformation. □

Definition 3.2: *The origin is called a centre for the real system* 1.1 *if there is a neighborhood of the origin such that every integral of* (1.1) *passing through a point of that neighborhood is closed.*

Observe that in a suitable neighborhood of the origin there is no point of equilibrium distinct from the origin.

Theorem 3.3: *The real system* (1.1) *has a centre in the origin if and only if it has an integral*

$$\begin{aligned}F(x,y) &= F_2(x,y)+F_3(x,y)+\ldots,\\ & \quad F_i \textit{ homogeneous polynomials in } x_y \textit{ of degree } i\end{aligned}$$

which is analytic in a neighborhood of the origin and starts with $F_2(x,y)=\frac{1}{2}(x^2+y^2)$*.*

Proof: Let (1.1) have a centre in the origin. Transforming (1.1) by using polar coordinates φ, r in $\mathbb{R}^2$ we obtain a single equation

$$\frac{dr}{d\varphi} = r' = r\frac{g_1(\varphi)r+\ldots}{1+h_1(\varphi)r+\ldots} \tag{3.1}$$

without singularity. Numerator and denominator are convergent power series in r with 2π-periodic coefficients $g_1, g_2, \ldots, h_1, h_2, \ldots$. The g_i, h_i are in fact polynomials in $\cos\varphi, \sin\varphi$. Since all solutions of (3.1) with

$$r_0 = r(0),\ 0 \le r_0 < \varepsilon,$$

are 2π-periodic it is easy to see that the origin is stable. According to Theorem 3.1 this is so for (2.3) and

$$\widetilde{F}(\widetilde{x},\widetilde{y}) = \widetilde{x}\widetilde{y} + \text{ higher order terms}$$

is *a* constant of motion being holomorphic in $|\widetilde{x}| < \varepsilon$, $|\widetilde{y}| < \varepsilon$. Inserting

$$\widetilde{x} = \tfrac{1}{\sqrt{2}}(x+iy)$$

$$\widetilde{y} = \tfrac{1}{\sqrt{2}}(x-iy)$$

we obtain

$$\widetilde{x}\widetilde{y} = \frac{1}{2}(x^2+y^2).$$

If

$$\widetilde{F}(\widetilde{x},\widetilde{y}) = \widetilde{x}\widetilde{y} + \sum_{\nu+\mu\ge 3} \widetilde{F}_{\nu\mu}\widetilde{x}^{\nu}\widetilde{y}^{\mu}$$

then with suitable coefficients $F_{\nu\mu}$ we obtain

$$\widetilde{F}(\frac{1}{\sqrt{2}}(x+iy), \frac{1}{\sqrt{2}}(x-iy)) = \frac{1}{2}(x^2+y^2) + \sum_{\nu+\mu\geq 3} F_{\nu\mu} x^\nu y^\mu,$$

and

$$F(x,y) = \frac{1}{2}(x^2+y^2) + \sum_{\nu+\mu\geq 3} (\mathcal{R}eF_{\nu\mu}) x^\nu y^\mu \tag{3.2}$$

is the desired real analytic constant of motion of (1.1). If conversely the convergent power series (3.2) is a constant of motion of (1.1) then the origin is a strict minimum of F and therefore a centre for (1.1). □

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