Generation of Centres by Adding Higher Order Terms in $y' = -\frac{x^{2n-1}+P(x,y)}{y^{2n-1}+Q(x,y)}$

Wolf von Wahl
Universität Bayreuth
Department of Mathematics
D-95440 Bayreuth, Germany

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Synopsis:
We systematically study the question how to convert a focus into a centre. This question was first raised by Frommer [1].

1 Introduction

Let $n \in \mathbb{N}$. $P(x,y), Q(x,y)$ are polynomials in x, y starting with terms of order $2n$ at least. If

$$y' = -\frac{x^{2n-1} + P(x,y)}{y^{2n-1} + Q(x,y)} = -\frac{\mathcal{A}(x,y)}{\mathcal{B}(x,y)} \tag{1.1}$$

has a focus at the critical point $(0,0)$ it is sometimes possible to convert $(0,0)$ into a centre by adding higher order polynomials in the numerator and denominator. Frommer [1] was the first to study the influence of higher order terms on the question whether (1.1) can be made a centre or not. Our work ist motivated by his contributions.

2 Systematic Approach

As announced we intend to convert a focus $y' = -\frac{\mathcal{A}(x,y)}{\mathcal{B}(x,y)}$ into a centre by replacing the preceding equation by $y' = -\frac{\mathcal{A}(x,y)+\mathcal{Z}_1(x,y)}{\mathcal{B}(x,y)+\mathcal{Z}_2(x,y)}$. The additional terms $\mathcal{Z}_1(x,y), \mathcal{Z}_2(x,y)$ vanish faster than $\mathcal{A}(x,y)$, $\mathcal{B}(x,y)$ at $(0,0)$.

We are needing a Eulerian multiplier. Since such a multiplier does not vanish it is close by to try it with an expression $\mu = ce^{\widetilde{p}}$ (c constant $\neq 0$, $\widetilde{p}$ an appropriate function). We start with $n \in \mathbb{N}$,

$$\mathcal{A}(x,y) = x^{2n-1} + P(x,y), \tag{2.1}$$

$$\mathcal{B}(x,y) = y^{2n-1} + Q(x,y), \tag{2.2}$$

P, Q homogeneous polynomials of one and the same degree $p \geq n$. With still unknown polynomials $\widetilde{q}, \widetilde{p}$ we try the ansatz

$$\begin{aligned}
\frac{x^{2n-1} + P + \frac{1}{2n}\widetilde{q}\widetilde{p}_x}{y^{2n-1} + Q + \frac{1}{2n}\widetilde{q}\widetilde{p}_y} &= \frac{2n(x^{2n-1} + P + \frac{1}{2n}\widetilde{q}\widetilde{p}_x)}{2n(y^{2n-1} + Q + \frac{1}{2n}\widetilde{q}\widetilde{p}_y)} \\
&= \frac{(2nx^{2n-1} + \widetilde{q}_x) + (x^{2n} + y^{2n} + \widetilde{q})\widetilde{p}_x}{(2ny^{2n-1} + \widetilde{q}_y) + (x^{2n} + y^{2n} + \widetilde{q})\widetilde{p}_y} \\
&= \frac{\partial_x[(x^{2n} + y^{2n} + \widetilde{q})e^{\widetilde{p}}]}{\partial_y[(x^{2n} + y^{2n} + \widetilde{q})e^{\widetilde{p}}]} \\
&= \frac{\partial_x F}{\partial_y F} \text{ with } F = (x^{2n} + y^{2n} + \widetilde{q})e^{\widetilde{p}},\ \mu = 2ne^{\widetilde{p}}.
\end{aligned} \tag{2.3}$$

If

$$grad\widetilde{q} \geq 2n+1 \tag{2.4}$$

the level lines of F are closed and the origin is a centre for $y' = -\frac{\mathcal{A}+\frac{1}{2n}\widetilde{q}\widetilde{p}_x}{\mathcal{B}+\frac{1}{2n}\widetilde{q}\widetilde{p}_y}$ provided

$$2nP = \widetilde{q}_x + (x^{2n}+y^{2n})\widetilde{p}_x, \tag{2.5}$$

$$2nQ = \widetilde{q}_y + (x^{2n}+y^{2n})\widetilde{p}_y. \tag{2.6}$$

The additional term in the denominator is $\frac{1}{2n}\widetilde{q}\widetilde{p}_y$ and in the numerator it is $\frac{1}{2n}\widetilde{q}\widetilde{p}_x$. Let us compare the degrees. We have

$$deg\,P = \deg\, Q = p \geq 2n. \tag{2.7}$$

Consequently

$$deg\,\widetilde{q} \;=\; p+1, \tag{2.8}$$

$$2n-1+deg\,\widetilde{p} \;=\; p,\; deg\,\widetilde{p} = p-(2n-1). \tag{2.9}$$

For the given $2(p+1)$ coefficients of P, Q we have at our disposal $2p+2+2-2n = 2p+4-2n$ coefficients of $\widetilde{q}$ and $\widetilde{p}$. If the coefficient vectors of $P, Q, \widetilde{q}, \widetilde{p}$ are $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}$ respectively we arrive at a linear system

$$\begin{pmatrix}\mathfrak{a}\\ \mathfrak{b}\end{pmatrix} = \mathcal{C}\begin{pmatrix}\mathfrak{c}\\ \mathfrak{d}\end{pmatrix}. \tag{2.10}$$

Here $\mathfrak{a}, \mathfrak{b}$ have $p+1$ rows each. $(\mathfrak{a}\,\mathfrak{b})^T$ is a column, $\mathcal{C}$ has $2(p+1)$ rows and $2(p+1)+2-2n$ columns, $\mathfrak{c}, \mathfrak{d}$ have $p+2$, $p-(2n-1)+1 = p-2n+2$ rows respectively and $(\mathfrak{c}\,\mathfrak{d})^T$ is a column. For $n=1$ the matrix $\mathcal{C}$ is quadratic. At most in the case (2.10) has a solution $(\mathfrak{c}\,\mathfrak{d})^T$ for any right hand side $(\mathfrak{a}\,\mathfrak{b})^T$. For $n \geq 2$ the system (2.10) is overdetermined. $\mathcal{C}$ has only nonnegative integer entries.

We achieve a considerable simplification if we exploit the structure of (2.5,6).

Theorem 2.1 *Let $n \in \mathbb{N}$ and P, Q homogeneous polynomials in x, y of degree $p \geq 2n$. Let $\widetilde{p}$ a homogeneous polynomial of degree $p-(2n-1)$. Let*

$$y^{2n-1}\widetilde{p}_x - x^{2n-1}\widetilde{p}_y = P_y - Q_x. \tag{2.11}$$

Then there is a homogeneous polynomial $\widetilde{q}$ of degree $p+1$ such that (2.5,6) are valid. This means

$$\left.\begin{aligned} 2nP &= \widetilde{q}x + (x^{2n}+y^{2n})\widetilde{p}_x, \\ 2nQ &= \widetilde{q}_y + (x^{2n}+y^{2n})\widetilde{p}_y. \end{aligned}\right\} \tag{2.12}$$

Proof: (2.11) implies that $(2nP-(x^{2n}+y^{2n})\widetilde{p}_x,\, 2nQ-(x^{2n}+y^{2n})\widetilde{p}_y)$ is a gradient. Set for instance

$$\partial_x f = \sum_{\nu+\mu=p} \widehat{p}_{\mu\nu}x^\mu y^\nu = 2nP-(x^{2n}+y^{2n})\widetilde{p}_x,$$

$$\partial_y f = \sum_{\nu+\mu=p} \widehat{q}_{\mu\nu}x^\mu y^\nu = 2nQ-(x^{2n}+y^{2n})\widetilde{p}_y.$$

Since we have on the right hand side the Taylor expansions for $\partial_x f, \partial_y f$ around the origin we obtain that f is a polynomial with degree $p+1$. Employing principal functions in x, y respectively we get

$$\begin{aligned} f &= \sum_{\substack{\mu+\nu=p+1,\\ \mu\geq 1,\ \nu\geq 1}}^{I} \widehat{p}_{\mu-1\nu}\frac{1}{\mu}x^\mu y^\nu + \sum_{\substack{\mu=\lambda+1,\\ \mu\geq 1}}^{II} \widehat{p}_{\mu-10}\frac{1}{\mu}x^\mu + \varphi(y), \\ &= \sum_{\substack{\mu+\nu=p+1\\ \mu\geq 1,\ \nu\geq 1}}^{III} \widehat{q}_{\mu\nu-1}\frac{1}{\nu}x^\mu y^\nu + \sum_{\substack{\nu=\lambda+1,\\ \nu\geq 1}}^{IV} \widehat{q}_{0\nu-1}\frac{1}{\nu}y^\nu + \psi(x) \end{aligned}$$

with $\nu\widehat{p}_{\mu-1\nu} = \mu\widehat{p}_{\mu\nu-1}$ for $\mu+\nu = p+1$, $\mu \geq 1$, $\nu \geq 1$. Setting $\varphi = \sum^{IV}$, $\psi = \sum^{II}$ we arrive at

$$\widetilde{q} = f = \sum^{I} + \sum^{II} + \sum^{III}$$

as the desired homogeneous polynomial of degree $p+1$. □

Evidently (2.12) implies (2.11). (2.11) furnishes a linear system for the $p+2-2n$ coefficients of $\widetilde{p}$. The right hand side $\mathfrak{h}$ of this system consists of the p coefficients of $P_y - Q_x$. We arrive at

$$\mathcal{D}\mathfrak{d} = \mathfrak{h}. \tag{2.13}$$

$\mathfrak{h}$ is a column with p rows, $\mathcal{D}$ is a matrix with p rows and $p+2-2n$ columns. $\mathcal{D}$ has only integer entries. We see that by (2.13) the Matrix $\mathfrak{C}$ in (2.10) is diminished. A detailed discussion of (2.11) can be found in [2].

3 Examples First Part

In our **first example** we deal with $n = 1$. Then $\mathcal{D}$ in (2.13) is quadratic and we are interested in $\det \mathcal{D}$. Let $p = 4$, thus $\widetilde{p}$ has degree 3. Set

$$\widetilde{p}(x,y) = ax^3 + bx^2y + cxy^2 + dy^3.$$

Then

$$\begin{aligned}
y\widetilde{p}_x - x\widetilde{p}_y &= y(ax^3 + bx^2y + cxy^2 + dy^3)_x - x(ax^3 + bx^2y + cxy^2 + dy^3)_y, \\
&= y(3ax^2 + 2bxy + cy^2) - x(bx^2 + 2cxy + 3dy^2), \\
&= 3ayx^2 + 2bxy^2 + cy^3 - bx^3 - 2cx^2y - 3dxy^2, \\
&= -bx^3 + (3a-2c)x^2y + (2b-3d)xy^2 + cy^3.
\end{aligned}$$

We can easily satisfy $P_y - Q_x = y\widetilde{p}_x - x\widetilde{p}_y$ by choosing a suitable $\widetilde{p}$ if for any column $\mathfrak{h} = (\alpha, \beta, \gamma, \delta)^T$ the system

$$\begin{aligned}
0 - b + 0 + 0 &= \alpha, \\
3a + 0 - 2c + 0 &= \beta, \\
0 + 2b + 0 - 3d &= \gamma, \\
0 + 0 + c + 0 &= \delta,
\end{aligned}$$

which means $\mathcal{D}\mathfrak{d} = \mathfrak{h}$, is solvable in the unknowns $\mathfrak{d} = (a,b,c,d)^T$. Since

$$\det \mathcal{D} = \det \begin{pmatrix} 0 & -1 & 0 & 0 \\ 3 & 0 & -2 & 0 \\ 0 & 2 & 0 & -3 \\ 0 & 0 & 1 & 0 \end{pmatrix} = -\det \begin{pmatrix} 0 & -1 & 0 \\ 3 & 0 & 0 \\ 0 & 2 & -3 \end{pmatrix} = -\det \begin{pmatrix} 3 & 0 \\ 0 & -3 \end{pmatrix} = 9$$

this in fact the case. Our **second example** treats $n = 1$, $p = 5$, this is degree $\widetilde{p} = 4$ and exhibits a characteristic difference between the cases ″$\widetilde{p}$ has odd degree, this is p is even ″ and ″$\widetilde{p}$ has even degree, this is p is odd ″ which has already been observed by Frommer [1]. The reason is that in case p odd the system (2.13) may not be solvable. For details cf. [2]. If it is solvable however then the first focal value d_1 in the expansion

$$\det \begin{pmatrix} F_x & F_y \\ \mathcal{A} & \mathcal{B} \end{pmatrix} = \sum_{j=1}^{\infty} d_j (x^{2j+2} + y^{2j+2}) \tag{3.1}$$

vanishes. $F = x^2 + y^2 + f_2(x,y) + f_3(x,y) + \ldots$ is a formal power series whose construction goes back to Poincaré. The oberservation on the disappearance of d_1 was already made by Frommer [1, p. 406]. We obtain

$$\begin{aligned}
y\widetilde{p}_x - x\widetilde{p}_y &= y(ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4) - \\
&\quad -x(ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4)y, \\
&= y(4ax^3 + 3bx^2y + 2cxy^2 + dy^3) - \\
&\quad -x(bx^3 + 2cx^2y + 3dxy^2 + 4ey^3), \\
&= 4ax^3y + 3bx^2y^2 + 2cxy^3 + dy^4 - \\
&\quad -bx^4 - 2cx^3y - 3dx^2y^2 - 4exy^3, \\
&= -bx^4 + (4a - 2c)x^3y + (3b - 3d)x^2y^2 + (2c - 4e)xy^3 + dy^4.
\end{aligned}$$

For arbitrary $\mathfrak{h} = (\alpha, \beta, \gamma, \delta, \epsilon)^T$ we consider the system $\mathfrak{h} = \mathfrak{D}\mathfrak{d}$, this is

$$\begin{aligned}
0 + -b + 0 + 0 + 0 &= \alpha, \\
4a + 0 - 2c + 0 + 0 &= \beta, \\
0 + 3b + 0 - 3d + 0 &= \gamma, \\
0 + 0 + 2c + 0 - 4c &= \delta, \\
0 + 0 + 0 + d + 0 &= \epsilon
\end{aligned}$$

in the variables $\mathfrak{d} = (a, b, c, d, e)^T$. Its determinant vanishes since

$$-\frac{1}{2}(\text{ first column of } \mathfrak{D}) - \frac{1}{2}(\text{fifth column of } \mathfrak{D}) = \text{third column of } \mathfrak{D}.$$

Since

$$\det \mathcal{D}' = 12,\ \mathcal{D}' = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 4 & 0 & -2 & 0 \\ 0 & 3 & 0 & -3 \\ 0 & 0 & 2 & 0 \end{pmatrix}$$

$\mathcal{D}'$ has rank 4.

Third example Let $n = 2$. We consider

$$y' = -\frac{x^3 + P(x,y)}{y^3 + Q(x,y)}$$

with homogeneous polynomials P, Q of degree 4 and look for a homogeneous polynomial $\widetilde{p}$ of degree 1; this means

$$\widetilde{p}(x,y) = ax + by$$

and

$$P_y - Q_x = ay^3 - bx^3.$$

In the simplest case we have

$$P(x,y) = \frac{a}{4}y^4,\ Q(x,y) = \frac{b}{4}x^4$$

and there exists a homogeneous polynomial $\widetilde{q}$ of degree 5 such that the origin is a centre for $y' = -\frac{x^3+P(x,y)+\frac{1}{4}\widetilde{q}\widetilde{p}_x}{y^3+Q(x,y)+\frac{1}{4}\widetilde{q}\widetilde{p}_y}$. Interest in this statement could be increased by showing that the origin is a focus for $y' = -\frac{x^3+P(x,y)}{y^3+Q(x,y)}$. To discuss this question is more difficult than in the case $n = 1$. We need to find a substitute for (3.1) and the focal values d_j. This was performed in [1, pp. 412,413] and we are going to explain the ideas and some open questions. Transforming (1.1) into plane polar coordinates we get

$$r' = \frac{dr}{d\varphi} = r\frac{\mathcal{A}(r\cos\varphi, r\sin\varphi)\sin\varphi - \mathcal{B}(r\cos\varphi, r\sin\varphi)\cos\varphi}{\mathcal{A}(r\cos\varphi, r\sin\varphi)\cos\varphi + \mathcal{B}(r\cos\varphi, r\sin\varphi)\sin\varphi} = \frac{\mathcal{Z}(\varphi, r)}{\mathcal{N}(\varphi, r)}$$

Now $r' = \mathcal{Z}/\mathcal{N}$ is compared with $r' = -\partial\varphi\mathcal{F}/\partial_r\mathcal{F}$ where $\mathcal{F}$ is a formal power series $\mathcal{F}(\varphi, r) = \sum_{\lambda\geq 1} f_\lambda(\varphi)r^\lambda$ in r with coefficient functions $f_\lambda : [0, 2\pi] \to \mathbb{R}$. The result corresponds to (3.1) and reads as follows.

Theorem 2: *There is a unique formal power series $\mathcal{F}(\varphi, r) = \sum_{\lambda\geq 2n} f_\lambda(\varphi)r^\lambda$ in r with continuously differentiable 2π-periodic functions f_λ, $f_\lambda(0) = 1$, and a unique sequence $c_{4n}, c_{4n+1}, \ldots$ such that*

$$\det\begin{pmatrix} \partial_\varphi\mathcal{F} & \partial_r\mathcal{F} \\ -\mathcal{Z} & \mathcal{N} \end{pmatrix} = \sum_{j=4n}^{\infty} c_j r^j$$

Proof: Set $\mathcal{Z}(\varphi, r) = \sum\limits_{\lambda\geq 2n} \mathcal{Z}_\lambda(\varphi)r^\lambda$, $\mathcal{N}(\varphi, r) = \sum_{\lambda\geq 2n-1}\mathcal{N}_\lambda(\varphi)r^\lambda$. If we compare the coefficients of the r-powers in

$$\mathcal{Z}\partial_r\mathcal{F} = -\mathcal{N}\partial_\varphi\mathcal{F} + \sum_{j=2n}^{\infty} c_j r^j$$

we arrive at

$$\begin{aligned}
\sum_{\lambda\geq 2n} Z_\lambda(\varphi)r^\lambda \sum_{\lambda\geq 2n}\lambda f_\lambda(\varphi)r^{\lambda-1} &= \sum_{\lambda\geq 2n} Z_\lambda(\varphi)r^\lambda \sum_{\lambda\geq 2n-1}(\lambda+1)f_{\lambda+1}(\varphi)r^\lambda \\
&= (\sum_{\lambda\geq 0} Z_{\lambda+2n}(\varphi)r^\lambda)(\sum_{\lambda\geq 0}(\lambda+2n)f_{\lambda+2n}(\varphi)r^\lambda)r^{4n-1} \\
&= \sum_{\lambda\geq 0}(\sum_{\kappa=0}^{\lambda}\mathcal{Z}_{\lambda+2n-\kappa}(\varphi)(\kappa+2n)f_{\kappa+2n}(\varphi))r^{\lambda+4n-1} \\
&= -\sum_{\lambda\geq 2n-1}\mathcal{N}_\lambda(\varphi)r^\lambda\sum_{\lambda\geq 2n}f'_\lambda(\varphi)r^\lambda + \sum_{j=2n}^{\infty}c_j r^j \\
&= -(\sum_{\lambda\geq 0}\mathcal{N}_{\lambda+2n-1}(\varphi)r^\lambda)(\sum_{\lambda\geq 0}f'_{\lambda+2n}(\varphi)r^\lambda)r^{4n-1} + \sum_{j=2n}^{\infty}c_j r^j \\
&= -\sum_{\lambda\geq 0}(\sum_{\kappa=0}^{\lambda}\mathcal{N}_{\lambda+2n-1-\kappa}(\varphi)f'_{\kappa+2n}(\varphi))r^{\lambda+4n-1} + \sum_{\lambda\geq 0}c_{\lambda+4n-1}\cdot r^{\lambda+4n-1}
\end{aligned}$$

with $c_{2n}, c_{2n+1}, \ldots, c_{4n-2} = 0$,

$$\sum_{\kappa=0}^{\lambda}\mathcal{N}_{\lambda+2n-1-\kappa}f'_{\kappa+2n} + \sum_{\kappa=0}^{\lambda}\mathcal{Z}_{\lambda+2n-\kappa}(\kappa+2n)f_{\kappa+2n} = c_{\lambda+4n-1}$$

Now $\mathcal{N}_{2n-1}(\varphi) = \cos^{2n}\varphi + \sin^{2n}\varphi$ is positive definite and we arrive at

$$f'_{\lambda+2n} + \frac{(\lambda+2n)\mathcal{Z}_{2n}}{\mathcal{N}_{2n-1}(\varphi)} - f_{\lambda+2n} + \sum_{\kappa=0}^{\lambda-1}\frac{1}{\mathcal{N}_{2n-1}(\varphi)}(\mathcal{Z}_{\lambda+2n-\kappa}(\kappa+2n)f_{\kappa+2n} + \\ +\mathcal{N}_{\lambda+2n-1-\kappa}f'_{\kappa+2n}) = c_{\lambda+4n-1} \tag{3.2}$$

for $\lambda \geq 1$ and

$$f'_\lambda + \frac{2n\mathcal{Z}_{2n}(\varphi)}{\mathcal{N}_{2n-1}(\varphi)}f_{2n} = c_{4n-1} \text{ for } \lambda = 0 \tag{3.3}$$

Since $\mathcal{Z}_{2n}(\varphi) = \cos^{2n-1}\varphi\sin\varphi - \sin^{2n-1}\varphi\cos\varphi$ the coefficient $\mathcal{Z}_{2n}/\mathcal{N}_{2n-1}$ is 2π-periodic. Moreover

$$\int_0^{2\pi}\frac{\mathcal{Z}_{2n}(\varphi)}{\mathcal{N}_{2n-1}(\varphi)}d\varphi = 0 \tag{3.4}$$

since $\mathcal{Z}_{2n}$ is odd. If (3.3) has a 2π-periodic solution the constant c_{4n-1} has to vanish. In this case each solution of (3.3) is 2π-periodic. As for $\lambda = 1$ we obtain

$$f'_{2n+1} + \frac{(1+2n)\mathcal{Z}_{2n}(\varphi)}{\mathcal{N}_{2n-1}(\varphi)} f_{2n+1} + \frac{1}{\mathcal{N}_{2n-1}(\varphi)}(2n\mathcal{Z}_{1+2n} f_{2n} + \mathcal{N}_{1+2n-1} f'_{2n}) = c_{4n} \tag{3.5}$$

If (3.5) has a 2π-periodic solution then c_{4n} is uniquely determined and every solution of (3.5) is 2π-periodic. This follows from (3.4). In general the situation is as follows: If $h, f : \mathbb{R} \to \mathbb{R}$ are continuous and 2π periodic with $\int_0^{2\pi} h d\psi = 0$ and if

$$y' + hy + f = c, \; c = \text{ constant}, \tag{3.6}$$

has a 2π-periodic solution then

$$c = \frac{\int_0^{2\pi} e^{\int_0^\varphi h d\psi} f d\varphi}{\int_0^{2\pi} e^{\int_0^\varphi h d\psi} d\varphi} \tag{3.7}$$

and every solution is 2π-periodic. On the other hand, c in (3.7) is the only constant such that every solution of $y' + hy + f = c$ is 2π periodic. In view of (3.2) it is now easy to prove the assertion by induction over λ. □

According to Frommer [1, p. 412] the constants c_j play the rôle of the focal values d_j in (1.1). Cf. also section 4 to follow. As for our example $y' = -\frac{x^3+P(x,y)}{y^3+Q(x,y)} = -\frac{x^3+(a/4)y^4}{y^3+(b/4)x^4}$ we set

$$\begin{aligned}
P_3(x,y) &= -x^3, \; Q_3 = y^3, \; p_4 = \sin\varphi P_3(\cos\varphi, \sin\varphi) + \cos\varphi Q_3(\cos\varphi,\sin\varphi), \\
P_4(x,y) &= -\frac{a}{4}y^4, \; Q_4 = \frac{b}{4}x^4, p_5 = \sin\varphi P_4(\cos\varphi,\sin\varphi) + \cos\varphi Q_4(\cos\varphi,\sin\varphi),
\end{aligned}$$

$$\begin{aligned}
q_4 &= \cos\varphi P_3(\cos\varphi,\sin\varphi) - \sin\varphi Q_3(\cos\varphi,\sin\varphi), \\
q_5 &= \cos\varphi P_4(\cos\varphi,\sin\varphi) - \sin\varphi Q_4(\cos\varphi,\sin\varphi).
\end{aligned}$$

Then

$$\begin{aligned}
\mathcal{Z}(\varphi, r) &= -r(\sin\varphi(-P_3)r^3 + \sin\varphi(-P_4)r^4) - r(\cos\varphi Q_3 r^3 + \sin\varphi Q_4 r^4), \\
&= -r(-p_4 r^3 - p_5 r^5) = p_4 r^4 + p_5 r^5, \\
\mathcal{N}(\varphi, r) &= -(\cos\varphi(-P_3)r^3 + \cos\varphi(-P_4)r^4 + \cos\varphi Q_3 r^3 + \cos\varphi Q_4 r^4), \\
&= -(-q_4 r^3 - q_5 r^4) = q_4 r^3 + q_5 r^4,
\end{aligned}$$

$$f'_4 + \frac{4\mathcal{Z}_4}{\mathcal{N}_3} f_4 = 0, \; f'_5 + \frac{5\mathcal{Z}_4}{\mathcal{N}_3} f_5 + \frac{1}{\mathcal{N}_3}(4\mathcal{Z}_5 f_4 + \mathcal{N}_4 f'_4) = c_{4n}$$

$$f'_6 + \frac{6\mathcal{Z}_4}{\mathcal{N}_3} f_6 + \frac{1}{\mathcal{N}_3}(5\mathcal{Z}_5 f_5 + \mathcal{N}_4 f'_5) = c_{4n+1}.$$

We intend to show that $c_{4n+1} \neq 0$ if a, b are chosen appropriately. In terms of the trigonometric polynomials p_i, q_i we have

$$f'_4 + \frac{4p_4}{q_4} f_4 = 0, \tag{3.8}$$

$$f'_5 + \frac{5p_4}{q_4} f_5 + \frac{1}{q_4}(4p_5 f_4 + q_5 f'_4) = c_{4n}. \tag{3.9}$$

$$f'_6 + \frac{6p_4}{q_4} f_6 + \frac{1}{q_4}(5p_5 f_5 + q_5 f'_5) = c_{4n+1}, \tag{3.10}$$

$$
\begin{aligned}
p_4 &= -\cos^3\varphi\sin\varphi + \sin^3\varphi\cos\varphi = \cos\varphi\sin\varphi(\sin^2\varphi - \cos^2\varphi),\\
p_5 &= -\frac{a}{4}\sin^5\varphi + \frac{b}{4}\cos^5\varphi,\\
q_4 &= -\cos^4\varphi - \sin^4\varphi,\\
q_5 &= -\frac{a}{4}\sin^4\varphi\cos\varphi - \frac{b}{4}\cos^4\varphi\sin\varphi,\\
f_4 &= \exp(-\int_0^\varphi (4p_4/q_4)d\psi)(f(0)=1).
\end{aligned}
$$

f_4 is even. p_4, q_4 have period π, p_4/q_4 is odd. Then $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(4p_4/q_4)d\psi = \int_0^\pi(4p_4/p_4)d\psi = 0$ and f_4 is π-periodic. Since p_5, q_5 have degree 5 as polynomials in $\sin\varphi, \cos\varphi$ we have $p_5(\varphi+\pi) = -p_5(\varphi)$, $q_5(\varphi+\pi) = -q_5(\varphi)$. Every solution to (3.9) with $c_{4n} = 0$ is 2π-periodic (cf. section 4). We thus remain with (3.10). It now turns out, after some tedious calculations, that $c_{4n+1} = 0$ for any choice of a, b. As we will show in the next section we have a focus if there is a coefficient $c_{\lambda+4n-1} \neq 0$. If on the contrary all $c_{\lambda+4n-1}$ vanish it should be conjectured that (0,0) is a center. The proof in [1] is not complete however since the lack of convergence of the $\mathcal{F}$-series requires a more detailed discussion. Thus a decision if at (0,0) there is a focus in our particular example is not yet possible. We are going to take up this question in the next section.

4 Examples Second Part

If in Theorem 2 the first nonvashing constant amongst $c_{4n}, c_{4n+1}, \ldots$ is c_{λ_0+4n-1} for some $\lambda_0 \geq 1$ we obtain with $\mathcal{F} = \sum_{\mu=0}^{\lambda_0} f_{\mu+2n} r^{\mu+2n}$

$$
\begin{aligned}
r' - r_1' = \frac{\mathcal{Z}}{\mathcal{N}} + \frac{\partial_\varphi\mathcal{F}}{\partial_r\mathcal{F}} = \frac{\mathcal{Z}\partial_r\mathcal{F} + \mathcal{N}\partial_\varphi\mathcal{F}}{\mathcal{N}\partial_r\mathcal{F}} = \frac{1}{\mathcal{N}\partial_r\mathcal{F}}\Bigg\{ c_{\lambda_0+4n-1} r^{\lambda_0+4n-1} - \sum_{\lambda\geq\lambda_0+1} \\
(\sum_{\kappa=0}^{\lambda_0} \mathcal{Z}_{\lambda+2n-\kappa}(\kappa+2n) f_{\kappa+2n} + \mathcal{N}_{\lambda+2n-1-\kappa} f'_{\kappa+2n}) r^{\lambda+4n} \Bigg\} \\
= \frac{1}{\mathcal{N}\partial_r\mathcal{F}}(c_{\lambda_0+4n-1} r^{\lambda_0+4n-1} + \mathcal{O}(r^{\lambda_0+4n})).
\end{aligned}
$$

Since $f_{2n}(\varphi) = e^{-\int_0^\varphi \frac{2n\mathcal{Z}_{2n}}{\mathcal{N}_{2n-1}} d\psi}$, $\partial_r\mathcal{F} = 2nf_{2n}r^{2n-1} + \ldots$, $\mathcal{N} = \mathcal{N}_{2n-1}r^{2n-1}$ the functions $\partial_r\mathcal{F}, \mathcal{N}$ have positive resp. negative definite lowest order coefficients and we obtain

$$
r' - r_1' = \frac{c_{\lambda_{0+4n}-1}}{2nf_{2n}\mathcal{N}_{2n-1}} r^{\lambda_0+1} + \ldots
$$

Thus the origin is a focus.

We now turn to a sharpened version of Theorem 2. It is due to Frommer [1, p. 413]. A remark on trigonometric polynomials

$$
p_l(\varphi) = \sum_{\alpha_1+\alpha_2=l} c_{\alpha_1\alpha_2} \cos^{\alpha_1}\varphi \sin^{\alpha_2}\varphi,\ c_{\alpha_1\alpha_2} \text{ constant},
$$

of degree l is in order. We have

$$
p_l(\varphi+\pi) = p_l(\varphi),\ l \text{ even}, p_l(\varphi+\pi) = -p_l(\varphi),\ l \text{ odd}. \tag{4.1}
$$

Let l be odd, $f, h : \mathbb{R} \to \mathbb{R}$ continuous and π-periodic with $\int_0^\pi h d\psi = 0$. Then every solution of $(*) y' + hy + p_l f = 0$ is 2π-periodic. This is seen as follows: We have

$$
\begin{aligned}
y'(\varphi+\pi)+hy(\varphi+\pi)+p_l f(\varphi+\pi) &= 0,\\
y'(\varphi+\pi)+h(\varphi)y(\varphi+\pi)-p_l(\varphi)f(\varphi) &= 0,\\
-y'(\varphi)-h(\varphi)y(\varphi)-p_l(\varphi)f(\varphi) &= 0.
\end{aligned}
$$

Thus $y(\varphi+\pi)+y(\varphi)$ solves the homogeneous problem. We obtain

$$
\begin{aligned}
y(\varphi+\pi)+y(\varphi) &= (y(\pi)+y(0))(\exp(-\int_0^{\varphi} hd\psi)),\\
y(\varphi+2\pi)+y(\varphi+\pi) &= (y(\pi)+y(0))(\exp(-\int_0^{\varphi+\pi} hd\psi),\\
&= (y(\pi)+y(0))(\exp(-\int_0^{\varphi} hd\psi).
\end{aligned}
$$

This clearly implies $y(\varphi+2\pi)=y(\varphi)$. Next we show that there is one and only one solution of $(*)$ with $y(\varphi+\pi)+y(\varphi)=0$. The formula for the solution of $(*)$ with initial value $y(-\frac{\pi}{2})$ is

$$
y(\varphi)=y(-\frac{\pi}{2})\exp(-\int_{\frac{\pi}{2}}^{\varphi} h(d\psi)-\int_{-\frac{\pi}{2}}^{\varphi}\exp(-\int_{\widetilde{\varphi}}^{\varphi} hd\psi)p_l f d\widetilde{\varphi}
$$

Thus the desired solution has initial value

$$
\begin{aligned}
y(-\frac{\pi}{2}) &= (1+\exp(-\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} hd\psi))^{-1}\cdot\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\exp(-\int_{\widetilde{\varphi}}^{\frac{\pi}{2}} hd\psi)p_l f d\widetilde{\varphi},\\
&= \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\exp(-\int_{\widetilde{\varphi}}^{\frac{\pi}{2}} hd\psi)p_l f d\widetilde{\varphi}.
\end{aligned}
\tag{4.2}
$$

It is clearly uniquely determined by the requirement $y(\varphi+\pi)+y(\varphi)=0$. Let l be even. f,h as above. Then $p_l f(\varphi+\pi)=p_l f(\varphi)$ and any 2π-periodic solution of $(*)y'+hy+p_l f=0$ is π-periodic. Namely, we have for any solution y the relations

$$y(\varphi)-y(\varphi+\pi) \text{ is } \pi\text{-periodic, thus}$$

$$y(\varphi+\pi)-y(\varphi+2\pi)=y(\varphi)-y(\varphi+\pi)$$

whence by $y(\varphi)=y(\varphi+2\pi)$ it follows

$$y(\varphi)=y(\varphi+\pi)$$

(19) holds correspondingly.
Theorem 3: *There are a uniquely determined even* $\Lambda\in\mathbb{N}\cap\{0,+\infty\}$*, uniquely determined continuously differentiable functions* $\widehat{f}_{2n},\ldots,\widehat{f}_{2n+\Lambda-2},\widehat{f}_{2n+\Lambda-1},\widehat{f}_{2n+1},\widehat{f}_{2n+\Lambda+1},\ldots:\mathbb{R}\to\mathbb{R}$ *and uniquely determined numbers* $\widehat{d}_{4n-1}=0,\ldots,\widehat{d}_{4n+\Lambda-3}=0,\widehat{d}_{4n+\Lambda-2}=0,\widehat{d}_{4n+\Lambda-1}\neq 0,\widehat{d}_{4n+\Lambda},\ldots$ *such that*

$$\widehat{f}_{2n} \textit{ is } \pi\textit{-periodic, } \widehat{f}_{2n}(-\frac{\pi}{2}) = 1 \tag{4.3}$$

$$\widehat{f}_{2n+1}(\varphi+\pi) + \widehat{f}_{2n+1}(\varphi) = 0,\ \widehat{f}_{2n+1} \textit{ is } 2\pi\textit{-periodic,} \tag{4.4}$$

$$\vdots$$

$$\widehat{f}_{2n+\Lambda-2} \textit{ is } \pi\textit{-periodic, } \widehat{f}_{2n+\Lambda-2}(-\frac{\pi}{2}) = 1, \tag{4.5}$$

$$\widehat{f}_{2n+\Lambda-1}(\varphi+\pi) + \widehat{f}_{2n+\Lambda-1}(\varphi) = 0,\ \widehat{f}_{2n+\Lambda-1} \textit{ is } 2\pi\textit{-periodic,} \tag{4.6}$$

$$\widehat{f}_{2n+\Lambda} \textit{ is } 2\pi\textit{-perodic with } \widehat{d}_{4n+\Lambda-1} \neq 0,\ \widehat{f}_{2n+\Lambda}(-\frac{\pi}{2}) = 1, \tag{4.7}$$

$$\widehat{f}_{2n+\Lambda+j} \textit{ is } 2\pi\textit{-periodic with } \widehat{f}_{2n+\Lambda+j}(-\frac{\pi}{2}) = 1,\ j \geq 1, \tag{4.8}$$

the formal power series $\widehat{\mathcal{F}}(\varphi, r) = \sum_{\lambda \geq 2n} \widehat{f}_\lambda(\varphi) r^\lambda$ *satisfies*

$$\det \begin{pmatrix} \partial\varphi\widehat{\mathcal{F}} & \partial_r\widehat{\mathcal{F}} \\ -\mathcal{Z} & \mathcal{N} \end{pmatrix} = \sum_{j=4n+\Lambda-1}^{\infty} \widehat{d}_j r^j \tag{4.9}$$

Proof: We employ (3.2) with $\widehat{d}_{4n+\lambda-1}$, $\widehat{f}_{2n+\lambda}$ instead of $c_{4n+\lambda-1}$, $f_{2n+\lambda}.\mathcal{Z}_{\lambda+2n-\kappa}$, $\mathcal{N}_{\lambda+2n-1-\kappa}$ are homogeneous polynomials in $\cos\varphi$ and $\sin\varphi$ of degree $\lambda + 2n - \kappa$. Then $\mathcal{R}(\varphi) = (\mathcal{N}_{2n-1}(\varphi))^{-1} \cdot \sum_{\kappa=0}^{\lambda-1}(\mathcal{Z}_{\lambda+2n-\kappa}(\varphi)(\kappa+2n)\ldots)$ in (3.2) has the following properties: Let $\kappa = 0, \ldots, \lambda-1$. If

$$\widehat{f}_{2n+\kappa}(\varphi+\pi) + \widehat{f}_{2n+\kappa}(\varphi) = 0,\ \kappa \text{ odd}, \tag{4.10}$$

$$\widehat{f}_{2n+\kappa} \text{ is } \pi\text{-periodic, } \kappa \text{ even}, \tag{4.11}$$

then for λ **odd** we have $\mathcal{R}(\varphi+\pi) + \mathcal{R}(\varphi) = 0$. Moreover there is one and only one constant $\widehat{d}_{\lambda+4n-1} = c_{\lambda+4n-1}$ such that every solution of (3.2) is 2π-periodic. This is in fact equivalent to (3.2) having one 2π-periodic solution. Cf. (3.7). $\widehat{d}_{\lambda+4n-1}$ vanishes if and only if there is an $\widehat{f}_{2n+\lambda}$ with $\widehat{f}_{2n+\lambda}(\varphi+\pi) + \widehat{f}_{2n+\lambda}(\varphi) = 0$ and this particular one is uniquely determined. Now let λ **be even**. Then (4.10, 4.11) imply $\mathcal{R}(\varphi+\pi) - \mathcal{R}(\varphi) = 0$ and there is a uniquely determined constant $\widehat{d}_{\lambda+4n-1} = c_{\lambda+4n-1}$ such that every solution (equivalent: one solution) of (3.2) is π-periodic. As it is evident, any solution $\widehat{f}_{2n}$ of (3.3) is π-periodic and $\widehat{d}_{4n-1} = 0$. Now let us consider (3.5). We have $\mathcal{R}(\varphi+\pi) = \mathcal{R}(\varphi) = 0$. Thus

$$\begin{aligned}
\int_0^{2\pi} e^{\int_0^\varphi \frac{(1+2n)\mathcal{Z}_{2n}(\psi)}{\mathcal{N}_{2n-1}(\psi)}d\psi}\mathcal{R}(\varphi)d\varphi &= \int_0^{\pi} e^{\int_0^\varphi \frac{(1+2n)\mathcal{Z}_{2n}(\psi)}{\mathcal{N}_{2n-1}(\psi)}d\psi}\mathcal{R}(\varphi)d\varphi + \int_\pi^{2\pi} e^{\int_0^\varphi \frac{(1+2n)\mathcal{Z}_{2n}(\psi)}{\mathcal{N}_{2n-1}(\psi)}d\psi}\mathcal{R}(\varphi)d\varphi, \\
&= \int_0^{\pi} e^{\int_0^\varphi \frac{(1+2n)\mathcal{Z}_{2n}(\psi)}{\mathcal{N}_{2n-1}(\psi)}d\psi}(\mathcal{R}(\varphi) + \mathcal{R}(\varphi+\pi))d\varphi, && (4.12) \\
&= 0, && (4.13) \\
\widehat{d}_{4n} &= 0
\end{aligned}$$

Now $\widehat{f}_{2n}$ with $\widehat{f}_{2n}(-\frac{\pi}{2}) = 1$ and f_{2n+1} with $f_{2n+1}(\varphi+\pi) + f_{2n+1}(\varphi) = 0$ are plugged in into (3.2) for f_{2n+2}. If $\widehat{d}_{2+4n-1} \neq 0$ the point $(0,0)$ is a focus and we proceed as indicated in the Theorem ($\Lambda = 2$). If $\widehat{d}_{2+4n-1} = 0$ we proceed with $\widehat{f}_{2n+3}$ and find as in (4.12, 4.13) that $\widehat{d}_{3+4n-1} = 0$. In general if λ is odd we have $\widehat{d}_{\lambda+4n-1} = 0$ if (4.10) and (4.11) are satisfied. Thus the first $\widehat{d}_j$ which does not vanish has the form $\widehat{d}_{4n+\Lambda-1}$ with Λ even. □

As in the beginning of the present section one can show that if there is a first $\widehat{d}_{4n+\Lambda-1} \neq 0$ then the origin is a focus for $y' = -\frac{x^{2n-1}+P(x,y)}{y^{2n-1}+Q(x,y)}$. Now we consider $y' = -\frac{x^3+\frac{a}{4}y^4}{y^3+\frac{b}{4}x^4}$. By some lengthy calculations we again end up with $\widehat{d}_{4n+\Lambda-1} = \widehat{d}_9 = 0$. $(0,0)$ is however likely a focus, at least for appropriate values of a, b. This can be seen from the computer-graphics to follow. They show the integral curves in the x, y-space for initial values $(0,2;0)$, $(0,1;0)$ and $(0,1;0,1)$.

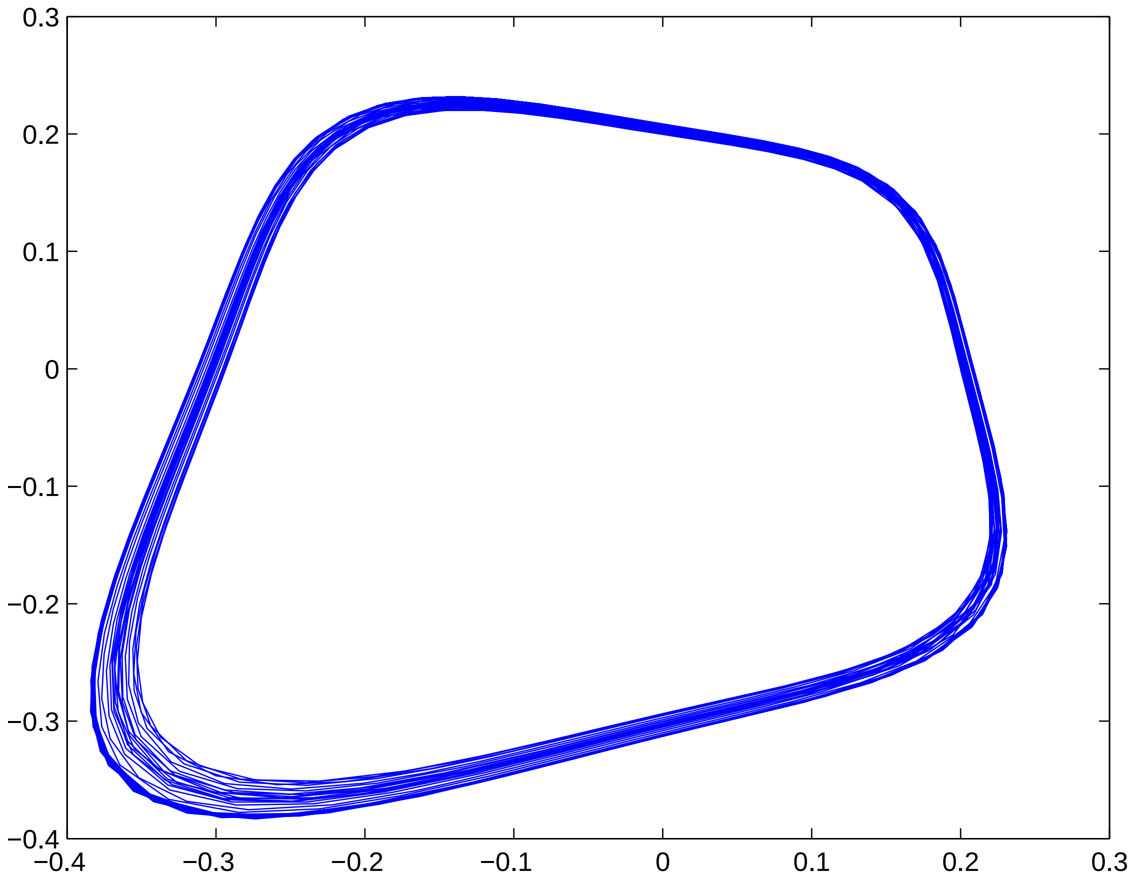

Figure 1: $y' = -\frac{x^3+y^4}{y^3+x^4}$ in $(0.2, 0)$

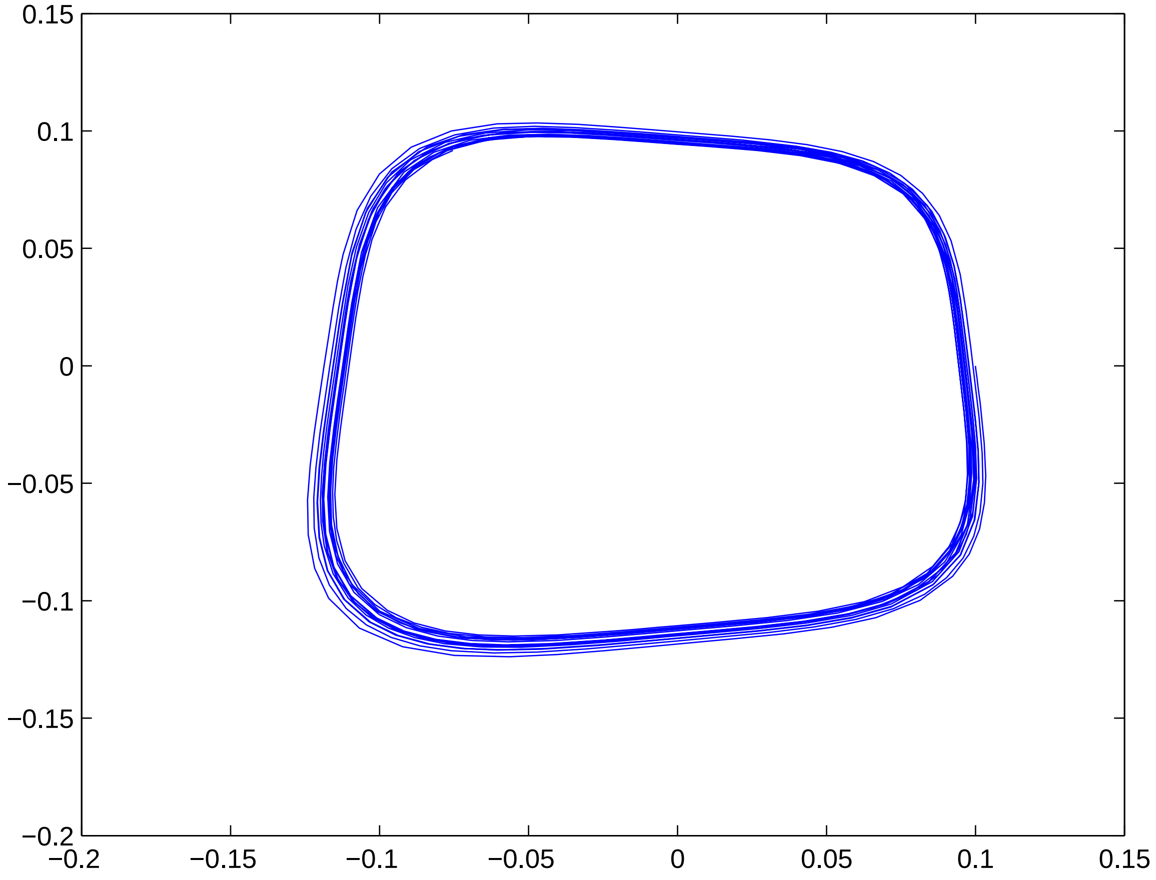

Figure 2: $y' = -\frac{x^3+y^4}{y^3+x^4}$ in $(0.1, 0)$

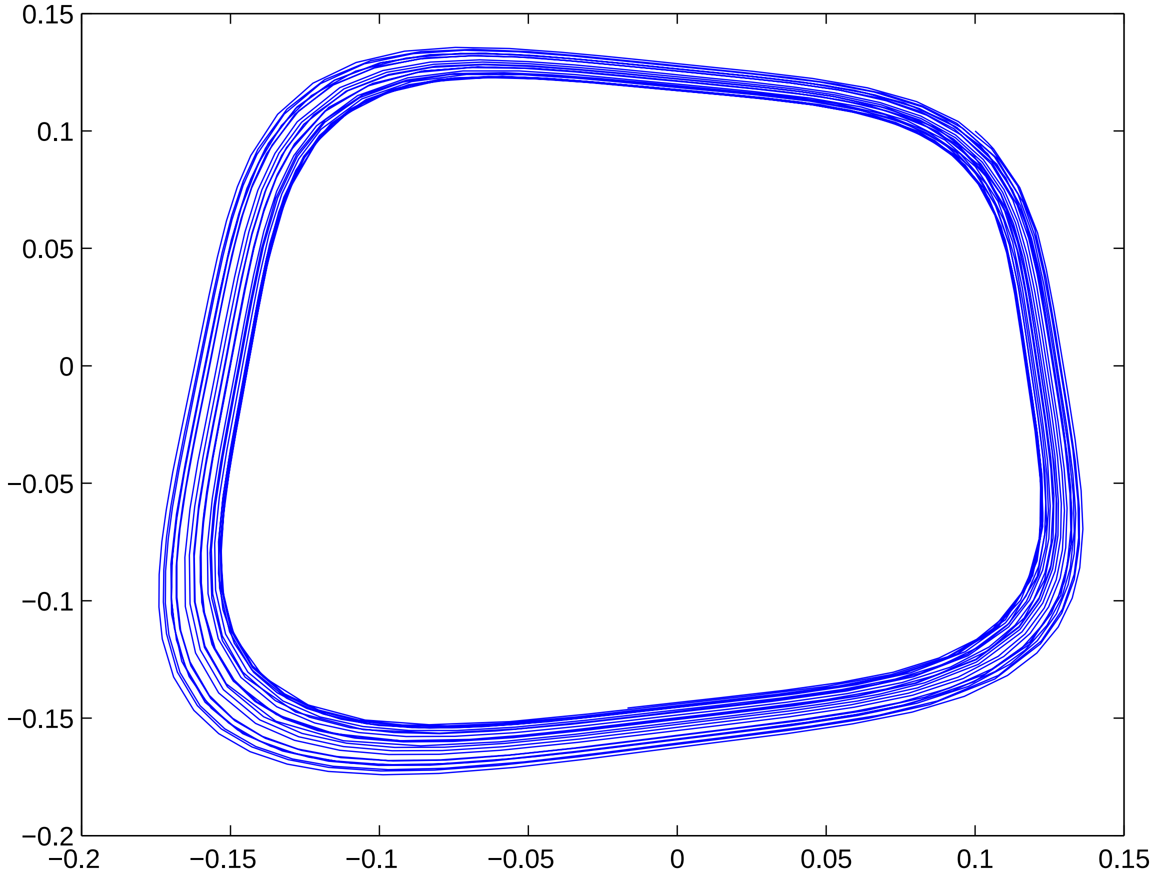

Figure 3: $y' = -\frac{x^3+y^4}{y^3+x^4}$ in $(0.1, 0.1)$

References

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