On the global existence of Euler's multiplier

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Abstract

We study the question of global existence of Euler's multiplier in any dimension under certain geometrical conditions.

1. Introduction

First order differential equations of the form

$$(1) \quad f(x,y)dx + g(x,y)dy = 0$$

sometimes can be solved by finding a so called *integrating factor,* i.e. some function $\lambda(x,y)$ such that there exists a C^1-function $Q(x,y)$ with the property

$$dQ(x,y) = \lambda(x,y)f(x,y)dx + \lambda(x,y)g(x,y)dy$$

with the consequence that equation (1) can also be written in the form

$$Q(x,y) = const,$$

which can also be considered as an implicit solution of (1).

It is well known that, if $f(x,y) \neq 0$ or $g(x,y) \neq 0$ there exists an open neighbourhood U of (x,y) and nontrivial functions $\lambda \in C^0(U,\mathbb{R})$, $Q \in C^1(U,\mathbb{R})$ such that

$$dQ(x,y) = \lambda(x,y)f(x,y)dx + \lambda(x,y)g(x,y)dy \quad \text{in } U,$$

see e.g. [von Westenholz]. The question of zeros of λ is treated in [Mařík].

In n dimensions the question of existence of the functions λ and Q is more complicated.

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Let $U \subset \mathbb{R}^n$, $A \in C^0(\overline{U}, \mathbb{R}^n)$ be any vector field. Now we are looking for nontrivial functions $Q \in C^1(\overline{U}, \mathbb{R})$, $\lambda \in C^0(\overline{U}, \mathbb{R})$ such that

$$\forall x \in U: \quad \nabla Q(x) = \lambda(x) A(x),$$

where λ is then called Euler's multiplier. It is easy to verify that, concerning the corresponding Pfaffian form $\omega = \sum_{j=1}^{n} A_j dx_j$, the condition

$$(2) \quad \omega \wedge d\omega = 0$$

is necessary for the existence of such functions. Conversely, if condition (2) is satisfied and $A(x) \neq 0$, Frobenius theory guarantees the local existence of λ and Q in the sense of above, see [Gerlich],[Grauert] and [Holmann]. But in general, this does not imply the global existence of λ and Q, even if $A(x) \neq 0$ and (2) are globally fulfilled.

In this article we apply the local result and present two geometrical conditions which are sufficient for the global existence of λ and Q considering the so called integral surfaces. These are surfaces whose tangential spaces are pointwise the orthogonal complement of the field vector $A(x)$. The geometrical conditions are the following:

(i) In the first case we assume that A has an isolated critical point in 0 and in some neighbourhood of 0 all integral surfaces are closed, more precisely, each integral surface Γ is the boundary of some domain $\mathcal{G}$ with $0 \in \mathcal{G}$. This situation is related to a problem of Poincaré who maintains the existence of Q and λ in the class of analytic functions around 0, which, in general, is not true (see [Frommer, §2a)]).

(ii) In the second case we suppose that one component of the vector field A is bounded from below. In more than two dimensions additionally we first assume U to be an infintite cylinder and the component A_{n+1} along the cylinder axis to be bounded from below. In a second step this can be carried over to the more general case of a tube which is C^2-diffeomorphic to the infinite cylinder. Under this condition we can show the global existence of integral surfaces in the sense above which then leads us to a globalization of Q and λ.

In both cases the condition (2) implies then the global existence of λ and Q. The two dimensional cases are considered particularly, because there the proofs are much easier.

Concerning the physical application of this result, we note that the global existence of Euler's multiplier for a given vector field A means that this field is everywhere parallel to the gradient of some function, i.e. the field line image of A then is the same as of some potential field. This may be useful in some questions of mathematical physics.

2. The 2-dimensional case

Existence of Euler's multiplier in the case of closed integral curves surrounding a critical point

2.1. Assumptions: *Let $U \subset \mathbb{R}^2$ be an open set, $0 \in U$ and $A = (a_1, a_2) \in C^1(\overline{U}, \mathbb{R}^2)$ with the properties:*

$$A(0) = 0, \quad \forall x \in \overline{U} \setminus \{0\} : A(x) \neq 0.$$

In addition to this we suppose all integral curves to be closed, i.e. all solutions $x = (x_1, x_2)$ of the differential equation system

$$(3) \quad \begin{array}{rcl} \dot{x}_1(t) & = & a_2(x(t)) \\ \dot{x}_2(t) & = & -a_1(x(t)) \end{array}$$

to be periodic in t. Without restriction we may assume ∂U to be an integral curve and all solutions of (3) to have winding number 1 with respect to 0.

2.2. Notations: In what follows let $x(.)$ denote a solution of equation (3), where $\dot{}$ means $\frac{d}{dt}$ or $\frac{\partial}{\partial t}$, respectively. On the other hand, any solution of the field line equation

$$(4) \quad \mathrm{X}'(s) = A(\mathrm{X}(s))$$

will be denoted by $\mathrm{X}(.)$, where $'$ means differentiation with respect to parameter s. Let $\langle .,. \rangle$ denote the Euklidian scalar product in $\mathbb{R}^2$.

Obviously we have $\langle \dot{x}(t), \mathrm{X}'(s) \rangle = 0$ wherever $x(t) = \mathrm{X}(s)$.

2.3. Lemma: *Under the assumptions 2.1. every field line cuts every integral curve in exactly one point.*

PROOF: (a) According to Jordan's theorem the plane $\mathbb{R}^2$ is decomposed uniquely into an interior and an exterior domain by each closed integral curve. The outer normal is given by $-\frac{A}{|A|}$. As every field line penetrating an integral curve has the orientation of the inner normal, a field line can never leave such an interior domain.
(b) Now let $\mathrm{X} :]\sigma_0, +\infty[\rightarrow U \setminus \{0\}$ be any field line of A. We are going to show that $\lim_{s \to \infty} \mathrm{X}(s) = 0$.
We assume the existence of a closed intergral curve $x : [0, T[\rightarrow U \setminus \{0\}$, $\Gamma_i := x([0, T[)$, with

$$(5) \quad \text{trace } \mathrm{X} \cap \Gamma_i = \emptyset \quad \text{but} \quad \lim_{s \to \infty} \text{dist}(\mathrm{X}(s), \Gamma_i) = 0.$$

Let $\mathcal{G}_i$ be the interior of Γ_i and Γ_h an integral line with interior domain $\mathcal{G}_h$, $\mathcal{G}_i \subset \mathcal{G}_h$, where $\text{dist}(\Gamma_i, \Gamma_h)$ is assumed to be sufficiently small. Then for an appropriate small $\eta > 0$ the mapping

$$\beta : [0, T[\times] - \eta, +\eta[\rightarrow U, \quad (t, \varrho) \mapsto x(t) + \varrho A(x(t))$$

is a unique parametrization of a neighbourhood of Γ_i, such that

$$S := \mathcal{G}_h \setminus \mathcal{G}_i$$

is a subset of $\beta\big([0, T[\times[-\eta, 0]\big)$. Moreover there exist some $\epsilon > 0$ satisfying

(6) $\forall x \in S : |A(x)| \geq \epsilon$

and some $\sigma \in \,]\sigma_0, +\infty[$ such that

$$\forall s \geq \sigma : \mathrm{X}(s) \in S.$$

Now let $t(s), \varrho(s)$ be the coordinates of $\mathrm{X}(s)$ with respect to β. Then we have

(7) $\varrho'(s) \leq 0$

and

(8)
$$\begin{aligned}
\mathrm{X}'(s) = \big(\beta(t, \varrho)\big)'(s) = \\
= \dot{x}(t(s))t'(s) + \varrho(s)DA\big(x(t(s))\big)\dot{x}(t(s))t'(s) + \varrho'(s)A\big(x(t(s))\big).
\end{aligned}$$

Because of eq. (5) and (6) we have

(9) $\varrho(s) \to 0, \ s \to +\infty$

and therefore

$$\varrho(s)DA\big(x(t(s))\big)\dot{x}(t(s))t'(s) \to 0, \ s \to +\infty.$$

From eq. (8) we conclude

$$\varrho'(s)\big\langle A\big(x(t(s))\big), A\big(\mathrm{X}(s)\big)\big\rangle - \big\langle \mathrm{X}'(s), A\big(\mathrm{X}(s)\big)\big\rangle \to 0, \ s \to +\infty,$$

and thus, applying eq. (5),

$$\lim_{s \to \infty} \varrho'(s) = \lim_{s \to \infty} \frac{|A\big(\mathrm{X}(s)\big)|^2}{\big\langle A\big(x(t(s))\big), A\big(\mathrm{X}(s)\big)\big\rangle} = 1,$$

which contradicts (7) and (9). $\square$

An immediate consequence of this is the following lemma.

2.4. Lemma: *Let the assumptions in 2.1. be satisfied. Let* $\mathrm{X} : \,]\sigma_0, +\infty[\,\to U$ *be any field line of* A, *where* $\mathrm{X}(\sigma_0 - 0) \in \partial U$. *Let* $x(., s)$ *be the unique solution of the initial value problem* (3), $x(0, s) = \mathrm{X}(s)$ *for* $s \in \,]\sigma_0, +\infty[$ *with periodicity length* $T(s) > 0$. *Then the mapping*

$$x(.,.) : [0, T(s)[\times \,]\sigma_0, +\infty[\,\to U \setminus \{0\}, \ (t, s) \mapsto x(t, s)$$

is bijective. Moreover, $x(.,.)$ *is periodically extendable with respect to* s *to some function* $x(.,.) \in C^1\big([\sigma_0, +\infty[\times\mathbb{R}, \overline{U}\big)$.

PROOF: The regularity result follows from the theory of ordinary differential equations. □

Now we can prove the first main result of this paragraph.

2.5. Theorem: *Let the assumptions of 2.1. be satisfied. Then there exist $Q \in C^1(\overline{U}, \mathbb{R})$ and $\lambda \in C^0(\overline{U}, \mathbb{R})$ with the properties*

$$\nabla Q(x) = \lambda(x) A(x) \ \text{in} \ U, \quad \lambda(x) \neq 0 \ \text{in} \ U \setminus \{0\}, \quad \lambda(0) = 0.$$

PROOF: (a) First we show that the perodicity length $T(s)$ is continuously differentiable with respect to parameter s.
$T(s)$ is the smallest value $t > 0$ with the property

(10) $\quad x(t, s) = x(0, s) = \mathrm{X}(s).$

Because of $\dot{x}(t, s) \neq 0$ in a sufficiently small neighbourhood of $t = T(s)$ there exists exactly one t with this property, where s is fixed. Now let $s_0 \in]\sigma_0, +\infty[$, $t_0 = T(s_0)$. Since $A(x) \neq 0$ where $x \neq 0$, we may assume without restriction

$$\dot{x}_1(t_0, s_0) = a_2\big(x(t_0, s_0)\big) \neq 0.$$

For (t, s) in some neighbourhood of (t_0, s_0) we define

$$\Phi(t, s) := x_1(t, s) - x_1(0, s).$$

Then Φ is continuously partially differentiable satisfying

$$\Phi(t_0, s_0) = 0, \quad \partial_t \Phi(t_0, s_0) = \dot{x}_1(t_0, s_0) \neq 0.$$

The implicit function theorem yields the existence of $\delta_1, \delta_2 > 0$ and some function $\tau \in C^1\Big( ]s_0 - \delta_2, s_0 + \delta_2[\, , \,]t_0 - \delta_1, t_0 + \delta_1[\Big)$ which fulfills the condition

$$\forall (t, s) \in \,]t_0 - \delta_1, t_0 + \delta_1[\times]s_0 - \delta_2, s_0 + \delta_2[\ : \ \Phi(t, s) = 0 \Leftrightarrow t = \tau(s).$$

As $t = \tau(s)$ is uniquely determined, in a small neighbourhood eq. (10) is equivalent to $\Phi(t, s) = 0$. Thus, for s near s_0 we have $T(s) = \tau(s)$ and $T(.)$ is continuously differentiable.
(b) Now we modify our parametrization of $\overline{U} \setminus \{0\}$ by scaling the periodicity length of x to 1. For $(t, s) \in \mathbb{R} \times]\sigma_0, +\infty[$ let us define

$$\tilde{x}(t, s) := x\big(T(s)t, s\big).$$

Thus $\tilde{x}(., s)$ is the unique solution of the initial value problem

(11) $\quad \dot{\tilde{x}}(t, s) = T(s) \begin{pmatrix} a_2(\tilde{x}(t, s)) \\ -a_1(\tilde{x}(t, s)) \end{pmatrix}, \quad \tilde{x}(0, s) = \tilde{x}(1, s) = \mathrm{X}(s)$

and

$$\tilde{x} : [0,1[\times]\sigma_0,+\infty[\ \to\ U\setminus\{0\}, \ \ (t,s)\ \mapsto\ \tilde{x}(t,s)$$

is a C^1-mapping with a periodic C^1-extension to $\mathbb{R}\times]\sigma_0,+\infty[$. Because of Lemma 2.4. s is uniquely determined by x or $\tilde{x}$, respectively, so we may define

$$\mathfrak{A}\big(\tilde{x}(t,s)\big) \ :=\ T(s)\begin{pmatrix} a_2\big(\tilde{x}(t,s)\big) \\ -a_1\big(\tilde{x}(t,s)\big) \end{pmatrix}.$$

Since $\mathfrak{A}$ is continuously differentiable, we know from the theory of ordinary differential equations that $\dot{\tilde{x}}(.,s)$ and $\tilde{x}'(.,s)$ are solutions of the linear differential equation

$$\dot{w}(t,s) \ =\ D\mathfrak{A}\big(\tilde{x}(t,s)\big)w(t,s)$$

and therefore we have

$$\big(\dot{\tilde{x}}_1\tilde{x}'_2-\dot{\tilde{x}}_2\tilde{x}'_1\big)(t,s) \ =\ \det\begin{pmatrix} \dot{\tilde{x}}_1 & \tilde{x}'_1 \\ \dot{\tilde{x}}_2 & \tilde{x}'_2 \end{pmatrix}(t,s) \ =$$

$$=\ \det\begin{pmatrix} \dot{\tilde{x}}_1 & \tilde{x}'_1 \\ \dot{\tilde{x}}_2 & \tilde{x}'_2 \end{pmatrix}(0,s)\cdot\exp\left(\int\limits_0^t \mathrm{trace}\,D\mathfrak{A}\big(\tilde{x}(\tau,s)\big)d\tau\right)\ \neq\ 0$$

since $\tilde{x}'(0,s)=\dot{x}(0,s)\,T'(s)+x'(0,s)$ and $x'(0,s)=\mathrm{X}'(s)=A\big(\mathrm{X}(s)\big)$. Thus also the corresponding inverse mapping

$$(12)\ \ \tilde{x}^{-1} : U\setminus\{0\} \ \to\ [0,1[\ \times\ ]\sigma_0,+\infty[$$

is continuously differentiable.

(c) In order to construct the desired function Q we choose any smooth function $\kappa : \ ]\sigma_0,+\infty[\ \to\ \mathbb{R}$ with the following properties

$$(13)\quad \forall s\in\,]\sigma_0,+\infty[\ :\ \kappa'(s)>0,\qquad \lim_{s\to+\infty}\kappa(s)=:c\in\mathbb{R}$$

$$\lim_{s\to+\infty}\frac{\kappa'(s)T(s)}{\big(\dot{\tilde{x}}_1\tilde{x}'_2-\dot{\tilde{x}}_2\tilde{x}'_1\big)(t,s)}=0,\quad \lim_{s\to+\infty}\frac{\kappa'(s)\dot{\tilde{x}}_j(t,s)}{\big(\dot{\tilde{x}}_1\tilde{x}'_2-\dot{\tilde{x}}_2\tilde{x}'_1\big)(t,s)}=0,\ j=1,2,$$

where the limites are supposed to be uniform with respect to $t\in\mathbb{R}$. According to (12) we can define

$$Q : \ U\setminus\{0\}\to\mathbb{R}, \ Q\big(\tilde{x}(t,s)\big):=\kappa(s).$$

Therefore $Q(\tilde{x}(.,.))$ is continuously partially differentiable and 1-periodic with respect to t, and thus $Q\in C^1\big(U\setminus\{0\},\mathbb{R}\big)$. The chain rule yields

$$\kappa'(s) = \frac{\partial Q}{\partial x_1}\big(\tilde{x}(t,s)\big)\tilde{x}'_1(t,s) + \frac{\partial Q}{\partial x_2}\big(\tilde{x}(t,s)\big)\tilde{x}'_2(t,s),$$

$$0=\frac{\partial\kappa(s)}{\partial t} = \frac{\partial Q}{\partial x_1}\big(\tilde{x}(t,s)\big)\dot{\tilde{x}}_1(t,s) + \frac{\partial Q}{\partial x_2}\big(\tilde{x}(t,s)\big)\dot{\tilde{x}}_2(t,s),$$

and thus

$$(14) \quad \begin{aligned} \frac{\partial Q}{\partial x_1}\big(\tilde{x}(t,s)\big) &= \frac{-\dot{\tilde{x}}_2(t,s)}{\big(\dot{\tilde{x}}_1\tilde{x}_2' - \tilde{x}_1'\dot{\tilde{x}}_2\big)(t,s)} \cdot \kappa'(s), \\ \frac{\partial Q}{\partial x_2}\big(\tilde{x}(t,s)\big) &= \frac{\dot{\tilde{x}}_1(t,s)}{\big(\dot{\tilde{x}}_1\tilde{x}_2' - \tilde{x}_1'\dot{\tilde{x}}_2\big)(t,s)} \cdot \kappa'(s), \end{aligned}$$

for $t \in \mathbb{R}$, $s \in \,]\sigma_0, +\infty[$. The assumptions (13) yield $\lim_{s\to\infty} \nabla Q(\tilde{x}(t,s)) = 0$ uniformly with respect to $t \in \mathbb{R}$, thus

$$\lim_{x\to 0} \nabla Q(x) = 0, \quad \text{and} \quad \lim_{x\to 0} Q(x) = c.$$

Q therefore has a C^1-extension to U with $Q(0) = c$, $\nabla Q(0) = 0$. Using this and (14) we obtain

$$\nabla Q\big(\tilde{x}(t,s)\big) = \frac{\kappa'(s)T(s)}{\big(\dot{\tilde{x}}_1\tilde{x}_2' - \tilde{x}_1'\dot{\tilde{x}}_2\big)(t,s)} \cdot A\big(\tilde{x}(t,s)\big).$$

Now define $\lambda : U \to \mathbb{R}$ by

$$\lambda\big(\tilde{x}(t,s)\big) := \frac{\kappa'(s)T(s)}{\big(\dot{\tilde{x}}_1\tilde{x}_2' - \tilde{x}_1'\dot{\tilde{x}}_2\big)(t,s)} \neq 0, \quad \lambda(0) := 0.$$

Because of eq. (13) λ is continuous and we arrive at

$$\forall x \in U: \quad \nabla Q(x) = \lambda(x)A(x). \qquad \square$$

The existence of Euler's multiplier in the case of one component which is bounded from below

Now we consider the case that one component of the vector field A is bounded from below. For our convenience we will change our notation and write (x, y) instead of (x_1, x_2) and (f, g) instead of (a_1, a_2).

2.6. Assumptions: *Let $G \subset \mathbb{R}^2$ be any domain, $f, g \in C^1(G, \mathbb{R})$. We consider the Pfaffian form*

$$\omega(x, y) = f(x, y)dx + g(x, y)dy.$$

The condition $\omega \wedge d\omega = 0$ is in two dimensions always satisfied.

2.7. Theorem: *Let $\varepsilon > 0$, let G_ε be an open subset of G with the property*

$$\forall (x, y) \in \overline{G_\varepsilon}: \quad g(x, y) \geq \varepsilon.$$

Then there exist $\lambda_\varepsilon \in C^0(\overline{G_\varepsilon}, \mathbb{R})$, $Q_\varepsilon \in C^1(\overline{G_\varepsilon}, \mathbb{R})$ satisfying

$$\forall (x, y) \in G_\varepsilon: \quad \begin{pmatrix} \partial_x Q_\varepsilon(x, y) \\ \partial_y Q_\varepsilon(x, y) \end{pmatrix} = \lambda_\varepsilon(x, y) \begin{pmatrix} f(x, y) \\ g(x, y) \end{pmatrix} \neq 0.$$

PROOF: (a) We consider the restrictions $f|_{\overline{G_\epsilon}}, g|_{\overline{G_\epsilon}} \in C^1(\overline{G_\epsilon}, \mathbb{R})$ and construct extensions $\bar{f}, \bar{g} \in C^1_{\text{unif}}(\mathbb{R}^2, \mathbb{R})$ with the property

$$(15) \quad \forall (x,y) \in \mathbb{R}^2 : \quad \bar{g}(x,y) \geq \frac{\varepsilon}{2}.$$

In particular, $\bar{f}, \bar{g}$ are globally bounded and uniformely Lipschitz-continuous. (Note that $\bar{f}|_{G_\epsilon} = f|_{G_\epsilon}$, $\bar{g}|_{G_\epsilon} = g|_{G_\epsilon}$, but in general $\bar{f}|_{G \backslash G_\epsilon} \neq f|_{G \backslash G_\epsilon}, \bar{g}|_{G \backslash G_\epsilon} \neq g|_{G \backslash G_\epsilon}$.) Those conditions guarantee the global existence of all solutions of

$$(16) \quad \begin{aligned} \dot{x}(t) &= \bar{g}(x(t), y(t)), \\ \dot{y}(t) &= -\bar{f}(x(t), y(t)). \end{aligned}$$

Because of (15) for every solution of (16) the function $x(.)$ is strictly montonically increasing in $\mathbb{R}$, where $0 < c_1 < \dot{x}(.) < c_2$ for some constants $c_1, c_2 \in \mathbb{R}_+$. Therefore $x(.) : \mathbb{R} \to \mathbb{R}, t \mapsto x(t)$ is bijective and the inverse mapping $x^{-1}(.)$ is differentiable.

Thus we can write $y(x) := y(t)$, where $x = x(t)$. Obviously the trace of the soulution of the initial value problem (16), $x(0) = x_0$, $y(0) = y_0$ is equal to the graph of the solution of the initial value problem

$$(17) \quad y'(x) = -\frac{\bar{f}}{\bar{g}}(x, y(x)), \quad y(x_0) = y_0,$$

where y' means $\frac{dy}{dx}$. For $(x_0, y_0) \in \mathbb{R}^2$ the unique solution of (17) will be denoted by $u(., x_0, y_0)$ and exists for all $x \in \mathbb{R}$.

(b) Now let $\kappa \in C^1(\mathbb{R}, \mathbb{R})$ be any smooth function with $\forall y \in \mathbb{R} : \kappa'(y) > 0$. For $(x, y) \in \mathbb{R}^2$ we define

$$Q(x, y) := \kappa(u(0, x, y)).$$

That means, if the solution $u(., x, y)$ cuts the y-axis in $(0, y_0)$, so we set $Q(x, y) := \kappa(y_0)$. From the theory of ordinary differential equations we know that $u(., x, y)$ is continuously differentiable with respect to x and y, and the chain rule thus yields $Q \in C^1(\mathbb{R}^2, \mathbb{R})$. The function Q is constant along the soulution curves of (16), (17), respectively.

Now we will show that for all $(x, y) \in \mathbb{R}^2$

$$\nabla Q(x,y) \perp \begin{pmatrix} \bar{g} \\ -\bar{f} \end{pmatrix} (x,y) \quad \text{and therefore} \quad \nabla Q(x,y) \parallel \begin{pmatrix} \bar{f} \\ \bar{g} \end{pmatrix} (x,y).$$

Suppose $(x_0, y_0) \in \mathbb{R}^2$. Then we have

$$\partial_x Q(x_0, y_0) \bar{g}(x_0, y_0) \; - \; \partial_y Q(x_0, y_0) \bar{f}(x_0, y_0) \; =$$

$$= \; \bar{g}(x_0, y_0) \Big(\partial_x Q(x_0, y_0) - \partial_y Q(x_0, y_0) \cdot \frac{\bar{f}}{\bar{g}}(x_0, y_0) \Big) \; =$$

$$= \; \bar{g}(x_0, y_0) \Big(\partial_x Q\big(x_0, u(x_0, x_0, y_0)\big) -$$

$$- \; \partial_y Q\big(x_0, u(x_0, x_0, y_0)\big) \cdot \frac{\bar{f}}{\bar{g}}\big(x_0, u(x_0, x_0, y_0)\big) \Big) \; =$$

$$= \; \bar{g}(x_0, y_0) \Big(\partial_x Q\big(x_0, u(x_0, x_0, y_0)\big) +$$

$$+ \; \partial_y Q\big(x_0, u(x_0, x_0, y_0)\big) \cdot u'(x_0, x_0, y_0) \Big) \; =$$

$$= \; \bar{g}(x_0, y_0) \cdot \frac{d}{dx} \Big(Q\big(x, u(x, x_0, y_0)\big) \Big) \Big|_{x=x_0} \; =$$

$$= \; \bar{g}(x_0, y_0) \cdot \frac{d}{dx} \Big(\kappa\big(u(0, x, u(x, x_0, y_0))\big) \Big) \Big|_{x=x_0} \; =$$

$$= \; \bar{g}(x_0, y_0) \cdot \frac{d}{dx} \Big(\kappa\big(u(0, x_0, y_0)\big) \Big) \Big|_{x=x_0} \; = \; 0.$$

Therefore and since $\bar{g}(x, y) \neq 0$ there exists some $\lambda(x, y) \in \mathbb{R}$ satisfying

$$(18) \quad \begin{aligned} \frac{\partial Q}{\partial x}(x, y) &= \lambda(x, y) \bar{f}(x, y), \\ \frac{\partial Q}{\partial y}(x, y) &= \lambda(x, y) \bar{g}(x, y). \end{aligned}$$

From $(18)_2$ follows $\lambda \in C^0(\mathbb{R}^2, \mathbb{R})$.

(c) Finally we have to show that $\partial_y Q(x, y) \neq 0$ and therefore $\lambda(x, y) \neq 0$. We have

$$(19) \quad \frac{\partial Q}{\partial y}(x, y) \; = \; \frac{\partial}{\partial y}\Big(\kappa\big(u(0, x, y)\big) \Big) \Big|_{x, y} \; = \; \kappa'\big(u(0, x, y)\big) \cdot \frac{\partial u}{\partial y_0}(0, x, y),$$

where $\frac{\partial u}{\partial y_0}(., x_0, y_0)$ is the unique solution of the linear initial value problem

$$w'(x) \; = \; \frac{\partial}{\partial y}\Big(-\frac{\bar{f}}{\bar{g}}\Big)\big(x, u(x, x_0, y_0)\big) \cdot w(x), \quad w(x_0) = 1.$$

Thus we have

$$\frac{\partial u}{\partial y_0}(x, x_0, y_0) \; = \; \underbrace{\frac{\partial u}{\partial y_0}(x_0, x_0, y_0)}_{1} \cdot \int_{x_0}^{x} \exp\Big\{ \frac{\partial}{\partial y}\Big(-\frac{\bar{f}}{\bar{g}}\Big)\big(\xi, u(\xi, x_0, y_0)\big) \Big\} \, d\xi,$$

and therefore

$$\frac{\partial u}{\partial y_0}(0, x, y) \; = \; \int_{x_0}^{0} \exp\Big\{ \frac{\partial}{\partial y}\Big(-\frac{\bar{f}}{\bar{g}}\Big)\big(\xi, u(\xi, x, y)\big) \Big\} \, d\xi \; \neq \; 0.$$

Now eq. (19) yields $\partial_y Q(x, y) \neq 0$, and $Q_\varepsilon := Q|_{\overline{G_\varepsilon}}$, $\lambda_\varepsilon := \lambda|_{\overline{G_\varepsilon}}$ have the desired property. $\qquad \square$

3. The n-dimensional case

The existence of Euler's multiplier in the case of closed integral surfaces inclosing a critical point

3.1. Assumptions: *Let $G \subset \mathbb{R}^n$ be any domain with $0 \in G$. Let $A = (A_1, ..., A_n) \in C^1(G, \mathbb{R}^n)$ with*

$$\forall x \in G: \quad A(x) \neq 0, \qquad A(0) = 0.$$

Let the Pfaffian form $\omega := \sum_{j=1}^n A_j dx_j$ satisfy

$$\omega \wedge d\omega = 0.$$

In some open neighbourhood $U \subset G$ with $0 \in U$ we assume for all $p_0 \in U \setminus \{0\}$

$$\mathfrak{F}_{p_0} := \{p \in G | \exists \text{ piecewise continuously differentiable curve } \alpha$$
$$\text{connecting } p_0 \text{ and } p, \text{ where trace } \alpha \subset G,$$
$$\text{and } \alpha \text{ is everywhere orthogonal to the field vector } A,$$
$$\text{i.e. } \langle \dot{\alpha}, A(\alpha) \rangle = 0\}$$

to be the smooth boundary of some simply connected domain. Without restriction, we assume $\partial U = \mathfrak{F}_{p_0}$ for some $p_0 \in \partial U$, and we suppose $\langle A, \nu \rangle|_{\partial U} < 0$, where ν is the outer normal on ∂U with respect to U.

Obviously, $\mathfrak{F}_{p_0}$ is pathwise connected and by $p \sim p_0 :\Leftrightarrow p \in \mathfrak{F}_{p_0}$ an equivalence relation is defined.

3.2. Notations: For any $q \in \overline{U}$ $\gamma(., q)$ shall denote the unique solution of the initial value problem

$$(20) \quad \dot{\gamma} = A(\gamma), \quad \gamma(0) = q.$$

Any local chart of ∂U will be denoted by

$$\psi = (\psi_1, ..., \psi_n) : W \subset \mathbb{R}^{n-1} \to V \subset \partial U, \; \zeta = (\zeta_1, ..., \zeta_{n-1}) \mapsto \xi = \psi(\zeta).$$

For any $\xi = (\xi_1, ..., \xi_n) = \psi(\zeta) \in \partial U$ we will also write

$$\tilde{\gamma}(t, \zeta) := \gamma(t, \psi(\zeta)) = \gamma(t, \xi).$$

Thus, $\tilde{\gamma}(., \zeta)$ is the unique solution of the initial value problem

$$\dot{\tilde{\gamma}}(t, \zeta) = A(\tilde{\gamma}(t, \zeta)), \quad \tilde{\gamma}(0, \zeta) = \psi(\zeta).$$

As usual, $\langle ., . \rangle$ denotes the Euklidian scalar product in $\mathbb{R}^n$, $\mathbb{R}^{n+1}$, respectively.

3.3. Lemma: *Let the assumptions in 3.1. hold. Let $q \in U \setminus \{0\}$. Then the corresponding field line $\gamma(.,q)$ penetrates every surface $\mathfrak{F}_{p_0} \subset U$ in exactly one point. Moreover, there holds*

$$\lim_{t \to +\infty} \gamma(t,q) = 0$$

and there exists some $\sigma(q) \leq 0$ with

$$\lim_{t \to \sigma(q)} \gamma(t,q) = \xi(q) \in \partial U.$$

The second condition means that every point $q \in U \setminus \{0\}$ lies on (exactly) one field line with origin on ∂U.

PROOF: Every surface $\mathfrak{F}_{p_0} \subset \overline{U}$ is the boundary of some bounded domain $\mathfrak{G}_{p_0} \subset U$, where $\mathfrak{G}_{p_0}$ is positively invariant with respect to the field line equation

$$\dot{\gamma} = A(\gamma),$$

and its outer normal in $z \in \mathfrak{F}_{p_0}$ is given by

$$\nu(z) = -\frac{A(z)}{|A(z)|}.$$

Obviously there holds $0 = \bigcap_{p_0 \in U} \overline{\mathfrak{G}_{p_0}}$. Supposed every solution $\gamma(.,q)$ of (20) penetrates every surface $\mathfrak{F}_{p_0} \subset \overline{U}$, then we have $\lim_{t \to \infty} \gamma(t,q) = 0$. Now we assume $\gamma(.,q)$ to be any solution of (20) and $\mathfrak{F}_{p'} \subset \mathfrak{G}_q$ to be such a surface satisfying

$$\forall t \geq 0: \quad \gamma(t,q) \notin \mathfrak{F}_{p'},$$

but

$$\lim_{t \to \infty} \text{dist}\big(\gamma(t,q), \mathfrak{F}_{p'}\big) = 0.$$

Then there exist some neighbourhood W of $\mathfrak{F}_{p'}$ and some surface $\mathfrak{F}_{q'} \subset W$ with $\mathfrak{F}_{p'} \subset \mathfrak{G}_{q'}$, $\mathfrak{G}_{q'} \setminus \mathfrak{G}_{p'} \subset W$ and some $\eta > 0$ such that

$$\beta: \mathfrak{F}_{p'} \times \,]-\eta, +\eta[\,\to W, \quad (z, \varrho) \mapsto z - \varrho \cdot \frac{A(z)}{|A(z)|}$$

is a bijective mapping. Here we may suppose $\mathfrak{G}_{q'} \setminus \mathfrak{G}_{p'} \subset \beta\big(\mathfrak{F}_{p'} \times [0, +\eta[\big)$. There exists some $\varepsilon > 0$ such that

$$\forall x \in W: \quad |A(x)| \geq \varepsilon.$$

For sufficiently large $t_0 > 0$ we have then

$$\forall t \geq t_0: \quad \gamma(t,q) \in \mathfrak{G}_{q'} \setminus \mathfrak{G}_{p'}.$$

Let $z(t)$, $\varrho(t)$ be the coordinates of $\gamma(t, q)$ with respect to β, i.e.

$$\forall t \geq t_0 : \ \beta\big(z(t), \varrho(t)\big) \ = \ \gamma(t, q).$$

We may assume that $z(t)$ in the sense of a local chart and $\varrho(t)$ are differentiable with respect to t. Then we have for $t \geq t_0$:

$$
\big|A\big(\gamma(t,q)\big)\big| \ = \ \big\langle \dot{\gamma}(t,q), A\big(\gamma(t,q)\big)\big\rangle \ =
$$
$$
= \ \big\langle \dot{z}(t), A\big(\gamma(t,q)\big)\big\rangle \ - \ \dot{\varrho}(t) \cdot \frac{\big\langle A(z(t)), A(\gamma(t,q))\big\rangle}{|A(z(t))|} \ -
$$
$$
- \ \varrho(t) \cdot \Big\langle \frac{d}{dt}\frac{A(z(t))}{|A(z(t))|}, A\big(\gamma(t,q)\big)\Big\rangle,
$$

where, because of $\langle \dot{z}(t), A(z(t))\rangle = 0$, we have

$$\big\langle \dot{z}(t), A\big(\gamma(t,q)\big)\big\rangle \ \to \ 0, \ \ t \to \infty,$$

$$\Big|\Big\langle \frac{d}{dt}\frac{A(z(t))}{|A(z(t))|}, A\big(\gamma(t,q)\big)\Big\rangle\Big| \ \leq \ \text{const}, \ \ \ \varrho(t) \to 0, \ t \to \infty,$$

$$0 \ < \ \frac{\big\langle A(z(t)), A\big(\gamma(t,q)\big)\big\rangle}{|A\big(z(t)\big)|} \ \leq \ \text{const},$$

and thus

$$\forall t \geq t_1 : \ \ -\dot{\varrho}(t) \ \geq \ \varepsilon'$$

for some $\varepsilon' > 0$ and some $t_1 \geq t_0$. From this we obtain

$$\varrho(t) \ = \ \varrho(t_1) + \int_{t_1}^{t} \dot{\varrho}(\tau)d\tau \ < \ 0$$

for sufficiently large $t > t_1$, which contradicts our assumption that the surface $\mathfrak{F}_{p'}$ is not penetrated by $\gamma(., q)$.

The exterior domain $\hat{\mathfrak{G}}_{p_0}$ of any surface $\mathfrak{F}_{p_0}$ is negatively invariant with respect to (20), and

$$\lim_{t \to \sigma(q)} \gamma(t, q) \ \in \ \partial U$$

can be shown in a corresponding way. $\hspace{2cm}\square$

3.4. Remark: As a consequence of Lemma 3.3. we note that the mapping

$$[0, +\infty[\ \times \ \partial U \ \to \ \overline{U} \setminus \{0\}, \ (t, \xi) \ \mapsto \ \gamma(t, \xi)$$

is bijective.

3.5. Theorem: *Let the assumptions in* 3.1. *hold. Then there exist functions* $\lambda \in C^0(\overline{U}, \mathbb{R})$, $Q \in C^1(\overline{U}, \mathbb{R})$ *such that*

$$\forall x \in U \setminus \{0\}: \quad \nabla Q(x) = \lambda(x) A(x).$$

In the proof of this theorem the local existence of Euler's multiplier is used.

PROOF: (a) *Construction of a suitable parametrization of* $U \setminus \{0\}$.

(i) We fix any point $\xi^0 \in \partial U$ and consider the field line $\gamma(., \xi^0)$ (see 3.2.). According to Lemma 3.3. for every $\xi \in \partial U$ there exists exactly one value $\tau(t, \xi^0, \xi) \geq 0$ satisfying

$$\gamma\big(\tau(t, \xi^0, \xi), \xi\big) \in \mathfrak{F}_{\gamma(t, \xi^0)}.$$

For fixed $\xi^0, \xi \in \partial U$ the function $\tau(., \xi^0, \xi)$ is monotonically increasing, and according to Remark 3.4. the mapping

$$\Phi: [0, +\infty[\times \partial U \to \overline{U} \setminus \{0\}, \quad \Phi(t, \xi) := \gamma\big(\tau(t, \xi^0, \xi), \xi\big)$$

is also bijective.

(ii) Now we show the differentiability of Φ in the sense of a local chart. From the theory of ordinary differential equations we know that $\gamma(., \xi)$ is continuously differentiable with respect to ξ. In order to investigate the differentiability of $\tau(t, \xi^0, .)$, let $\zeta_0 \in W \subset \mathbb{R}^{n-1}$, $\xi \in V \subset \partial U$, $\psi(\zeta_0) = \xi$, where ψ is a local chart (see 3.2.). Furthermore, let $\mathcal{G} \subset U$ be an open neighbourhood of $\gamma\big(\tau(t, \xi^0, \xi), \xi\big) \in U$ and suppose $\varphi \in C^1(\mathcal{G}, \mathbb{R})$, $\mu \in C^0(\mathcal{G}, \mathbb{R})$ satisfying

(21) $\quad \forall x \in \mathcal{G}: \quad \nabla \varphi(x) = \mu(x) A(x), \quad \mu(x) > 0.$

This is possible, since $A(x) \neq 0$ and $\omega \wedge d\omega = 0$. Obviously we have

(22) $\quad \{x \in \mathcal{G}: \varphi(x) = \varphi(x_0)\} \subset \mathfrak{F}_{x_0}$

for any $x_0 \in \mathcal{G}$, if $\mathcal{G}$ is sufficiently small.

Set $\tau := \tau(t, \xi^0, \xi)$. Then for sufficiently small $\delta > 0$ the mapping

$$\varphi\big(\gamma(., \psi(.))\big) = \varphi\big(\tilde{\gamma}(., .)\big): \ ]\tau - \delta, \tau + \delta[\times W \to \mathbb{R}$$

is continuously partially differentiable with the property

(23) $\quad \dfrac{\partial}{\partial t}\Big(\varphi\big(\tilde{\gamma}(t, \zeta)\big)\Big) = \big\langle \nabla \varphi\big(\tilde{\gamma}(t, \zeta)\big), \dot{\tilde{\gamma}}(t, \zeta) \big\rangle = \mu\big(\tilde{\gamma}(t, \zeta)\big) \big| A\big(\tilde{\gamma}(t, \zeta)\big) \big|^2 > 0.$

The implicit function theorem yields, after shrinking W and δ if necessary, some C^1-function

$$\tilde{\tau}: \ W \to]\tau - \delta, \tau + \delta[$$

satisfying

$$\forall \tau' \in]\tau - \delta, \tau + \delta[, \ \zeta \in W: \ \Big(\varphi\big(\tilde{\gamma}(\tau', \zeta)\big) = \varphi\big(\gamma(\tau, \xi^0)\big) \Leftrightarrow \tau' = \tilde{\tau}(\zeta)\Big).$$

From the definition of $\tau(t,\xi^0,\xi)$ and (22) we see that for $\zeta \in W$ and the considered value t

$$\bar{\tau}(\zeta) = \tau\big(t,\xi^0,\psi(\zeta)\big).$$

From this we can conlude the continuous differentiability of $\tau(t,\xi^0,.)$ with respect to ξ and, using the chain rule, the continuous differentiability of Φ with respect to ξ in the sense of a local chart.

(iii) Now we prove the differentiability of Φ with respect to t in any point $(\hat{t},\xi)$.
(α) First we assume $\gamma(\hat{t},\xi^0)$ and $\gamma\big(\tau(\hat{t},\xi^0,\xi),\xi\big)$ to lie in a common domain $\mathcal{G}$, where φ and μ exist as described in (ii). Set $\hat{\tau} := \tau(\hat{t},\xi^0,\xi)$. Choosing $\delta_1,\delta_2,\delta_1',\delta_2' > 0$ in a suitable way (and sufficiently small) we can obtain that

$$\gamma\big(\,]\hat{t}-\delta_1,\hat{t}+\delta_2[\,,\xi^0\big) \subset \mathcal{G}, \quad \gamma\big(\,]\hat{\tau}-\delta_1',\hat{\tau}+\delta_2'[\,,\xi\big) \subset \mathcal{G}$$

and, since $\varphi\big(\gamma(.,\xi)\big), \varphi\big(\gamma(.,\xi^0)\big)$ are strictly monotonically increasing (cf. (23)),

$$\varphi\Big(\gamma\big(\,]\hat{t}-\delta_1,\hat{t}+\delta_2[\,,\xi^0\big)\Big) = \varphi\Big(\gamma\big(\,]\hat{\tau}-\delta_1',\hat{\tau}+\delta_2'[\,,\xi\big)\Big).$$

In particular, we have

$$\forall t \in\,]\hat{t}-\delta_1,\hat{t}+\delta_2[\; : \quad \frac{d}{dt}\varphi\big(\gamma(t,\xi^0)\big) > 0,$$

$$\forall \tau \in\,]\hat{\tau}-\delta_1',\hat{\tau}+\delta_2'[\; : \quad \frac{d}{d\tau}\varphi\big(\gamma(\tau,\xi)\big) > 0,$$

and the function $\varphi\big(\gamma(.,\xi)\big)$ is invertible. Because of eq. (21) φ is constant on every pathwise connected subset of $\mathfrak{F}_p \cap \mathcal{G}$. Thus, taking into account the definition of $\tau(t,\xi^0,\xi)$, there arises

$$(24) \quad \forall t \in\,]\hat{t}-\delta_1,\hat{t}+\delta_2[\; : \quad \varphi\Big(\gamma\big(\tau(t,\xi^0,\xi),\xi\big)\Big) = \varphi\big(\gamma(t,\xi^0)\big),$$

and therefore

$$\forall t \in\,]\hat{t}-\delta_1,\hat{t}+\delta_2[\; : \quad \tau(t,\xi^0,\xi) = \Big(\varphi\big(\gamma(.,\xi)\big)\Big)^{-1}\big(\varphi(\gamma(t,\xi^0))\big),$$

where the right hand side means the inverse mapping of $\varphi(\gamma(.,\xi))$ applied to the argument $\varphi(\gamma(t,\xi^0))$. From this we can conclude the continuous differentiability of $\tau(.,\xi^0,\xi)$ with respect to t and

$$\forall t \in\,]\hat{t}-\delta_1,\hat{t}+\delta_2[\; : \quad \dot{\tau}(t,\xi^0,\xi) > 0.$$

(β) Now we consider the case of $\gamma(\hat{t},\xi^0)$ and $\gamma\big(\tau(\hat{t},\xi^0,\xi),\xi\big) = \gamma(\hat{\tau},\xi)$ not lying in a common domain $\mathcal{G}$ as described in (α). There exists a curve connecting $\gamma(\hat{t},\xi^0)$ and $\gamma(\hat{\tau},\xi)$ whose trace is a subset of $\mathfrak{F}_{\gamma(\hat{t},\xi^0)}$ and there also exists a finit number of points $x^0,...,x^N \in \mathfrak{F}_{\gamma(\hat{t},\xi^0)}$ with the following properties:

- $x^0 = \gamma(\hat{t}, \xi^0)$, $x^N = \gamma(\hat{\tau}, \xi)$,

- For $j = 0, ..., N-1$ the points x^j, x^{j+1} and the connecting piece of curve are included in a common domain $\mathcal{G}_j$ in which $\varphi = \varphi_j$ and $\mu = \mu_j$ exist according to (ii).

The corresponding boundary points will be denoted by $\xi^0, ..., \xi^N$, i.e.

$$\forall j \in \{0, ..., N\}: \quad x^j = \gamma\big(\tau(\hat{t}, \xi^0, \xi^j), \xi^j\big),$$

where $\xi^N = \xi$. The argumentation of (i) applied to ξ^j instead of ξ^0 yields the existence of some value $\tau(t, \xi^j, \xi^{j+1}) > 0$ such that

$$\gamma\big(\tau(t, \xi^j, \xi^{j+1}), \xi^{j+1}\big) \in \mathfrak{F}_{\gamma(t,\xi^j)},$$

and thus

$$\varphi_j\Big(\gamma\big(\tau(t, \xi^j, \xi^{j+1}), \xi^{j+1}\big)\Big) = \varphi_j\big(\gamma(t, \xi^j)\big)$$

for $t \in \,]\hat{t} - \delta, \hat{t} + \delta[$, $j = 1, ..., N-1$, where $\delta = \delta_j > 0$ is sufficiently small. From the definition of $\tau(., ., .)$ we easily see that

$$\tau(t, \xi^0, \xi^{j+1}) = \tau\big(\tau(t, \xi^0, \xi^j), \xi^j, \xi^{j+1}\big), \quad j = 1, ..., N-1.$$

By induction with respect to j we can conclude the continuous differentiability of $\tau(., \xi^0, \xi^j)$ by applying the argumentation of (α) to $\tau(., \xi^{j-1}, \xi^j)$ and the chain rule.

From (α) and (β) we can conclude the continuous differentiability of $\tau(.)$ with respect to t. The chain rule finally yields the continuous differentiability of Φ with respect to t. We have

$$\dot{\Phi}(t, \xi) = \dot{\tau}(t, \xi^0, \xi) \cdot \dot{\gamma}\big(\tau(t, \xi^0, \xi), \xi\big) = \dot{\tau}(t, \xi^0, \xi) \cdot A\Big(\gamma\big(\tau(t, \xi^0, \xi), \xi\big)\Big)$$

and therefore

$$(25) \quad \forall \xi \in \partial U, t \in [0, +\infty[: \quad \dot{\Phi}(t, \xi) = \dot{\tau}(t, \xi^0, \xi) \cdot A\big(\Phi(t, \xi)\big) \neq 0.$$

(b) *Construction of Euler's multiplier.*
For a local chart $\psi: W \subset \mathbb{R}^{n-1} \to V \subset \partial U$ of ∂U we set

$$(26) \quad \Theta(t, \zeta) := \Phi\big(t, \psi(\zeta)\big), \qquad t \geq 0, \zeta \in W.$$

Then Θ is continuously partially differentiable, cf. (a)(iii).
(i) In a first step we show that the Jacobi-matrix $D\Theta$ has an inverse. From the definition of Φ in (a)(i) we know

$$(27) \quad \Theta(t, \zeta) = \gamma\Big(\tau\big(t, \xi^0, \psi(\zeta)\big), \psi(\zeta)\Big) = \tilde{\gamma}\Big(\tau\big(t, \xi^0, \psi(\zeta)\big), \zeta\Big),$$

thus

$$\dot{\Theta}(t,\zeta) = \dot{\tau}(t,\xi^0,\psi(\zeta)) \cdot \dot{\tilde{\gamma}}\Big(\tau(t,\xi^0,\psi(\zeta)),\zeta\Big),$$

(28) $\quad \partial_{\zeta_j}\Theta(t,\zeta) = \dfrac{\partial\tilde{\gamma}}{\partial\zeta_j}\Big(\tau(t,\xi^0,\psi(\zeta)),\zeta\Big) +$

$$+ \Big\{ \sum_{k=1}^{n} \frac{\partial\tau}{\partial\xi_k}(t,\xi^0,\psi(\zeta)) \cdot \partial_{\zeta_j}\psi_k(\zeta) \Big\} \cdot \dot{\tilde{\gamma}}\Big(\tau(t,\xi^0,\psi(\zeta)),\zeta\Big),$$

for $j = 1,...,n-1$. Thus we arrive at

$$\det\Big(\partial_t\Theta\Big|\partial_{\zeta_1}\Theta\Big|...\Big|\partial_{\zeta_{n-1}}\Theta\Big)(t,\zeta) =$$

$$= \dot{\tau}(t,\xi^0,\psi(\zeta))\det\Big(\dot{\tilde{\gamma}}\Big|\partial_{\zeta_1}\tilde{\gamma}\Big|...\Big|\partial_{\zeta_{n-1}}\tilde{\gamma}\Big)\Big|_{\tau(t,\xi^0,\psi(\zeta)),\zeta} =$$

$$= \underbrace{\dot{\tau}(t,\xi^0,\psi(\zeta))}_{>0}\det\Big(\dot{\tilde{\gamma}}\Big|\partial_{\zeta_1}\tilde{\gamma}\Big|...\Big|\partial_{\zeta_{n-1}}\tilde{\gamma}\Big)\Big|_{0,\zeta}^{\tau(t,\xi^0,\psi(\zeta))} \cdot \exp\int_0^{\tau(t,\xi^0,\psi(\zeta))}\mathrm{trace}DA(\tilde{\gamma}(t',\zeta))dt'$$

$$\neq 0,$$

because $\dot{\tilde{\gamma}}$ and $\partial_{\zeta_j}\tilde{\gamma}$ are solutions of the linear differential equation

$$\dot{w}(t) = DA(\tilde{\gamma}(t,\zeta))w(t)$$

and $\partial_{\zeta_j}\tilde{\gamma}(0,\zeta) = \partial_{\zeta_j}\psi(\zeta)$.

(ii) Now let $u \in C^1(\mathbb{R},\mathbb{R}_+)$ be an arbitrary function satifying

(29) $\quad \forall t \in \mathbb{R} : \dot{u}(t) < 0, \qquad \lim\limits_{t\to+\infty}\dot{u}(t) = 0.$

For $(t,\xi) \in [0,+\infty[\times \partial U$ we define

(30) $\quad Q(\Phi(t,\xi)) := u(t), \quad$ i.e. $\quad Q(\Theta(t,\zeta)) := u(t)$

for $\xi = \psi(\zeta), \zeta \in W \subset \mathbb{R}^{n-1}$. Then we have

$$\dot{u}(t) = \sum_{k=1}^{n} \frac{\partial Q}{\partial x_k}(\Theta(t,\zeta)) \cdot \frac{\partial\Theta_k}{\partial t}(t,\zeta),$$

$$0 = \sum_{k=1}^{n} \frac{\partial Q}{\partial x_k}(\Theta(t,\zeta)) \cdot \frac{\partial\Theta_k}{\partial\zeta_j}(t,\zeta), \quad j = 1,...,n-1,$$

therefore

(31) $\quad (D\Theta(t,\zeta))^{\mathrm{T}}\nabla Q(\Theta(t,\zeta))^{\mathrm{T}} = (\dot{u}(t),0,...,0)^{\mathrm{T}} = \dot{u}(t)\cdot(1,0,...,0)^{\mathrm{T}}.$

But there also arises

$$\sum_{k=1}^{n}\partial_t\Theta_k(t,\zeta)A_k(\Theta(t,\zeta)) \overset{(25)}{\underset{(26)}{=}} \underbrace{\dot{\tau}(t,\xi^0,\psi(\zeta))}_{>0}\underbrace{|A(\Theta(t,\zeta))|^2}_{>0},$$

$$\sum_{k=1}^{n} \partial_{\zeta_j} \Theta_k(t,\zeta) A_k\big(\Theta(t,\zeta)\big) \underset{(21)}{=} \partial_{\zeta_j} \Theta_k(t,\zeta)\, \mu\big(\Theta(t,\zeta)\big) \frac{\partial \varphi}{\partial x_k}\big(\Theta(t,\zeta)\big) =$$

$$= \mu\big(\Theta(t,\zeta)\big) \cdot \frac{\partial \varphi(\Theta(t,\zeta))}{\partial \zeta_j} \underset{(27)}{\overset{(24)}{=}} 0,$$

where μ, φ are locally as in (ii), i.e.

$$(32) \quad \big(D\Theta(t,\zeta)\big)^{\mathrm{T}} A\big(\Theta(t,\zeta)\big)^{\mathrm{T}} = \dot{\tau}(t,\xi^0,\psi(\zeta))\, \big|A(\Theta(t,\zeta))\big|^2 \cdot (1,0,...,0)^{\mathrm{T}}.$$

Defining

$$\lambda\big(\Phi(t,\xi)\big) := \frac{\dot{u}(t)}{\dot{\tau}(t,\xi^0,\xi)|A(\Phi(t,\xi))|^2},$$

or, locally,

$$\lambda\big(\Theta(t,\zeta)\big) := \frac{\dot{u}(t)}{\dot{\tau}(t,\xi^0,\psi(\zeta))|A(\Theta(t,\zeta))|^2},$$

where $\xi = \psi(\zeta)$, we obtain, using equations (31), (32) and the inverse matrix of $(D\Theta)^{\mathrm{T}}$,

$$(33) \quad \begin{aligned} \nabla Q\big(\Theta(t,\zeta)\big) &= \lambda\big(\Theta(t,\zeta)\big) A\big(\Theta(t,\zeta)\big), \\ \nabla Q\big(\Phi(t,\xi)\big) &= \lambda\big(\Phi(t,\xi)\big) A\big(\Phi(t,\xi)\big) \end{aligned}$$

for $t \in [0,+\infty[$, $\xi = \psi(\zeta) \in \partial U$. Note that $(33)_1$ is a local, $(33)_2$ the corresponding global formulation. The local formulation is necessary to treat the questions of regularity. As ∂U is compact, for every $t \in [0,+\infty[$ we have

$$\chi(t) := \min_{\xi \in \partial U} \dot{\tau}(t,\xi^0,\xi)\, \big|A\big(\Phi(t,\xi)\big)\big|^2 > 0.$$

Thus, supposing in addition to (29)

$$\lim_{t \to +\infty} \frac{\dot{u}(t)}{\chi(t)} = 0$$

and setting $\lambda(0) := 0$ and $\nabla Q(0) := 0$, λ can be extended to a $C^0(\overline{U},\mathbb{R})$-function and Q to a $C^1(\overline{U},\mathbb{R})$-function, and we obtain finally

$$\forall x \in U : \quad \nabla Q(x) = \lambda(x)A(x). \qquad \square$$

The existence of Euler's multiplier in the case of one component bounded from below in an infinite cylinder

3.6. Assumptions: *Let $G \subset \mathbb{R}^n$ be a simply connected domain, let*

$$Z := G \times \mathbb{R} \subset \mathbb{R}^{n+1},$$

$A = (A_1, ..., A_n, A_{n+1}) \in C^1_{\text{unif}}(\overline{Z}, \mathbb{R}^{n+1})$ *with the corresponding Pfaffian form*

$$\omega := \sum_{k=1}^{n+1} A_j dx_j, \text{ and the properties}$$

(34) $A_{n+1} \geq \varepsilon > 0$ *in $\overline{Z}$,*

(35) $\omega \wedge d\omega = 0$ *in $\overline{Z}$.*

3.7. Theorem: *Let the assumptions in 3.6. hold. Then there exist functions $\lambda \in C^0(\overline{Z}, \mathbb{R})$ and $Q \in C^1(\overline{Z}, \mathbb{R})$ satisfying*

$$\begin{aligned} \lambda(x) &\neq 0 & in \ \overline{Z}, \\ dQ &= \lambda\omega, & i.e. \\ \nabla Q &= \lambda A & in \ \overline{Z}. \end{aligned}$$

PROOF: In this proof we will make use of the local existence of Euler's multiplier, which is guaranteed by eq. (35).

(a) *Preliminaries and further notations.* Let $\mathcal{M} \subset Z$ be any open and connected subset of Z, $p_0 \in \mathcal{M}$. Similar to 3.1. we define

$$\mathfrak{F}^{\mathcal{M}}_{p_0} := \big\{ p \in G \big| \ \exists \text{ piecewise continuously differentiable curve } \alpha$$
$$\text{connecting } p_0 \text{ and } p, \text{ where trace } \alpha \subset \mathcal{M},$$
$$\text{and } \alpha \text{ is everywhere orthogonal to the field vector } A,$$
$$\text{i.e. } \langle \dot{\alpha}, A(\alpha) \rangle = 0 \big\}$$

Obviously, $\mathfrak{F}^{\mathcal{M}}_{p_0}$ is pathwise connected. By setting

$$p \underset{\mathcal{M}}{\sim} p_0 \quad :\Leftrightarrow \quad p \in \mathfrak{F}^{\mathcal{M}}_{p_0}$$

an equivalence relation is defined. From Frobenius theory it is well known, that for every $p_0 \in Z$ there exists an open neighbourhood $U \subset Z$ of p_0 and functions $\mu \in C^0(\overline{U}, \mathbb{R})$, $\varphi \in C^1(\overline{U}, \mathbb{R})$ satisfying

(36) $\forall x \in U: \quad \nabla\varphi(x) = \mu(x)A(x), \quad \mu(x) \neq 0 \quad \text{in } U.$

If U is sufficiently small, $\{x \in U | \ \varphi(x) = \varphi(p_0)\}$ is pathwise connected, and we have

$$\mathfrak{F}^{U}_{p_0} := \big\{ x \in U \big| \ \varphi(x) = \varphi(p_0) \big\}.$$

Because of $A_{n+1}(x) \neq 0$ we also have $\partial_{x_{n+1}}\varphi(x) \neq 0$, and the implicit function theorem yields, after shrinking U if necessary, the existence of some neighbourhood $V \subset \mathbb{R}^n$ of $(p_{01}, ..., p_{0n}) \in \mathbb{R}^n$ and some C^1-function $\Psi : V \to \mathbb{R}$ satisfying

$$\mathfrak{F}^{U}_{p_0} = \big\{ x \ \big| \ (x_1, ..., x_n) \in V, \ x_{n+1} = \Psi(x_1, ..., x_n) \big\},$$

where $x = (x_1, ..., x_n, x_{n+1})$.

Now we are going to show, that for each $p_0 \in Z$ the set $\mathfrak{F}_{p_0}^Z$ is a *global surface*, i.e. that for each $\hat{x} = (x_1, ..., x_n) \in G$ there exists exactly one $x_{n+1} \in \mathbb{R}$ such that $(\hat{x}, x_{n+1}) = (x_1, ..., x_n, x_{n+1}) \in \mathfrak{F}_{p_0}^Z$. Note that until now it is not clear that $\mathfrak{F}_{p_0}^Z \cap \mathcal{M} = \mathfrak{F}_{p_0}^\mathcal{M}$ for any $\mathcal{M} \subset Z$. A family $(U_\iota)_{\iota \in I}$ of open sets U_ι will be said to be *admissible*, if for each $\iota \in I$ functions $\mu = \mu_\iota$ and $\varphi = \varphi_\iota$ in the sense of (36) exist in U_ι.

(b) $\mathfrak{F}_p^Z$ *is a global surface.* Let $p = (\hat{p}, p_{n+1}) = (p_1, ..., p_n, p_{n+1}) \in Z$.

(i) First we will prove that for each $\hat{x} \in G$ there exists at least one $x_{n+1} \in \mathbb{R}$ such that $(\hat{x}, x_{n+1}) \in \mathfrak{F}_p^Z$: Without restriction let $p = (0, ..., 0, p_{n+1})$. Since G is open and pathwise connected there exists a curve $\alpha \in C^1([0, 1], G)$ with $\alpha(0) = 0$ and $\alpha(1) = \hat{x}$. Now let $y(.)$ be the unique solution of the initial value problem

$$(37) \quad \dot{y}(t) = -\sum_{k=1}^n \dot{\alpha}_k(t) \cdot \frac{A_k}{A_{n+1}}(\alpha(t), y(t)), \quad y(0) = p_{n+1},$$

where $\alpha = (\alpha_1, ..., \alpha_n)$. Since $\frac{A_k}{A_{n+1}}$ is uniformly bounded, the solution $y(.)$ of (37) exists for all $t \in [0, 1]$. Thus, $\gamma := (\alpha, y) : [0, 1] \to Z$ is a C^1-curve which is always orthogonal to the field vector A:

$$\langle \dot{\gamma}, A(\gamma) \rangle = \sum_{k=1}^n \dot{\alpha}_k A_k(\alpha, y) - A_{n+1}(\alpha, y) \sum_{k=1}^n \dot{\alpha}_k \cdot \frac{A_k}{A_{n+1}}(\alpha, y) = 0.$$

Therefore $(\hat{x}, y(1)) \in \mathfrak{F}_p^Z$.

(ii) Now we will prove the uniqueness of this x_{n+1}. Assume, without restriction, $(0, ..., 0, y_1), (0, ..., 0, y_2) \in \mathfrak{F}_p^Z$, where $y_1 \leq y_2$. Let $\gamma = (\alpha, y) : [0, 1] \to Z$ be a piecewise continuously differentiable curve satisfying

$$\langle \dot{\gamma}, A(\gamma) \rangle = 0, \qquad \gamma(0) = (0, y_1), \quad \gamma(1) = (0, y_2).$$

Consequently, $\gamma = (\alpha_1, ..., \alpha_n, y)$ satisfies (piecewise) the equation

$$(38) \quad \dot{\gamma} = \begin{pmatrix} \dot{\alpha}_1 \\ \vdots \\ \dot{\alpha}_n \\ \dot{y} \end{pmatrix} = \dot{\alpha}_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ -\frac{A_1(\gamma)}{A_{n+1}(\gamma)} \end{pmatrix} + \ldots + \dot{\alpha}_n \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ -\frac{A_n(\gamma)}{A_{n+1}(\gamma)} \end{pmatrix}.$$

Since $\alpha(0) = \alpha(1) = 0$ and G is simply connected, α is null homotopic in G. Let

$$h_\alpha = (h_{\alpha 1}, ..., h_{\alpha n}) : [0, 1] \times [0, 1] \to G$$

an appropriate homotopy, i.e. $h_\alpha(1,t) = \alpha(t)$, $h_\alpha(0,t) = 0$. We may assume $h_\alpha(.,t) \in C^1([0,1], G)$, $h_\alpha(s,.)$ to be piecewise continuously differentiable for each $s \in [0,1]$ and

$$(39) \quad \lim_{s \to 0} \sup_{t \in [0,1]} |\dot{h}_\alpha(s,t)| = 0.$$

Now let $y_h(s,.)$ be the unique solution of the initial value problem

$$(40) \quad \dot{y}_h(s,t) = -\sum_{k=1}^{n} \dot{h}_{\alpha k}(s,t) \cdot \frac{A_k}{A_{n+1}} (h_\alpha(s,t), y_h(s,t)), \quad y_h(s,0) = y_1,$$

where $\dot{\ }$ means differentiation with respect to t. The solution $y_h(s,.)$ exists for all $t \in [0,1]$ and depends continuously differentiably on parameter s. Because of the regularity of h_α and (40) $y_h(.,t)$ is a C^1-function of parameter s for each $t \in [0,1]$, and since $\dot{h}_\alpha(1,t) = \dot{\alpha}(t)$ the mapping

$$h_\gamma = (h_\alpha, y_h) : [0,1] \times [0,1] \to Z$$

is a homotopy satisfying

$$\langle \dot{h}_\gamma(s,t), A(h_\gamma(s,t)) \rangle = 0 \quad \text{and} \quad h_\gamma(1,t) = \gamma(t).$$

Because of eq. (39) and (40) there arises $y_h(0,1) = y_1$. Since $y_h(.,1)$ is continuous and $y_h(1,1) = y(1) = y_2$ we can conclude

$$[y_1, y_2] \subset y_h([0,1], 1) \quad \text{and thus} \quad \{(0,y) \mid y_1 \le y \le y_2\} \subset \mathfrak{F}^Z_{(0,y_1)}.$$

Assume that $y_1 < y_2$ and let $s_0 \in [0, 1[$ be the largest value satisfying

$$\forall s \in [0, s_0] : \quad y_h(s,1) = y_1.$$

In particular, for each $s \in [0, s_0]$ the curve $h_\gamma(s,.)$ is closed. Now we choose a finite admissible family $(U_j)_{j=1,...,N}$, $U_j \subset Z$, with the following property: There exist $t_1, ..., t_N$ with $0 < t_1 < t_2 < ... < t_N < 1$ satisfying

$$\begin{aligned}
&\forall t \in [0, t_1] : & &h_\gamma(s_0, t) \in U_1, & &h_\gamma(s_0, t_1) \in U_1 \cap U_2, \\
&\forall t \in [t_{j-1}, t_j] : & &h_\gamma(s_0, t) \in U_j, & &h_\gamma(s_0, t_j) \in U_j \cap U_{j+1}, \ j = 2, ..., N-1, \\
&\forall t \in [t_{N-1}, t_N] : & &h_\gamma(s_0, t) \in U_N, & &h_\gamma(s_0, t_N) \in U_N \cap U_1, \\
&\forall t \in [t_N, 1] : & &h_\gamma(s_0, t) \in U_1.
\end{aligned}$$
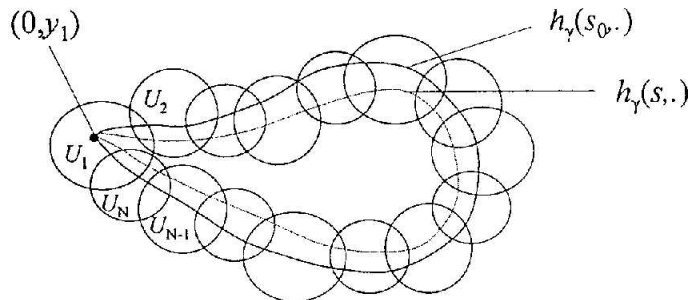
Since U_1 and $U_1 \cap U_2$ are open and $h_\gamma(.,.)$ is continuous there exists some $\delta_1 > 0$ such that

$$\begin{aligned}\forall s \in [s_0, s_0 + \delta_1], \; t \in [0,1]: && h_\gamma(s,t) &\in U_1,\\ \forall s \in [s_0, s_0 + \delta_1]: && h_\gamma(s,t_1) &\in U_1 \cap U_2\end{aligned}$$

and because of

$$\forall s \in [s_0, s_0 + \delta_1], \; t \in [0,t_1]: \quad \langle \dot{h}_\gamma(s,t), A(h_\gamma(s,t)) \rangle = 0$$

we arrive at

$$\forall s \in [s_0, s_0 + \delta_1], \; t \in [0,t_1]: \quad \varphi_1(h_\gamma(s,t)) = \varphi_1(0, y_1).$$

In particular,

$$[s_0, s_0 + \delta_1] \to U_1 \cap U_2, \; s \mapsto h_\gamma(s,t_1)$$

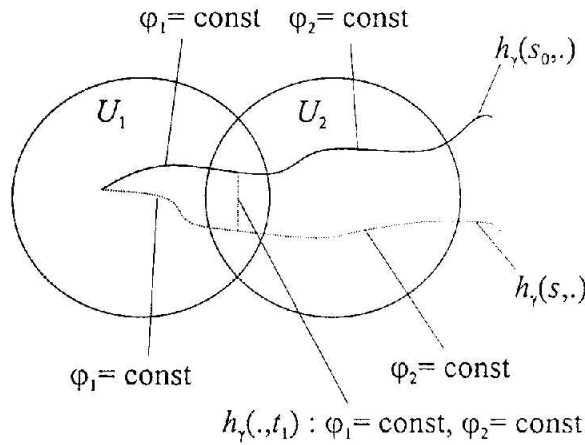is a C^1-curve along which φ_1 is constant. But this implies that this curve is orthogonal to A, i.e.

$$\forall s \in [s_0, s_0 + \delta_1]: \quad \langle h'_\gamma(s,t_1), A(h_\gamma(s,t_1)) \rangle = 0,$$

where $'$ means differentiation with respect to s. Thus, φ_2 is also constant along $h_\gamma(.,t_1)\big|_{[s_0,s_0+\delta_1]}$. Since each curve $h_\gamma(s,.)$ is orthogonal to A, φ_2 is constant along

$$h_\gamma(s,.)\big|_{[t_1,t_2]}: \quad [t_1,t_2] \to U_2, \; t \mapsto h_\gamma(s,t)$$

for all $s \in [s_0, s_0 + \delta_1]$ with the consequence

$$\forall t \in [t_1, t_2], \; s \in [s_0, s_0 + \delta_1]:$$
$$\varphi_2(h_\gamma(s,t)) = \varphi_2(h_\gamma(s,t_1)) = \varphi_2(h_\gamma(s_0,t_1)).$$

$h_\gamma(.,t_1): \varphi_1 = \text{const}, \; \varphi_2 = \text{const}$

By induction we can choose δ_j, $j = 1, \ldots, N$, such that $0 < \delta_j \le \delta_{j-1}$ and

$$\begin{aligned}\forall s \in [s_0, s_0 + \delta_j], \; t \in [t_{j-1}, t_j]: && h_\gamma(s,t) &\in U_j,\\ \forall s \in [s_0, s_0 + \delta_j]: && h_\gamma(s,t_j) &\in U_j \cap U_{j+1}, \quad j = 1, \ldots, N,\end{aligned}$$

where $U_{N+1} := U_1$. Applying the corresponding argument we obtain

$$\forall t \in [t_{j-1}, t_j], \ s \in [s_0, s_0 + \delta_{j-1}]:$$
$$\varphi_j\big(h_\gamma(s, t)\big) = \varphi_j\big(h_\gamma(s, t_{j-1})\big) = \varphi_j\big(h_\gamma(s_0, t_{j-1})\big)$$

for $j = 2, \ldots, N$, and finally

$$\forall t \in [t_N, 1], \ s \in [s_0, s_0 + \delta_N]:$$
$$\varphi_1\big(h_\gamma(s, t)\big) = \varphi_1\big(h_\gamma(s, t_N)\big) = \varphi_1\big(h_\gamma(s_0, t_N)\big).$$

Since $h_\gamma(s_0, 1) = (0, y_1) = h_\gamma(s_0, 0)$ we have in particular

$$\forall s \in [s_0, s_0 + \delta_N]: \quad \varphi_1\big(h_\gamma(s, 1)\big) = \varphi_1(0, y_1).$$

According to the definition of s_0 there arises

$$h_\gamma(s, 1) = (0, y) \subset U_1 \ \text{ where } \ y_1 \neq y \to y_1, \ s \searrow s_0$$

for sufficiently small $s > s_0$. Applying the implicit function theorem to a sufficiently small neighbourhood $U \subset U_1$ of $(0, y_1)$ as in (a), we obtain

$$\big\{(\hat{x}, y) \in U \ \big| \ \varphi_1(\hat{x}, y) = \varphi_1(0, y_1)\big\} = \big\{(\hat{x}, y) \in V \times \mathbb{R} \ \big| \ y = \Psi(\hat{x})\big\},$$

thus $y = \Psi(0) = y_1$ for sufficiently small $s > s_0$ which contradicts our assumption.

(c) *Construction of Euler's multiplier in a circular cylinder.* First we construct the desired functions Q and λ in a circular cylinder included in Z.
Let $\hat{x}^0 \in G$,

$$K_R^n(\hat{x}^0) := \big\{\hat{x} \in \mathbb{R}^n \ \big| \ |\hat{x} - \hat{x}^0| < R\big\}, \quad \overline{K_R^n(\hat{x}^0)} \subset G,$$
$$Z_{\hat{x}^0, R} := K_R^n(\hat{x}^0) \times \mathbb{R}.$$

Without restriction we may assume $\hat{x}^0 = 0$. Let $(\hat{x}, y) \in Z_{0,R}$. Let $u(., 1, y, \hat{x})$ denote the unique solution of the initial value problem

$$(41) \quad \dot{u}(t) = -\sum_{k=1}^n \frac{A_k}{A_{n+1}} \big(t\hat{x}, u(t)\big) \, \hat{x}_k, \quad u(1) = y.$$

Since $\frac{A_k}{A_{n+1}}$ is uniformely bounded, this solution exists at least in $\,]-\frac{R}{|\hat{x}|}, +\frac{R}{|\hat{x}|}[$. The curve

$$(42) \quad \gamma = (\gamma_1, \ldots, \gamma_{n+1}) : \ \Big] -\frac{R}{|\hat{x}|}, +\frac{R}{|\hat{x}|} \Big[\ \to Z_{0,R}, \quad \begin{cases} \gamma_k(t) & := t\hat{x}_k, \ k = 1, \ldots, n, \\ \gamma_{n+1}(t) & := u(t, 1, y, \hat{x}) \end{cases}$$

is always orthogonal to the field vector A and we have trace $\gamma \subset Z_{0,R}$. Now let $\kappa \in C^1_{\text{unif}}(\mathbb{R}, \mathbb{R})$ be an arbitrary function satisfying

$$\forall y \in \mathbb{R}: \quad \kappa'(y) > 0.$$

We define

(43) $\quad \forall(\hat{x}, y) \in Z_{0,R}: \quad Q(\hat{x}, y) := \kappa\big(u(0, 1, y, \hat{x})\big).$

From the theory of ordinary differential equations we know that $u(0, 1, ., .)$ is a C^1-function of y and $\hat{x}_1, \ldots, \hat{x}_n$, where

$$\frac{\partial u}{\partial y}(0, 1, y, \hat{x}) \neq 0,$$

and the chain rule then yields $Q \in C^1\big(Z_{0,R}, \mathbb{R}\big)$ and

$$\forall(\hat{x}, y) \in Z_{0,R}: \quad \frac{\partial Q}{\partial y}(\hat{x}, y) \neq 0.$$

We are now going to show that ∇Q is everywhere parallel to A. Since the curve γ defined above connects the points $(\hat{x}, y)$ and $\big(0, u(0, 1, y, \hat{x})\big)$ and is always orthogonal to A, we have

(44) $\quad (\hat{x}, y) \underset{Z_{0,R}}{\sim} \big(0, u(0, 1, y, \hat{x})\big)$

in the sense of part (a) of the proof. From part (b) we know that each point $(\hat{x}, y) \in Z_{0,R}$ is equivalent to exactly one point of the form $(0, .)$ with respect to $Z_{0,R}$ and also with respect to Z. Let $(\hat{x}^1, y^1) \in \mathfrak{F}^{Z_{0,R}}_{(\hat{x},y)}$, i.e.

(45) $\quad (\hat{x}^1, y^1) \underset{Z_{0,R}}{\sim} (\hat{x}, y).$

According to (43) with respect to $(\hat{x}^1, y^1)$ instead of $(\hat{x}, y)$ we have

$$Q(\hat{x}^1, y^1) = \kappa\big(u(0, 1, y^1, \hat{x}^1)\big)$$

and, since γ connects the following points, (cf. (41), (42))

(46) $\quad (\hat{x}^1, y^1) \underset{Z_{0,R}}{\sim} \big(0, u(0, 1, y^1, \hat{x}^1)\big).$

From (44), (45) and (46) we can conclude

$$\big(0, u(0, 1, y, \hat{x})\big) \underset{Z_{0,R}}{\sim} \big(0, u(0, 1, y^1, \hat{x}^1)\big),$$

and therefore, applying the uniqueness result of (b),

$$u(0, 1, y, \hat{x}) = u(0, 1, y^1, \hat{x}^1)$$

and thus (cf. (43))

$$Q(\hat{x}, y) = Q(\hat{x}^1, y^1).$$

This implies that Q is constant on each surface $\mathfrak{F}_p^{Z_0,R}$ and thus

$$\nabla Q \perp \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ -\frac{A_1}{A_{n+1}} \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ -\frac{A_2}{A_{n+1}} \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ -\frac{A_n}{A_{n+1}} \end{pmatrix},$$

since these n vectors span the tangential space of $\mathfrak{F}_p^{Z_0,R}$. From this we can conclude the existence of some $\lambda = \lambda(\hat{x}, y) \in \mathbb{R}$ satisfying

$$\nabla Q = \lambda A \quad \text{in } Z_{0,R}.$$

Because of $\partial_y Q(\hat{x}, y) \neq 0$ and $A_{n+1}(\hat{x}, y) \neq 0$ we have

$$\lambda(\hat{x}, y) = \frac{\frac{\partial Q}{\partial y}(\hat{x}, y)}{A_{n+1}(\hat{x}, y)} \neq 0,$$

and $\lambda \in C^0(Z_{0,R}, \mathbb{R})$.

(d) *Globalization of λ and Q.* Now we intend to globalize these functions. Assume $(W_j)_{j \in \mathbb{N}_0}$ to be a complete covering of G in $\mathbb{R}^n$ with the following properties

$$W_0 = K_R^n(\hat{x}^0),$$
$$W_j = K_{R_j}^n(\hat{x}^j) := \left\{ \hat{x} \in \mathbb{R}^n \mid |\hat{x} - \hat{x}^j| < R_j \right\}, \quad j \in \mathbb{N},$$
$$\hat{x}^j \in \bigcup_{\iota=0}^{j-1} W_\iota, \quad R_j > 0.$$

We construct λ and Q by induction with respect to j. Assume λ and Q with $\nabla Q = \lambda A$ to be constructed on $\left(\bigcup_{\iota=0}^{j-1} W_\iota \right) \times \mathbb{R}$. We redefine

$$\kappa(.) := Q(\hat{x}^j, .), \quad \hat{x}^0 := \hat{x}^j, \quad R := R_j$$

and make the corresponding construction on $Z_{\hat{x}^0,R} := W_j \times \mathbb{R}$ as in (c).
To verify that Q and λ are well defined in $\left[\left(\bigcup_{\iota=0}^{j-1} W_\iota \right) \times \mathbb{R} \right] \cap \left(W_j \times \mathbb{R} \right)$ let

$$p \in \left[\left(\bigcup_{\iota=0}^{j-1} W_\iota \right) \times \mathbb{R} \right] \cap \left(W_j \times \mathbb{R} \right), \quad p \underset{W \times \mathbb{R}}{\sim} (\hat{x}^j, y)$$

for $W = \bigcup_{\iota=0}^{j-1} W_\iota$ or $W = W_j$, respectively. According to the uniqueness result of (b) there exists exactly one point on the axis $(\hat{x}^j, .)$ which is equivalent to p with respect to Z or $\left(\bigcup_{\iota=0}^{j-1} W_\iota \right) \times \mathbb{R}$ or $W_j \times \mathbb{R}$, respectively. Since $\hat{x}^j \in \left(\bigcup_{\iota=0}^{j-1} W_\iota \right) \cap W_j$ we have

$$p \underset{W \times \mathbb{R}}{\sim} (\hat{x}^j, y)$$

for $W = \bigcup_{\iota=0}^{j-1} W_\iota$ and $W = W_j$ and $W = G$. Thus, each construction of Q yields

$$Q(p) = Q(\hat{x}^j, y),$$

and this means that Q, and therefore also λ, is well defined.
Since $\mathfrak{F}_{p_0}^Z \cap \left[\left(\bigcup_{\iota=1}^{j-1} W_\iota \right) \times \mathbb{R} \right]$ is connected and Q is constant on each $\mathfrak{F}_{p_0}^Z \cap (W_\iota \times \mathbb{R})$, Q is constant on $\mathfrak{F}_{p_0}^Z \cap \left[\left(\bigcup_{\iota=1}^{j-1} W_\iota \right) \times \mathbb{R} \right]$. By the definition

$$\lambda(\hat{x}, y) := \frac{\frac{\partial Q}{\partial y}(\hat{x}, y)}{A_{m+1}(\hat{x}, y)}$$

we also obtain a well defined globalization of λ. $\qquad\square$

3.8. Corollary: *Let $G \subset \mathbb{R}^n$ be a simply connected domain, $Z := G \times \mathbb{R} \subset \mathbb{R}^{n+1}$, let $\mathfrak{Z} \subset \mathbb{R}^{n+1}$ and*

$$\Phi = \left(\Phi_1, \ldots, \Phi_{n+1} \right)^{\mathrm{T}} : \quad \overline{Z} \to \overline{\mathfrak{Z}}$$

be a global C^2-diffeomorphism. Further let $B \in C^1\left(\overline{\mathfrak{Z}}, \mathbb{R}^{n+1} \right)$, $\mathrm{w} := \sum_{k=1}^{n+1} B_k d\xi_k$ satisfying

$$\mathrm{w} \wedge d\mathrm{w} = 0 \quad and \quad \left\langle B, \frac{\partial \Phi}{\partial x_{n+1}} \circ \Phi^{-1} \right\rangle \geq \varepsilon > 0 \quad in \; \overline{\mathfrak{Z}}.$$

Then there exist functions $\mathfrak{Q} \in C^1\left(\overline{\mathfrak{Z}}, \mathbb{R} \right)$ and $\mu \in C^0\left(\overline{\mathfrak{Z}}, \mathbb{R} \right)$ satisfying

$$\forall \xi \in \mathfrak{Z} : \quad \mu(\xi) \neq 0$$

and

$$d\mathfrak{Q} = \mu\mathrm{w}, \quad i.e. \quad \nabla\mathfrak{Q} = \mu B \quad in \; \overline{\mathfrak{Z}}.$$

PROOF: In what follows for $x \in Z$ we will write

$$\xi := \Phi(x) \in \mathfrak{Z},$$

$$T(x) := D\Phi(x) = \begin{pmatrix} \partial_{x_1}\Phi_1(x) & \ldots & \partial_{x_{n+1}}\Phi_1(x) \\ \vdots & & \vdots \\ \partial_{x_1}\Phi_{n+1}(x) & \ldots & \partial_{x_{n+1}}\Phi_{n+1}(x) \end{pmatrix},$$

$$S(\xi) := T^{-1}\left(\Phi^{-1}(\xi) \right) = \left(D\Phi \right)^{-1}(x) = D(\Phi^{-1})(\xi).$$

Thus, we have

$$d\xi_k(\xi) = \sum_{j=0}^{n+1} T_{kj}(x) dx_j(x), \quad dx_j(x) = \sum_{k=0}^{n+1} S_{jk}(\xi) d\xi_k(\xi).$$

For $x \in Z$ we define furthermore

$$A(x) := \big(D\Phi\big)^{\mathrm{T}}(x)\, B\big(\Phi(x)\big) \;=\; T^{\mathrm{T}}(x)\, B\big(\Phi(x)\big),$$

$$\omega(x) := \sum_{j=1}^{n+1} A_j(x) dx_j(x).$$

Then there holds $\omega \wedge d\omega = 0$:

$$\omega(x) \;=\; \sum_{j,k=1}^{n+1} T_{kj}(x) B_k\big(\Phi(x)\big) dx_j \;=\; \sum_{j,k=1}^{n+1} B_k\big(\Phi(x)\big)\frac{\partial \Phi_k}{\partial x_j}(x) dx_j \;=$$

$$=\; \sum_{k=1}^{n+1} B_k(\xi) d\xi_k \;=\; \mathsf{w}(\xi),$$

$$d\omega(x) \;=\; \sum_{j=1}^{n+1} dA_j(x) \wedge dx_j \;=\; \sum_{j,l=1}^{n+1} \frac{\partial A_j}{\partial x_l}(x)\, dx_l \wedge dx_j \;=$$

$$=\; \sum_{j,k,l=1}^{n+1} \Big\{ \frac{\partial T_{kj}}{\partial x_l}(x) B_k\big(\Phi(x)\big) \;+$$

$$+\; T_{kj}(x) \sum_{q=1}^{n+1} \frac{\partial B_k}{\partial \xi_q}\big(\Phi(x)\big)\frac{\partial \Phi_q}{\partial x_l}(x) \Big\}\, dx_l \wedge dx_j \;=$$

$$=\; \sum_{k,j=1}^{n+1} \sum_{l=1}^{j-1} \underbrace{\Big(\frac{\partial T_{kj}}{\partial x_l}(x) - \frac{\partial T_{kl}}{\partial x_j}(x)\Big)}_{=0} B_k\big(\Phi(x)\big)\, dx_l \wedge dx_j \;+$$

$$+\; \sum_{j,k,p,q=1}^{n+1} S_{jp}(\xi) T_{kj}(x)\frac{\partial B_k}{\partial \xi_q}(\xi)\, d\xi_q \wedge d\xi_p \;=$$

$$=\; \sum_{k,q=1}^{n+1} \frac{\partial B_k}{\partial \xi_q}(\xi)\, d\xi_q \wedge d\xi_k \;=\; d\mathsf{w}(\xi).$$

There arises

$$\forall x \in Z: \quad \omega(x) \wedge d\omega(x) \;=\; \mathsf{w}(\xi) \wedge d\mathsf{w}(\xi) \;=\; 0.$$

Moreover, we have for all $x \in Z$:

$$A_{n+1}(x) = \big\langle \big(D\Phi(x)\big)^{\mathrm{T}} B\big(\Phi(x)\big), e_{n+1} \big\rangle = \big\langle B\big(\Phi(x)\big), D\Phi(x) e_{n+1} \big\rangle =$$

$$= \Big\langle B\big(\Phi(x)\big), \frac{\partial \Phi}{\partial x_{n+1}}(x) \Big\rangle = \Big\langle B(\xi), \frac{\partial \Phi}{\partial x_{n+1}}\big(\Phi^{-1}(\xi)\big) \Big\rangle \geq \varepsilon.$$

Theorem 3.7. yields the existence of $Q \in C^1\big(\overline{Z},\mathbb{R}\big)$, $\lambda \in C^0\big(\overline{Z},\mathbb{R}\big)$ satisfying

$$\forall x \in Z: \quad \nabla Q(x) = \lambda(x) A(x), \quad \lambda(x) \neq 0.$$

Therefore we have

$$\big(D\Phi^{-1}\big)^{\mathrm{T}}(\xi)\, \nabla_x Q\big(\Phi^{-1}(\xi)\big) \;=\; \lambda\big(\Phi^{-1}(\xi)\big)\big(D\Phi^{-1}\big)^{\mathrm{T}}(\xi)\, A\big(\Phi^{-1}(\xi)\big),$$

thus

$$\forall \xi \in \mathfrak{F}: \quad \nabla_\xi \big(Q \circ \Phi^{-1}\big)(\xi) \;=\; \big(\lambda \circ \Phi^{-1}\big)(\xi)\, B(\xi).$$

Now $\mathfrak{Q} := Q \circ \Phi^{-1}$ and $\mu := \lambda \circ \Phi^{-1}$ have the desired property. $\qquad\square$

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