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Introduction to the Cosserat problem

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The study of the Cosserat spectrum started more than 100 years ago with a series of papers [2]–[10] published between 1898 and 1901 by the French scientists Eugène and François Cosserat. Their motivation was to expand the solutions of certain basic problems of static elasticity into eigenvectors. Let $B_R := \{x \in \mathbb{R}^n : |x| < R\}$ for $n \geq 2$. In case $n = 3$ they tried to solve the following boundary value problem: If

$$\underline{u}_0 = (u_{01}, u_{02}, u_{03}) \in \left[C^\infty(B_R) \cap C^0(\bar{B}_R)\right]^3$$

satisfying $\Delta \underline{u}_0 = 0$ in B_R is given, find for $\sigma \in \mathbb{R}$ a solution of

$$\Delta \underline{u} + \sigma \nabla \operatorname{div} \underline{u} = 0 \text{ in } B_R, \quad \underline{u} \mid_{\partial B_R} = \underline{u}_0 \mid_{\partial B_R}. \tag{0.1}$$

They made use of an interesting formula. If $f \in C^\infty(B_R)$ is a harmonic homogeneous polynomial of degree $j \in \mathbb{N}_0$, then the solution of the Dirichlet problem

$$\Delta u = f \text{ in } B_R, \quad u \mid_{\partial B_R} = 0 \tag{0.2}$$

is given by

$$u(x) = \frac{1}{2(n+2j)} \big(|x|^2 - R^2\big) f(x). \tag{0.3}$$

If $p^{(k)}$ is a harmonic homogeneous polynomial of degree $k \in \mathbb{N}$, then a solution $\underline{v}^{(k)} = (v_1^{(k)}, \dots, v_n^{(k)}) \in \left[C^\infty(B_R) \cap C^0(\bar{B}_R)\right]^n$ of the Dirichlet problem

$$\Delta \underline{v}^{(k)} = \nabla p^{(k)} \text{ in } B_R, \quad \underline{v}^{(k)} \mid_{\partial B_R} = 0, \tag{0.4}$$

is given by

$$v_i^{(k)}(x) := \frac{1}{2(n-2+2k)} \big(|x|^2 - R^2\big) \partial_i p^{(k)}(x), \quad i = 1, \dots, n. \tag{0.5}$$

This is an easy consequence of (0.2) and (0.3). One readily calculates for $k \geq 1$

$$\operatorname{div} \underline{v}^{(k)} = \frac{k}{n-2+2k} p^{(k)}(x)$$

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and therefore

$$\Delta \underline{v}^{(k)} = \lambda_k \nabla \operatorname{div} \underline{v}^{(k)} \text{ in } B_R \text{ with } \lambda_k = \frac{n-2+2k}{k}. \tag{0.6}$$

It seems remarkable that for $n = 2$ all $\lambda_k = 2$ and that $\lambda_k > 2$, $\lambda_k \to 2$ $(k \to \infty)$ for $n \geq 3$. The solution of (0.1) is then easily constructed as follows: Let $p := \operatorname{div} \underline{u}_0$. Then there exist uniquely determined harmonic homogeneous polynomials $p^{(k)}$ such that

$$p(x) = \sum_{k=0}^{\infty} p^{(k)}(x) \text{ for } x \in B_R,$$

where the series converges absolutely and uniformly on every compact subset $C \subset B_R$ (see e.g. [1, Corollary 5.23, p. 84]). Let $\underline{v}^{(k)}$ be the solution of (0.4) given by (0.5) such that (0.6) holds true. Then it is readily seen that if $\sigma \neq -\lambda_k$ for all $k \in \mathbb{N}$ the solution of (0.1) is (at least formally) represented by

$$\underline{u}(x) = \underline{u}_0(x) - \sum_{k=1}^{\infty} \frac{\lambda_k \sigma}{\lambda_k + \sigma} \underline{v}^{(k)}(x), \quad x \in B_R. \tag{0.7}$$

At this point we have to observe that at the year of publication (1898) no other methods than explicit calculations were available to solve problems like (0.1). E.g. Fredholm's method of integral equations was developed later. Clearly the Cosserat brothers tried to extend their method to more general domains like an ellipsoid or a spherical shell $\{x \in \mathbb{R}^3 : 0 < r < |x| < R\}$. They did so in further papers. More general, if $G \subset \mathbb{R}^n$ $(n \geq 2)$ is a bounded domain, a number $\lambda \in \mathbb{R}$ is called a Cosserat eigenvalue if there exists a non-trivial $\underline{v} = (v_1, \ldots, v_n) \in \left[C^2(G) \cap C^0(\bar{G})\right]^n$ such that

$$\Delta \underline{v} = \lambda \nabla \operatorname{div} \underline{v} \text{ in } G, \qquad \underline{v}\,|_{\partial G} = 0. \tag{0.8}$$

More than 65 years later this problem was again studied by S. G. Mikhlin, with completely different methods and for more general domains. He published several papers between 1966 and 1973 [18], one in 1967 together with V. G. Maz'ya [17]. For a detailed history of the Cosserat problem we refer to A. Kozhevnikov's review article [16]. Using the method of pseudo-differential operators, A. Kozhevnikov investigated in several papers between 1993 and 2000 [13]–[15] the Cosserat spectrum for the four boundary value problems of static elasticity theory. A good knowledge of the Cosserat spectrum has a lot of applications as well in theoretical as in numerical analysis. E.g. W. Velte pointed out [25]–[27] that the optimal constants in certain inequalities are related to the Cosserat eigenvalues. As an example from numerical analysis we refer to M. Crouzeix's paper [11] concerning Uzawa's algorithm.

In the subsequent papers a weak L^q-version $(1 < q < \infty)$ of (0.8) is regarded. Let us briefly describe that procedure in the case of a bounded domain G with boundary $\partial G \in C^2$. A vector field $\underline{v} = (v_1, \ldots, v_n) \in \underline{H}_0^{1,q}(G) := \left[H_0^{1,q}(G)\right]^n$ is called a weak L^q-Cosserat eigenvector to the Cosserat eigenvalue $\lambda \in \mathbb{R}$ if $\underline{v} \neq 0$ and if

$$\langle \nabla \underline{v}, \nabla \underline{\phi} \rangle = \lambda \langle \operatorname{div} \underline{v}, \operatorname{div} \underline{\phi} \rangle \qquad \forall \underline{\phi} \in \underline{H}_0^{1,q'}(G), \quad q' := \frac{q}{q-1}. \tag{0.9}$$

Here $H_0^{1,s}(G)$ denotes the "usual" Sobolev space equipped with norm $\|\nabla.\|_s$ $(1 < s < \infty)$ with the properties $C_0^\infty(G) \subset H_0^{1,s}(G)$ and $H_0^{1,s}(G) = \overline{C_0^\infty(G)}^{\|\nabla.\|_s}$. For $\underline{u} \in \underline{H}_0^{1,q}(G)$, $\underline{\phi} \in \underline{H}_0^{1,q'}(G)$

$$\langle \nabla \underline{u}, \nabla \underline{\phi} \rangle = \sum_{i,k=1}^{n} \int_G \partial_i u_k \partial_i \phi_k dx$$

and for $f \in L^q(G)$, $g \in L^{q'}(G)$ we set

$$\langle f, g \rangle := \int_G fg dx.$$

Let $L_0^q(G) := \left\{ p \in L^q(G) : \int_G p dx = 0 \right\}$. The procedure in the subsequent papers is as follows: For $p \in L_0^q(G)$ there exists a unique $\underline{v} \in \underline{H}_0^{1,q}(G)$ such that (see [21])

$$\langle \nabla \underline{v}, \nabla \underline{\phi} \rangle = \langle p, \operatorname{div} \underline{\phi} \rangle \qquad \forall \underline{\phi} \in \underline{H}_0^{1,q'}(G). \tag{0.10}$$

Let $Z_q : L_0^q(G) \to L_0^q(G)$ be defined by

$$Z_q(p) := \operatorname{div} \underline{v} \tag{0.11}$$

where $\underline{v}$ is the solution of (0.10). Then (0.9) is equivalent to $\lambda Z_q(p) = p$. Therefore it suffices to investigate the operator Z_q. The authors make essential use of a direct decomposition ($q = 2$: orthogonal decomposition) being equivalent to weak L^q-solvability of the Dirichlet problem for the Bilaplacian Δ^2:

$$\begin{aligned} &L^q(G) = A^q(G) \oplus B^q(G), \text{ where} \\ &A^q(G) := \left\{ \Delta s : s \in H_0^{2,q}(G) \right\} \\ &B^q(G) := \left\{ p \in L^q(G) : \langle p, \Delta s \rangle = 0 \quad \forall s \in H_0^{2,q'}(G) \right\} \end{aligned} \tag{0.12}$$

and there is a constant $K_q > 0$ such that

$$\|\Delta s\|_q + \|p\|_q \le K_q \|\Delta s + p\|_q \qquad \forall s \in H_0^{2,q}(G), \forall p \in B^q(G).$$

Here $H_0^{2,q}(G) := \overline{C_0^\infty(G)}^{\|\cdot\|_{2,q}}$ equipped with norm $\|u\|_{2,q} := \left(\sum_{i,k=1}^n \|\partial_i \partial_k u\|_q^q \right)^{\frac{1}{q}}$. An equivalent norm on $H_0^{2,q}(G)$ is given by $\|\Delta.\|_q$. This decomposition holds true for bounded domains G with boundary $\partial G \in C^2$. If $G \subset \mathbb{R}^n$ is an unbounded domain with $\partial G \in C^2$ (e.g. a half-space or an exterior domain), the spaces $H^{2,q}(G)$ have to be replaced by slightly bigger spaces $\hat{H}_0^{2,q}(G)$ resp. $\hat{H}_\bullet^{2,q}(G)$ ((0.12) was shown in [20] for bounded domains and in [19] for exterior domains too). For $s \in H_0^{2,q}(G)$ it follows immediately $\Delta s \in L_0^q(G)$, whence

$$L_0^q(G) = A^q(G) \oplus B_0^q(G) := \left\{ p \in B^q(G) : \int_G p dx = 0 \right\}. \tag{0.13}$$

For $p = \Delta s \in A^q(G)$ it is readily seen that $\underline{v} = \nabla s \in \underline{H}_0^{1,q}(G)$ is the solution of (0.10) and $\operatorname{div} \underline{v} = \Delta s = p$, whence $Z_q(p) = p \; \forall p \in A^q(G)$, and $\lambda = 1$ is an eigenvalue of infinite multiplicity. For a complete characterization of the Cosserat spectrum by (0.12) it suffices to study $Z_q \mid_{B_0^q(G)}$.

For the case of the half-space $H := \{x \in \mathbb{R}^n : x_n > 0\}$ this was performed in [22]. Then it turned out that

$$Z_q(p) = 2p \qquad \forall p \in B^q(H).$$

Therefore in H for all $n \geq 2$ and all $1 < q < \infty$ the only Cosserat eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 2$, each of infinite multiplicity. In that paper the decomposition (0.12) is used and the solution $\underline{v}$ of (0.10) is constructed with the classical method of images. Then, for $p \in B^q(H)$ the value of $Z_q(p)$ was calculated by use of the explicitly known reproducing kernel for harmonic L^q-functions in the half-space. Only results for scalar equations had been used but no results on elliptic systems. In addition for $n = 3$ an eigenvalue problem similar to (0.8) is studied (replace $\nabla \operatorname{div} \underline{v}$ by $\operatorname{rot} \operatorname{rot} \underline{v}$ on the right hand side of (0.8)). The results of [22] had been sufficient to build up a complete theory for equation (0.10) in bounded and exterior domains $G \subset \mathbb{R}^n$ and all $1 < q < \infty$ (cf. M. Stark [24]). But until now it was not possible to extend the spectral results from [22] even to the case of a "slightly perturbed" half-space

$$H_w := \left\{x = (x', x_n) : x_n > w(x')\right\}$$

(where $w \in C_0^\infty(\mathbb{R}^{n-1})$, $w(0) = 0$, $\nabla' w(0) = 0$ and $\|\nabla' w\|_{\infty, \mathbb{R}^{n-1}}$ "small").

In the case of a bounded domain $G \subset \mathbb{R}^n$ $(n = 2, 3)$ with Lipschitz boundary ∂G and for $q = 2$ M. Crouzeix used an ingenious ansatz ([11, Theorem 3, p. 245/246]) for the study of $Z_2 \mid_{B_0^2(G)}$ and he sketched the proofs. St. Weyers [30] succeeded to prove that Crouzeix's ansatz can be extended to the case of all $n \geq 2$, $1 < q < \infty$ and to bounded as well as to exterior domains $G \subset \mathbb{R}^n$ with sufficiently smooth boundaries ∂G. Following Crouzeix's ansatz and using regularity results from [20] and [21] he was able to show the existence of a constant $C_q > 0$ such that $\left[Z_q(p) - \frac{1}{2}p\right] \in H^{1,q}(G)$ and

$$\left\| Z_q(p) - \frac{1}{2}p \right\|_{H^{1,q}(G)} \leq C_q \|p\|_{L^q(G)} \qquad \forall p \in B_{(0)}^q(G) \tag{0.14}$$

(where $B_{(0)}^q(G) = B_0^q(G)$ if G is bounded and $B_{(0)}^q(G) := B^q(G)$ if G is an exterior domain). Here $H^{1,q}(G) := \{f \in L^q(G) : \exists \partial_i f \in L^q(G) \text{ (weakly)}\}$ is equipped with the full norm

$$\|f\|_{H^{1,q}(G)} := \left(\|f\|_{L^q(G)}^q + \sum_{i=1}^n \|\partial_i f\|_{L^q(G)}^q \right)^{\frac{1}{q}}$$

and denotes in any case the "usual" Sobolev space. If G is bounded, the compactness of the embedding $H^{1,q}(G) \hookrightarrow L^q(G)$ implies the compactness of the operator $\left(Z_q - \frac{1}{2}I\right) \mid_{B_0^q(G)}: B_0^q(G) \to B_0^q(G)$. In the case of an exterior domain G the embedding $H^{1,q}(G) \hookrightarrow L^q(G)$ is continuous, but no longer compact. To overcome this difficulty Weyers proved decay estimates [30, Theorems 8.7, 8.8] for L^q-functions ($1 \leq q < \infty$) being harmonic in the complement of a ball. Then the embedding $H^{1,q}(G) \cap B^q(G) \hookrightarrow B^q(G)$ turns out to be compact also for exterior domains. This result readily implies compactness of the operator $\left(Z_q - \frac{1}{2}I\right)|_{B^q(G)}\colon B^q(G) \to B^q(G)$ for exterior domains $G \subset \mathbb{R}^n$. Decay estimates also form the basis for the proof that all Cosserat eigenfunctions to Cosserat eigenvalues $\lambda \notin \{1, 2\}$ (each of finite multiplicity) have gradients integrable to any power $1 < r < \infty$ [30, Theorem 8.12]. Therefore the spectrum of Z_q, without the values 1 and 2, does not depend on $1 < q < \infty$.

Another very interesting fact concerning a relation between Green's function and the reproducing kernel for the Laplacian in bounded domains $G \subset \mathbb{R}^n$ is proved in [30, Theorem 1.5]. It is still an open question if that result could be proved directly by careful consideration of Green's function for the Laplacian and the reproducing kernel for harmonic functions. The result of [30, Theorem 1.5] is formally quite analogous to a recent result found by M. Englis, D. Lukkassen, J. Peetre and L.-E. Persson [12, Theorem 4.3, p. 113].

The results of [30] allow a lot of applications. Some of them are summarized in [23]. If $G \subset \mathbb{R}^n$ is bounded for $p \in L^q_0(G)$, let the unique solution $\underline{v} \in \underline{H}^{1,q}_0(G)$ of (0.10) be denoted by $\mathrm{T}_q(p)$ and let $M^q(G) := \mathrm{T}_q\left(L^q_0(G)\right)$. Then $\operatorname{div} : M^q(G) \to L^q_0(G)$ is a bijective continuous map with continuous inverse ([23, Theorems 3.1–3.5]). Let

$$D^{1,q}_0(G) := \left\{\underline{u} \in \underline{H}^{1,q}_0(G) : \operatorname{div} \underline{u} = 0\right\}.$$

Then the direct decomposition

$$\underline{H}^{1,q}_0(G) = D^{1,q}_0(G) \oplus M^q(G), \qquad 1 < q < \infty$$

readily follows [23, Theorem 3.6]. Analogous results hold for exterior domains if $\underline{H}^{1,q}_0(G)$ is replaced by the "larger" space $\hat{\underline{H}}^{1,q}_{\bullet}(G)$. Nearly trivial consequences then are the solvability of Stokes' equation (Theorem 4.4) and of the Lamé–Navier equation (Theorem 4.3). In section 6 the authors study the problem whether for the Cosserat eigenvalue $\lambda = 1$ there exist eigensolutions such that $\Delta \operatorname{div} \underline{v} = 0$. If G is either a bounded or an exterior domain with smooth boundary ∂G, then $\mathbb{R}^n \setminus \bar{G}$ has at most $N \in \mathbb{N}$ connected components (Lemma 5.1). If and only if $N \geq 2$ there is a $(N-1)$-dimensional space of eigensolutions $\underline{v}$ to the eigenvalue $\lambda = 1$ such that $\Delta \operatorname{div} \underline{v} = 0$ (Theorem 6.1). This space is spanned by gradients of $(N-1)$ solutions of certain Dirichlet problems for the Bilaplacian Δ^2 (Theorem 5.7).

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