On the differential equation $yp_x - xp_y = R$ for real analytic functions with unknown p

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Synopsis The differential equation $yp_x - xp_y = R$ is investigated. We are looking for solutions being analytic in a neighborhood of the origin.

I. Introduction

When studying Poincaré's centre problem

$$y' = -\frac{x+P}{y+Q} = -\frac{\mathcal{A}(x,y)}{\mathcal{B}(x,y)}$$

around $(0,0)$ a major rôle is played by the differential equation

$$yp_x - xp_y = R \tag{1}$$

with given right hand side R. It serves to construct recursively a formal power series p with

$$\det \begin{pmatrix} p_x & p_y \\ \mathcal{A} & \mathcal{B} \end{pmatrix} = \sum_{j=1}^{\infty} d_j (x^{2j+2} + y^{2j+2}). \tag{A}$$

In the present paper we derive necessary and sufficient conditions on R under which (1) has as solution a formal power series around $(0,0)$. Then convergence is studied.

II. Recursion

Let $R = r_1 + r_2 + \ldots$ be a formal power series around $(0,0)$ with homogeneous parts r_i of degree i. For p we use the ansatz $p = p_1 + p_2 + \ldots$ with homogeneous parts of degree i again. The equation in question is solved for each degree l separately. This means we solve

$$yp_{lx} - xp_{ly} = r_l, \ l \geq 1.$$

For convenience we write occasionally p, r instead of p_l, r_l. Let

$$\begin{aligned} p &= \sum_{\nu+\mu=l} p_{\nu\mu} x^\nu y^\mu, \\ r &= \sum_{\nu+\mu=l} r_{\nu\mu} x^\nu y^\mu. \end{aligned}$$

Then

$$\begin{aligned} p_x &= \sum_{\nu+\mu=l} \nu p_{\nu\mu} x^{\nu-1} y^\mu = \sum_{\nu+\mu=l-1} (\nu+1) p_{\nu+1\mu} x^\nu y^\mu, \\ yp_x &= \sum_{\nu+\mu=l-1} (\nu+1) p_{\nu+1\mu} x^\nu y^{\mu+1} = \sum_{\nu+\mu=l} (\nu+1) p_{\nu+1\mu-1} x^\nu y^\mu, \\ p_y &= \sum_{\nu+\mu=l} \mu p_{\nu\mu} x^\nu y^{\mu-1} = \sum_{\nu+\mu=l-1} (\mu+1) p_{\nu\mu+1} x^\nu y^\mu, \\ xp_y &= \sum_{\nu+\mu=l} (\mu+1) p_{\nu-1\mu+1} x^\nu y^\mu. \end{aligned}$$

For the coefficients of p, r we thus obtain

$$(2) \qquad (\nu+1) p_{\nu+1\mu-1} - (\mu+1) p_{\nu-1\mu+1} = r_{\nu\mu}, \ \nu+\mu = l.$$

Proposition 1: *Let $\nu + \mu = l$ be odd. Then (2) has a unique solution $p_{0l}, p_{1l-1}, \ldots, p_{l0}$.*

Proof: (2) reads as follows.

$$\begin{aligned} 1 \cdot p_{1l-1} &= r_{0l}, \\ 2 \cdot p_{2l-2} - l \cdot p_{0l} &= r_{1l-1}, \\ 3 \cdot p_{3l-3} - (l-1) p_{1l-1} &= r_{2l-2}, \\ &\vdots \\ l \cdot p_{l0} - 2 p_{l-22} &= r_{l-11}, \\ -p_{l-11} &= r_{l0}. \end{aligned}$$

First we consider $p_{\nu+1\mu-1}$, $p_{\nu-1\mu+1}$ with $\nu+1, \nu-1$ odd. Then we can solve

$$\begin{aligned}
&1 \cdot p_{1l-1} = r_{0l}, \\
&3 \cdot p_{3l-3} - (l-1)p_{1l-1} = r_{2l-2}, \\
&5 \cdot p_{5l-3} - (l-3)p_{3l-3} = r_{4l-4}, \\
&\vdots \\
&l \cdot p_{l0} - 2p_{l-22} = r_{l-11}
\end{aligned}$$

succesively and see that the $p_{\lambda\mu}$ with λ odd are determined uniquely. As for $\nu+1, \nu-1$ even we obtain

$$\begin{aligned}
2p_{2l-2} - lp_{0l} &= r_{1l-1}, \\
4p_{4l-4} - (l-2)p_{2l-2} &= r_{3l-3}, \\
&\vdots \\
(l-1)p_{l-11} - 3p_{l-33} &= r_{l-22}, \\
-p_{l-11} &= r_{l0}.
\end{aligned}$$

Starting backward with p_{l-11} we arrive at the uniquely determined $p_{\lambda\mu}$ with even λ. □

If l is even the situation is more complicated.

Proposition 2: *Let $\nu+\mu = l$ be even. Then (2) has a solution if and only if a compatibility condition holds. This is*

$$\sum_{i=0}^{l/2} a_{2i}^{(l)} r_{2il-2i} = 0 \quad \textit{for } l \textit{ even} \tag{3}$$

with certain uniquely determined coprime positive integers $a_0^{(l)}, a_2^{(l)}, \cdots, a_l^{(l)}$.

Proof: Let $\nu+1, \nu-1$ be odd. We obtain

$$\begin{cases}
1 \cdot p_{1l-1} = r_{0l}, \\
3 \cdot p_{3l-3} - (l-1)p_{1l-1} = r_{2l-2}, \\
\vdots \\
(l-1)p_{l-11} - 3p_{l-33} = r_{l-22},
\end{cases} \tag{4}$$

$$-p_{l-11} = r_{l0}. \tag{5}$$

(4) determines $p_{1l-1}, \ldots, p_{l-11}$ uniquely. To fulfill (5) we need

$$\begin{aligned}
3p_{3l-3} &= (l-1)p_{1l-1} + r_{2l-2}, \\
&= (l-1)r_{0l} + r_{2l-2}.
\end{aligned}$$

Now

$$5p_{5l-5} = (l-3)p_{3l-3} + r_{4l-4}$$

If we insert for p_{3l-3} and continue this way we arrive at

$$(l-1)p_{l-11} = \sum_{i=1}^{(l-2)/2} q_{2i}^{(l)} r_{2il-2}$$

with positive rational numbers $q_{2i}^{(l)}$. Employing (5) yields the assertion (3). If $\nu+1, \nu-1$ are even we obtain

$$(6) \qquad \begin{cases} 2p_{2l-2} - lp_{0l} = r_{1l-1} \\ 4p_{4l-4} - (l-2)p_{2l-2} = r_{3l-3} \\ \quad\vdots \\ lp_{l0} - 2p_{l-22} = r_{l-11} \end{cases}$$

(6) is underdetermined and has infinitely many solutions which can be parametrized with respect to $\lambda_l = p_{0l}$. □

We arrive now at

Theorem 3:

1. *Let*

$$R = \sum_{\nu+\mu\geq 1} r_{\nu\mu} x^\nu y^\mu$$

be a formal power series. Then there is a formal power series

$$(7) \qquad p = \textstyle\sum_{\nu+\mu\geq 1} p_{\nu\mu} x^\nu y^\mu$$

such that

$$(8) \qquad yp_x - xp_y = R$$

provided (3) is satisfied. If conversely (8) is satisfied for a formal power series (7) then (3) holds.

2. *Let p be a formal power series as in 1. In particular (8) is satisfied. If $\mathfrak{x} = (x,y)^T$ and*

$$i_0 = \min\{i | r_i \not\equiv 0\} \geq 2$$

then we may have $p = 0(|\mathfrak{x}|^{i_0})$ formally, this is: p starts with r_{i_0} only.

Proof: The first part follows from Propositions 1 and 2 and the necessity of (4,5). As for the second part we remark that

$$r_{\nu\mu} = 0 \text{ for } \nu + \mu \le i_0 - 1.$$

Thus we can set $p_{\nu\mu} = 0$, $\nu + \mu \le i_0 - 1$. □

Some examples may in order now.

Example 1. We consider the question if in

$$y' = -\frac{x + 4x^2 y + y^3}{y - 2x^3 + xy^2}$$

the origin can be made a center by adding higher order polynomials in the numerator and the denominator. This is example 3 in [1, p. 406]. Moreover it is shown in [1] that $d_1 = 0$, $d_2 \neq 0$ in the expansion (A). In fact we prove in [2] that our question can be answered positively if

$$\begin{aligned} yp_x - xp_y &= R = P_y - Q_x \text{ with} \\ P &= 4x^2 y + y^3 \\ Q &= 2x^3 + xy^2 \end{aligned}$$

is solvable in the sense of Theorem 3. Since degree $(P_y - Q_x) = 2$ is even we have to employ (3). Since (4,5) read

$$p_{11} = r_{02},$$

$$-p_{11} = r_{20},$$

this is

$$r_{02} + r_{20} = 0,$$

and since

$$R = P_y - Q_x = 4x^2 + 3y^2 - bx^2 - y^2 = -2x^2 + 2y^2$$

thus satisfies (3), the present example can be subsumed under Theorem 3.

Example 2. Let R be a homogeneous polynomial of degree 4. Then (4,5) read

$$\begin{aligned} p_{13} &= r_{04}, \\ 3p_{31} - 3p_{13} &= r_{22}, \\ -p_{31} &= r_{40}. \end{aligned}$$

The necessary and sufficient condition for the solvability of this system is

$$3r_{04} + r_{22} + 3r_{40} = 0,$$

this is

$$\sum_{i=0}^{2} a_{2i}^{(4)} r_{2il-2i} = 0 \text{ with}$$

$$a_0^{(4)} = 3,\ a_2^{(4)} = 1, a_4^{(4)} = 3.$$

III. Convergence

In this section we are going to show that convergence of the power series R implies convergence of the power series p provided (3) is satisfied.

Let l be odd. According to the proof of Proposition 1 the first system we have to solve is

$$\begin{array}{rcl} p_{1l-1}+0+0+0+\ldots+0+0 &=& r_{0l} \\ -(l-1)p_{1l-1}+3p_{3l-3}+0+0+\ldots+0+0 &=& r_{2l-2} \\ 0-(l-3)p_{3l-3}+5p_{5l-5}+0+\ldots+0+0 &=& r_{4l-4} \\ &\vdots& \\ 0+0+0+0+\ldots-2p_{l-22}+lp_{l_0} &=& r_{l-11} \end{array}$$

with the $\frac{l+1}{2}\times\frac{l+1}{2}$-matrix

$$\mathcal{M}_1(l)=\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ -(l-1) & 3 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -(l-3) & 5 & 0 & \cdots & 0 & 0 \\ & & & \vdots & & & \\ 0 & 0 & 0 & 0 & & -2 & l \end{pmatrix}=(a_{ik}).$$

Thus $\mathcal{M}_1(l)$ has nonvanishing elements only in the diagonal and the subdiagonal. If M_{ik} originates from $\mathcal{M}_1(l)$ by cancelling the i-th row and the k-th column we prove now that $\det M_{ik}=0$ for $i\geq k+1$. Thus $(\det M_{ik})$ is upper triangular. To see this, let $n=\frac{l+1}{2}$. Then

$$\det M_{ik}=\sum_{\substack{1\leq\nu_1,\nu_2,\ldots,\nu_{i-1},\nu_{i+1},\ldots,\nu_n\leq n \\ (\nu_1,\ldots,\nu_{i-1},\nu_{i+1},\ldots,\nu_n)\text{ is a} \\ \text{permutation of }(1,\ldots,k-1,k+1,\ldots,n)}}\pm a_{1\nu_1}\ldots a_{i_1\nu_{i-1}}a_{i+1\nu_{i+1}}\ldots a_{n\nu_n}.$$

Let us assume that $\nu_j\leq j$, $1\leq j\leq i-1$, $i-1\geq k$ for some member of the last sum. The $\nu_1,\ldots,\nu_{i-1}$ are pairwise distinct. Therefore $\nu_1=1,\nu_2=2,\ldots,\nu_{i-1}=i-1$, and in particular $\nu_k=k$, which is a contradiction. Consequently there exists a j with $1\leq j\leq i-1$ and $j<\nu_j$. But then $a_{j\nu_j}=0$. Observe that we have used only that (a_{ik}) is lower triangular. A subexample may be in order. Take

$$(a_{ik}) = \begin{pmatrix} a & 0 & 0 \\ \alpha & b & 0 \\ \delta & \beta & c \end{pmatrix},$$

then $\det M_{21} = 0$, $\det M_{31} = 0$, $\det M_{32} = 0$. As for M_{ik} let us for instance assume that $k > i$, i. e. $k-1 \geq i$. We thus cancel the i-th row and the k-th column. This corresponds to the element a_{ik} in $\mathcal{M}_1(l) = (a_{pq})_{1 \leq p,q \leq n=(l+1)/2}$. Moreover we have with respect to the norm $||\mathfrak{x}|| = \sum_{k=1}^{m} |x_k|$ in $\mathbb{R}^m$ for a linear mapping $B = (b_{qp})_{1 \leq p,q \leq m} : \mathbb{R}^m \to \mathbb{R}^m$ the inequality

$$||B|| \leq \max_{1 \leq q \leq m} \sum_{p=1}^{m} |b_{pq}|.$$

In what follows we still employ these norms. Considering the last tow points we *first* have

$$M_{ik} = i \left(\begin{array}{c|c|c} \begin{matrix} a_{11} & I' = \text{lower triangular} \; 0 \\ a_{21} & \\ 0 & \\ a_{i-1\,i-2} & a_{i-1\,i-1} \end{matrix} & 0 & \overset{k}{0} \\ \hline 0 & \begin{matrix} a_{i+1\,i} & a_{i+1\,i+1} & II' = \text{upper triangular} \; 0 \\ & & a_{k-1\,k-1} \\ 0 & & a_{k\,k-1} \end{matrix} & 0 \\ \hline 0 & 0 & \begin{matrix} a_{k+1\,k+1} & 0 & \text{II} = \text{lower triangular} \\ a_{k+2\,k+1} & & \\ 0 & a_{n\,n-1} \, a_{nn} & \end{matrix} \end{array} \right).$$

Employing Laplace's Theorem we obtain

$$\begin{aligned} \det M_{ik} &= \det I' \cdot \det I'' \cdot \det II \text{ where} \\ & \quad I' \text{ is an } (i-1) \times (i-1)\text{-matrix} \\ & \quad I'' \text{ is a } (k-i) \times (k-i)\text{-matrix,} \\ & \quad II \text{ is a } (n-k) \times (n-k)\text{-matrix,} \\ \det I' &= 1 \cdot \ldots \cdot [2(i-1)-1] \text{ with odd factors} \\ \det I'' &= (l-(2i-1)) \cdot \ldots \cdot (l-(2(k-1)-1))(-1)^{k-i} \text{ with even factors} \\ \det II &= (2(k+1)-1) \cdot \ldots l \text{ with odd factors} \end{aligned}$$

Thus

$$|\det M_{ik}| \leq l^{\frac{l+1}{2}-1}.$$

It is easily seen now that this estimate is correct also for $k \leq i$. Set $\mathfrak{p} = (p_{1l-1}, p_{3l-3}, \ldots, p_{l0})^T \in \mathbb{R}^{(l+1)/2}$. $\mathcal{M}_1(l)$ defines an isomorphism from $\mathbb{R}^{(l+1)/2}$ onto itself which is also denoted by $\mathcal{M}_1(l)$. With $\mathfrak{r} = (r_{0l}, r_{2l-2}, \ldots, r_{l-11})^T \in \mathbb{R}^{(l+1)/2}$ we obtain

$$\mathfrak{p} = \mathcal{M}_1^{-1}(l)\mathfrak{r}$$

$$||\mathfrak{p}||_{(l+1)/2} \leq ||\mathcal{M}_1^{-1}(l)||||\mathfrak{r}||_{(l+1)/2},$$

$$\begin{aligned}
\mathcal{M}_1^{-1}(l) &= \frac{1}{\det \mathcal{M}_1(l)} \begin{pmatrix} \det M_{11} & & & 0 \\ (-1)\det M_{12} & \det M_{22} & & \\ \det M_{13} & & \ddots & \\ \vdots & & & \det M_{\frac{l+1}{2}\frac{l+1}{2}} \end{pmatrix} \\
&= \frac{1}{1 \cdot 3 \cdot \ldots \cdot l}((-1)^{i+k} \det M_{ik})^T
\end{aligned}$$

Thus we **secondly** have

$$||\mathcal{M}_1^{-1}(l)|| \leq \frac{1}{1 \cdot 3 \cdot \ldots \cdot l} \frac{l+1}{2} l^{\frac{l+1}{2}-1}$$

According to the proof of Proposition 1 we have as second linear system

$$\begin{aligned}
-lp_{0l} + 2p_{2l-2} + 0 + 0 + \ldots + 0 + 0 + 0 &= r_{1l-1} \\
0 - (l-2)p_{2l-2} + 4p_{4l-4} + 0 + \ldots + 0 + 0 + 0 &= r_{3l-3} \\
&\vdots
\end{aligned}$$

$$0 + \ldots + 0 - 3p_{l-33} + (l-1)p_{l-11} = r_{l-22}$$

$$0 + \ldots + 0 + 0 - p_{l-11} = r_{l0}$$

with the $(l+1)/2 \times (l+1)/2$ matrix

$$\mathcal{M}_2(l) = \begin{pmatrix} -l & 2 & 0 & 0 & \ldots & 0 & 0 \\ 0 & -(l-2) & 4 & 0 & \ldots & 0 & 0 \\ & & & \vdots & & & \\ & 0 & & 0 & \ldots & -3 & l-1 \\ & & & 0 & & 0 & -1 \end{pmatrix}$$

As before we arrive at

$$||\mathcal{M}_2^{-1}(l)|| \leq \frac{1}{1\cdot 3\cdot \ldots \cdot l}\frac{l+1}{2}l^{\frac{l+1}{2}-1}$$

Now let l be even. Since the calculations for the inverses of the matrices in (4,6) are very much the same as for $\mathcal{M}_1^{-1}(l)$ only a brief discussion is necessary. We assume that the compatibility condition (3) holds. As for (6) we set $p_{0\lambda} = 0$.

Then we have in (4) the $l/2 \times l/2$-matrix

$$\mathcal{M}_3(l) = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 & 0 \\ -(l-1) & 3 & 0 & \ldots & 0 & 0 \\ 0 & & & & & 0 \\ 0 & 0 & 0 & \ldots & -3 & l-1 \end{pmatrix}$$

with

$$||\mathcal{M}_3^{-1}(l)|| \leq \frac{1}{1\cdot 3\cdot \ldots \cdot (l-1)}\frac{l}{2}l^{\frac{1}{2}-1}.$$

As for (6) we obtain the $l/2 \times l/2$-matrix

$$\mathcal{M}_4(l) = \begin{pmatrix} 2 & 0 & 0 & \ldots & 0 & 0 \\ -(l-2) & 4 & 0 & \ldots & 0 & 0 \\ 0 & -(l-4) & 6 & \ldots & 0 & 0 \\ & 0 & & & 0 & \\ 0 & 0 & 0 & \ldots & -2 & l \end{pmatrix}$$

with

$$||\mathcal{M}_4^{-1}(l)|| \leq \frac{1}{2\cdot 4\cdot \ldots \cdot l}\frac{1}{2}l^{\frac{l}{2}-1}$$

We conclude with

Theorem 4: *Let $\rho > 0$,*

$$R = \sum_{\nu+\mu\geq 1} r_{\nu\mu} x^\nu y^\mu$$

be a convergent power series in $|x| < \rho, |y| < \rho$. *The coefficients* $r_{\nu\mu} \in \mathbb{R}$ *are supposed to satisfy the compatibility condition* (3) *if* $\nu + \mu = l$ *is even.*

Then the partial differential equation

$$yp_x - xp_y = R$$

has a formal solution

$$\begin{aligned} p &= \sum_{\nu+\mu\geq 1} p_{\nu\mu} x^\nu y^\mu \\ &= \sum_{l=1}^{\infty} \sum_{\substack{\nu+\mu=l, \\ \nu\geq 1 \text{if } l \text{ even}}} p_{\nu\mu} x^\nu y^\mu + \sum_{l=2, l \text{ even}}^{\infty} (p_{0l} = \lambda_l) y^l \end{aligned}$$

for any values $\lambda_2, \lambda_4, \ldots$. *The series for* p *is convergent in* $|x| < \frac{\rho}{\sqrt{e}}$, $|y| < \frac{\rho}{\sqrt{e}}$ *if the* $\lambda_2, \lambda_4, \ldots$ *are chosen in such a way that* $\lambda_2 y^2 + \lambda_4 y^4 + \ldots$ *converges in* $|y| < \frac{\rho}{\sqrt{e}}$.

Proof: In what follows $\widetilde{\nu}$ is a multiindex of $\mathbb{R}^2$. With $\mathfrak{x} = (x, y)^T$, $p = p(\widetilde{\mathfrak{x}}) = \sum_{|\widetilde{\nu}|\geq 1} p_{\widetilde{\nu}} \mathfrak{x}^{\widetilde{\nu}}$, $R = R(\widetilde{\mathfrak{x}}) = \sum_{|\widetilde{\nu}|} r_{\widetilde{\nu}} \mathfrak{x}^{\widetilde{\nu}}$ we obtain

$$\begin{aligned} \sum_{|\widetilde{\nu}|=l} |p_{\widetilde{\nu}}| &\leq \frac{l+1}{2} \frac{l^{\frac{l+1}{2}-1}}{1 \cdot 3 \cdot \ldots \cdot l} \sum_{|\widetilde{\nu}|=l} |r_{\widetilde{\nu}}|, \ l \text{ odd}, \\ \sum_{\substack{|\widetilde{\nu}|=l \\ \widetilde{\nu}\neq(0,l)}} |p_{\widetilde{\nu}}| &\leq \frac{l}{2} \max\left(\frac{l^{\frac{l}{2}-1}}{1 \cdot 3 \cdot \ldots \cdot (l-1)}, \frac{l^{\frac{l}{2}-1}}{2 \cdot 4 \cdot \ldots \cdot l} \right) \sum_{\substack{|\widetilde{\nu}|=l \\ \widetilde{\nu}\neq(0,l)}} |r_{\widetilde{\nu}}|, l \text{ even}. \end{aligned}$$

For l odd we have

$$\begin{aligned} l! &= 1 \cdot 3 \cdot \ldots \cdot l \cdot 2 \cdot 4 \cdot \ldots \cdot (l-1), \\ &\leq (1 \cdot 3 \cdot \ldots \cdot l)^2, \end{aligned}$$

and for l even

$$\begin{aligned} l! &= 1 \cdot 3 \cdot \ldots \cdot (l-1) \cdot 2 \cdot 4 \cdot \ldots \cdot l, \\ &\leq (2 \cdot 4 \cdot \ldots \cdot l)^2, \end{aligned}$$

$$l(l-1)\cdot\ldots\cdot 3\cdot 1 \geq l(l-2)\cdot\ldots\cdot 2.$$

Stirling's forumula now shows

$$\sum_{\substack{|\tilde{\nu}|=l\\ \tilde{\nu}\neq(0,l)\\ \text{for } l \text{ even}}} |p_{\tilde{\nu}}| \leq c\sqrt{l}e^{\frac{l}{2}} \sum_{|\tilde{\nu}|=l} |r_{\tilde{\nu}}|,$$

c being a constant which does not depend on l. This estimate completes the proof. □

References

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