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A GENERALIZED ENERGY FUNCTIONAL FOR PLANE COUETTE FLOW*

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Abstract. We present a generalized energy functional $\mathcal{E}$ for plane Couette flow providing conditional nonlinear stability for Reynolds numbers Re below $\mathrm{Re}_{\mathcal{E}} := 177.2$, which is larger than the ordinary energy stability limit. The method allows the explicit calculation of so-called stability balls in the $\mathcal{E}^{1/2}$-norm; i.e., the system is stable with respect to any perturbation with $\mathcal{E}^{1/2}$-norm in this ball.

Key words. hydrodynamical stability, plane Couette flow

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1. Introduction. Plane Couette flow is a paradigm with a long history of scientific investigation for a whole class of hydrodynamic stability problems, viz. plane parallel shear flows (cf. [DR, SH]). For these flows there is no obvious physical mechanism which triggers instability as, e.g., in circular Couette flow or in convection problems. Instead, viscous stresses seem to play the dominant role at the onset of instability. So, despite the striking simplicity of the set-up of these systems the onset of instability is up to the present insufficiently understood and its nature is the subject of ongoing research [TTRD, Gr].

The classical methods which yield rigorous stability results are the method of linearized stability and the energy method. The former method provides the critical value Re_c of the Reynolds number Re, below which the system is conditionally stable and above which it is unstable. In the case of Couette flow[1] it turns out that $\mathrm{Re}_c = \infty$, i.e., the system is linearly completely stable [Ro]. The second method provides global asymptotic stability below some value Re_E. For Couette flow $\mathrm{Re}_E = 82.6$ if Re is defined with the separation of the walls and their velocity difference [Jo]. This has to be compared with the experimentally observed onset of instability, which occurs at $\mathrm{Re} \approx 1300$ [Gr, DD]. Thus, none of the classical methods describes the instability behavior of Couette flow satisfactorily.

A more recent method which has successfully been applied to a couple of hydrodynamic stability problems uses generalized energy functionals which are better adjusted to the specific problems under consideration [J1, J2, GP, St]. A generalized energy functional $\mathcal{E}$ is a bilinear form of the dynamic variables of the problem. In comparison with the ordinary energy these variables are, however, differently weighted by additional coupling parameters and appear possibly in the form of higher derivatives. A first part $\mathcal{E}_1$ of the functional determines (analogously to the energy method) via a variational problem the stability boundary $\mathrm{Re}_{\mathcal{E}}$. The coupling parameters are chosen such that $\mathrm{Re}_{\mathcal{E}}$ becomes as large as possible—in particular, larger than Re_E.

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[1]Henceforth Couette flow always means *plane* Couette flow.

Contrary to the energy balance the nonlinear terms in general do not drop from the generalized energy balance. Therefore, a second part $\mathcal{E}_2$ involving higher derivatives of the dynamic variables is needed in $\mathcal{E}$ in order to dominate these terms. If the method works one obtains conditional stability for all Reynolds numbers below $\mathrm{Re}_{\mathcal{E}}$ together with explicit stability balls in the $\mathcal{E}^{1/2}$-norm. This is different from the method of linearized stability which gives no estimates for stability balls.

Generalized energy functionals have already been applied to plane parallel shear flows, however, under the assumption of stress-free boundary conditions for the perturbations [RM]. This assumption clearly overestimates the stability of these flows since wall induced stresses are neglected. In fact, the authors find conditional stability for all Reynolds numbers not only for Couette flow but also for Poiseuille flow, a system with finite critical Reynolds number if rigid boundary conditions are used.

If rigid boundary conditions are used no generalized energy functionals have been found so far, neither in plane parallel shear flows nor in any other hydrodynamical system with nontrivial basic flow and unrestricted (three-dimensional) perturbations. Moreover, in the case of Couette flow it has been argued that the generalized energy method as applied to systems with stress-free boundary conditions is incompatible with rigid boundary conditions [KT].

We present in this paper a generalized energy functional $\mathcal{E}$ for Couette flow (with correct rigid boundary conditions), which provides conditional stability for all Reynolds numbers below $\mathrm{Re}_{\mathcal{E}} = 177.2$. This number is still far below what is desired. However, there is now hope that still more appropriate functionals can be found which cover a larger stability region. The crucial point which allows the treatment of rigid boundary conditions is a more refined calculus inequality that takes advantage of the special geometry of the system.

The following point is of some historical interest: For Couette flow $\mathrm{Re}_{\mathcal{E}} = 177.2$ is just the two-dimensional energy stability limit, where perturbations are not allowed to vary in the spanwise direction. Following Orr [Or], early researchers in the field took this number for the correct energy stability limit. It came as a surprise when Joseph [Jo] showed that the complementary two-dimensional problem provided a considerably lower limit. Busse proved subsequently that the latter limit is in fact the correct energy stability limit [Bu]. Thus, our result may be viewed as a late justification of (a weakened version of) Orr's original claim.

The paper is organized as follows: Section 2 sets the mathematical framework for the subsequent analysis. In particular, we introduce the so-called poloidal-toroidal decomposition of divergence-free vector fields. This decomposition eliminates the divergence constraint and provides appropriate building blocks for generalized energy functionals. In section 3 a linear auxiliary problem is solved, viz. the variational problem associated with $\mathcal{E}_1$ which determines the stability limit $\mathrm{Re}_{\mathcal{E}}$. Section 4 provides estimates of the remaining terms in the energy balance of $\mathcal{E}_1 + \mathcal{E}_2$, the nonlinear terms in particular, and it formulates the basic stability result. Some well-known inequalities as well as the refined calculus inequality are collected in Appendix A, and the results of a numerical computation related to the variational problem are contained in Appendix B.

2. Mathematical setting. The Couette system is appropriately modeled by an infinite layer $\mathbb{R} \times (-\frac{1}{2}, \frac{1}{2})$ of thickness 1 with horizontal coordinates x, y and vertical coordinate z. The basic flow in this system takes the dimensionless form

$$\mathbf{U}_0 = \mathbf{U}_0(z) = \mathrm{Re} \begin{pmatrix} -z \\ 0 \\ 0 \end{pmatrix} \tag{2.1}$$

with Re being the Reynolds number based on the distance between bottom and top boundaries of the layer and their velocity difference. In order to investigate the stability of $\mathbf{U}_0$ we impose perturbations $\mathbf{u} = (u_x, u_y, u_z)$. These are governed by the system

$$\begin{aligned} \partial_t \mathbf{u} - \Delta \mathbf{u} - \mathrm{Re}(z\,\partial_x \mathbf{u} + u_z \mathbf{e}_x) + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p = 0, \\ \nabla\cdot\mathbf{u} = 0 \end{aligned} \tag{2.2}$$

in $\mathbb{R}^2 \times (-\frac{1}{2}, \frac{1}{2}) \times (0, T)$, $T > 0$, and satisfy the boundary conditions

$$\mathbf{u}(x, y, z, t) = 0 \qquad \text{for} \quad (x, y, z) \in \mathbb{R}^2 \times \left\{-\frac{1}{2}, \frac{1}{2}\right\},\ t > 0. \tag{2.3}$$

Here $\mathbf{e}_x = (1, 0, 0)^{\mathrm{T}}$. The initial value $\mathbf{u}(\cdot, \cdot, \cdot, 0) = \mathbf{u}_0$ is assumed to be given (and of course solenoidal). $\mathbf{u}$ corresponds to the velocity field of the perturbation and p denotes the pressure perturbation. Both $\mathbf{u}$ and ∇p are assumed to be x, y-periodic with respect to a rectangle $\mathcal{P} = (-\frac{\pi}{\alpha}, \frac{\pi}{\alpha}) \times (-\frac{\pi}{\beta}, \frac{\pi}{\beta})$ with wave numbers $(\alpha, \beta) \in \mathbb{R}^2_+$. In the following it suffices, therefore, to consider functions over the box

$$\Omega = \mathcal{P} \times \left(-\frac{1}{2}, \frac{1}{2}\right) = \left(-\frac{\pi}{\alpha}, \frac{\pi}{\alpha}\right) \times \left(-\frac{\pi}{\beta}, \frac{\pi}{\beta}\right) \times \left(-\frac{1}{2}, \frac{1}{2}\right).$$

As basic function space we take $L^2(\Omega)$. In what follows, $\|\cdot\|$ is always the norm in $L^2(\Omega)$ except in the case when applied to a function defined on $(-\frac{1}{2}, \frac{1}{2})$. Then, $\|\cdot\|$ means the norm in $L^2(-\frac{1}{2}, \frac{1}{2})$; the correct notion should be clear from the context. $(\cdot, \cdot)$ always denotes the scalar product associated with $\|\cdot\|$.

In order to cope with the divergence constraint $(2.2)_2$ we make use of the poloidal-toroidal decomposition [SW]:

$$\begin{aligned} \mathbf{u} &= \nabla \times (\nabla \times (\varphi\,\mathbf{e}_z)) + \nabla \times (\psi\,\mathbf{e}_z) + \mathbf{F} \\ &=: \boldsymbol{\delta}\varphi + \boldsymbol{\varepsilon}\psi + \mathbf{F}. \end{aligned} \tag{2.4}$$

Here $\mathbf{e}_z = (0, 0, 1)^{\mathrm{T}}$. The functions φ and ψ are determined uniquely if one requires them to be periodic with respect to $\mathcal{P}$ and to fulfill $\int_{\mathcal{P}} \varphi(x, y, z)\,dx\,dy = \int_{\mathcal{P}} \psi(x, y, z)\,dx\,dy = 0$ for every $z \in (-\frac{1}{2}, \frac{1}{2})$. The first part in (2.4) is called the poloidal part of $\mathbf{u}$ and the second part the toroidal one. The third part, the mean flow, depends only on z and has a constant third component. These three parts are mutually orthogonal in $L^2(\Omega)^3$. The vector operators $\boldsymbol{\delta}$ and $\boldsymbol{\varepsilon}$ have the form

$$\boldsymbol{\delta}\varphi = \begin{pmatrix} \partial_x\partial_z\varphi \\ \partial_y\partial_z\varphi \\ (-\Delta_2)\varphi \end{pmatrix}, \qquad \boldsymbol{\varepsilon}\psi = \begin{pmatrix} \partial_y\psi \\ -\partial_x\psi \\ 0 \end{pmatrix},$$

where $\Delta_2 = \partial_x^2 + \partial_y^2$ is the horizontal Laplacian. The boundary conditions (2.3) for $\mathbf{u}$ transform into

$$\varphi = \partial_z\varphi = 0, \quad \psi = 0, \quad F_x = F_y = 0 \qquad \text{for} \quad z = \pm\frac{1}{2}, \tag{2.5}$$

and $F_z(z) \equiv 0$. Applying the operators $\boldsymbol{\delta}$ and $\boldsymbol{\varepsilon}$ to $(2.2)_1$ as well as taking the mean with respect to $\mathcal{P}$, the system (2.2) can equivalently be formulated in terms of the new variables $(\varphi, \psi, F_x, F_y)$:

$$
\begin{aligned}
(-\Delta)(-\Delta_2)\partial_t\varphi + \Delta^2(-\Delta_2)\varphi - \operatorname{Re} z\,(-\Delta)(-\Delta_2)\partial_x\varphi + \boldsymbol{\delta}\cdot(\mathbf{u}\cdot\nabla\mathbf{u}) &= 0,\\
(-\Delta_2)\partial_t\psi + (-\Delta)(-\Delta_2)\psi - \operatorname{Re} z\,(-\Delta_2)\partial_x\psi + \operatorname{Re}(-\Delta_2)\partial_y\varphi - \boldsymbol{\varepsilon}\cdot(\mathbf{u}\cdot\nabla\mathbf{u}) &= 0,\\
\partial_t F_x + (-\partial_z^2)F_x + \tfrac{1}{|\mathcal{P}|}\textstyle\int_{\mathcal{P}} \tilde{\mathbf{u}}\cdot\nabla\tilde{u}_x\,dxdy &= 0,\\
\partial_t F_y + (-\partial_z^2)F_y + \tfrac{1}{|\mathcal{P}|}\textstyle\int_{\mathcal{P}} \tilde{\mathbf{u}}\cdot\nabla\tilde{u}_y\,dxdy &= 0.
\end{aligned}
\tag{2.6}
$$

$\tilde{\mathbf{u}} := \boldsymbol{\delta}\varphi + \boldsymbol{\varepsilon}\psi$ is that part of $\mathbf{u}$ which has vanishing mean value over $\mathcal{P}$, and $|\mathcal{P}| := \frac{4\pi^2}{\alpha\beta}$ denotes the volume of $\mathcal{P}$.

With $\Phi := (\varphi, \psi, F_x, F_y)^{\mathrm{T}}$ a neat matrix notation can be used for system (2.6):

$$
\mathcal{B}\,\partial_t\Phi + \mathcal{A}\Phi - \operatorname{Re}\mathcal{C}\Phi + \mathcal{M}(\Phi,\Phi) = 0. \tag{2.7}
$$

Here, $\mathcal{B}$ and $\mathcal{A}$ are diagonal matrix operators, $\mathcal{C}$ is a nonnormal interaction matrix, and $\mathcal{M}$ is a bilinear form. The operator $\mathcal{A}$, for example, has the form

$$
\mathcal{A} = \operatorname{diag}\big(\Delta^2(-\Delta_2), (-\Delta)(-\Delta_2), (-\partial_z^2), (-\partial_z^2)\big)
$$

acting in the Hilbert space

$$
\mathcal{H} := L^2_M(\Omega)\times L^2_M(\Omega)\times L^2\left(\left(-\frac{1}{2},\frac{1}{2}\right)\right)\times L^2\left(\left(-\frac{1}{2},\frac{1}{2}\right)\right),
$$

where $L^2_M(\Omega)$ denotes the space $\{f \in L^2(\Omega) \mid \int_{\mathcal{P}} f(x,y,z)\,dxdy = 0$ for a.e. $z \in (-\frac{1}{2},\frac{1}{2})\}$. The domain $D(\mathcal{A})$ is most easily described in terms of a Fourier mode expansion for φ and ψ with respect to the horizontal variables x and y:

$$
\varphi(x,y,z) = \frac{1}{\sqrt{|\mathcal{P}|}}\sum_{\boldsymbol{\kappa}\in\mathbb{Z}^2\setminus\{0\}} a_{\boldsymbol{\kappa}}(z)e^{i(\alpha\kappa_1 x+\beta\kappa_2 y)}, \tag{2.8}
$$

$$
\psi(x,y,z) = \frac{1}{\sqrt{|\mathcal{P}|}}\sum_{\boldsymbol{\kappa}\in\mathbb{Z}^2\setminus\{0\}} b_{\boldsymbol{\kappa}}(z)e^{i(\alpha\kappa_1 x+\beta\kappa_2 y)}. \tag{2.9}
$$

We then define (cf. [KS, Wa])

$$
D(\mathcal{A}) = D(\Delta^2(-\Delta_2))\times D((-\Delta)(-\Delta_2))\times D(-\partial_z^2)\times D(-\partial_z^2),
$$

where

$$
\begin{aligned}
D(\Delta^2(-\Delta_2)) = \Bigg\{\varphi\;\Bigg|\;&\varphi \text{ expanded as in (2.8)},\\
&a_{\boldsymbol{\kappa}} \in H^4\left(\left(-\frac{1}{2},\frac{1}{2}\right)\right),\quad a_{\boldsymbol{\kappa}} = \partial_z a_{\boldsymbol{\kappa}} = 0 \quad\text{at}\quad z = \pm\frac{1}{2},\\
&\sum_{\boldsymbol{\kappa}\in\mathbb{Z}^2\setminus\{0\}} (\alpha^2\kappa_1^2+\beta^2\kappa_2^2)^2\int_{-1/2}^{1/2} |(-\partial_z^2+\alpha^2\kappa_1^2+\beta^2\kappa_2^2)^2 a_{\boldsymbol{\kappa}}(z)|^2\,dz < \infty\Bigg\},
\end{aligned}
$$

$$D((-\Delta)(-\Delta_2)) = \left\{ \psi \;\middle|\; \begin{array}{l} \psi \text{ expanded as in (2.9),} \\ b_{\boldsymbol{\kappa}} \in H^2\left(\left(-\frac{1}{2}, \frac{1}{2}\right)\right), \quad b_{\boldsymbol{\kappa}} = 0 \quad \text{at} \quad z = \pm\frac{1}{2}, \\ \sum_{\boldsymbol{\kappa} \in \mathbb{Z}^2 \setminus \{0\}} (\alpha^2\kappa_1^2 + \beta^2\kappa_2^2)^2 \int_{-1/2}^{1/2} |(-\partial_z^2 + \alpha^2\kappa_1^2 + \beta^2\kappa_2^2) b_{\boldsymbol{\kappa}}(z)|^2 \, dz < \infty \end{array} \right\},$$

and

$$D(-\partial_z^2) = H^2\left(\left(-\frac{1}{2}, \frac{1}{2}\right)\right) \cap \mathring{H}^1\left(\left(-\frac{1}{2}, \frac{1}{2}\right)\right).$$

With these definitions $\mathcal{A}$ is a self-adjoint and strictly positive operator. Thus, fractional powers of $\mathcal{A}$ make sense and can analogously be explained in terms of the expansions (2.8) and (2.9). Similar definitions apply to the operators $\mathcal{B}$ and $\mathcal{C}$.

A natural class of vector fields within which (2.7) can locally be uniquely solved is given by (cf. [Wa])

$$\Phi \in L^2((0,T), \mathcal{D}(\mathcal{A})), \qquad \partial_t \Phi \in L^2((0,T), \mathcal{D}(\mathcal{B})), \tag{2.10}$$

and, as a consequence,

$$\Phi \in C^0([0,T], I)$$

with I being an appropriate interpolation space between $D(\mathcal{A})$ and $D(\mathcal{B})$. Going back to (2.2) we obtain from (2.10) at least a solution $(\mathbf{u}, p)$ with

(2.11)
$$\mathbf{u} \in L^2((0,T), D(-\Delta)) \cap C^0([0,T], D((-\Delta)^{1/2})), \qquad \partial_t \mathbf{u} \in L^2((0,T), L^2(\Omega)),$$
$$\nabla p \in L^2((0,T), L^2(\Omega)),$$

where $D(-\Delta) = \{\mathbf{u} \in (H^2(\Omega))^3 \mid \mathbf{u}$ periodic in x and y, $\mathbf{u} = 0$ at $z = \pm\frac{1}{2}\}$. This is the usual notion of a strong solution. On the other hand, strong solutions have further regularity properties. In particular, decomposing $\mathbf{u}$ from the class (2.11) in its poloidal and toroidal part and the mean flow, Φ can be shown to lie in the class (2.10). In the following we work with solutions within this class. All manipulations with $\mathbf{u}$ (or Φ) and its horizontal derivatives $\partial_x \mathbf{u}, \partial_y \mathbf{u}$ in the subsequent sections are then justified.

The energy of the system (in the volume Ω) becomes in the new variables[2]

$$E = \frac{1}{2}\|\mathbf{u}\|^2 = \frac{1}{2}\left\{\|\boldsymbol{\delta}\varphi\|^2 + \|\boldsymbol{\varepsilon}\psi\|^2 + |\mathcal{P}|\|\mathbf{F}\|^2\right\}, \tag{2.12}$$

and the variational expression determining Re_E takes the form (cf. [KS])

$$\frac{|\Re(u_x, u_z)|}{\|\nabla\mathbf{u}\|^2} = \frac{|\Re((-\Delta_2)\varphi, \partial_x\partial_z\varphi + \partial_y\psi + F_x)|}{\|(-\Delta)\boldsymbol{\varepsilon}\varphi\|^2 + \|\boldsymbol{\delta}\psi\|^2 + |\mathcal{P}|\|\partial_z\mathbf{F}\|^2}. \tag{2.13}$$

[2] We use the usual notation for L^2-scalar products of vector- or tensor-type quantities. Thus, there is, e.g., $\|\mathbf{u}\|^2 = (\mathbf{u}, \mathbf{u}) = \sum_{i=1}^3 (u_i, u_i)$ or $\|\nabla\mathbf{u}\|^2 = (\nabla\mathbf{u}, \nabla\mathbf{u}) = \sum_{i,j=1}^3 (\partial_i u_j, \partial_i u_j)$. Note that $\nabla\mathbf{u}$ is understood in the sense of a tensor product, whereas $\mathbf{u}\cdot\nabla = \sum_{i=1}^3 u_i\partial_i$ means the scalar product in $\mathbb{R}^3$.

For later convenience we admit here complex valued velocity fields. Thus, the real part (denoted by $\Re$) of the interaction term appears in the numerator of (2.13). Re_E is then given by

$$\mathrm{Re}_E^{-1} = \sup_{(\alpha,\beta)\in\mathbb{R}_+^2} \sup_{(\varphi,\psi)\in\mathcal{V}_{\alpha\beta}} \frac{|\Re((-\Delta_2)\varphi, (\partial_x\partial_z\varphi + \partial_y\psi))|}{\|(-\Delta)\boldsymbol{\varepsilon}\varphi\|^2 + \|\boldsymbol{\delta}\psi\|^2}. \tag{2.14}$$

Note that $\mathbf{F}$ does not depend on x or y and, therefore, drops from the numerator of (2.13). Thus, $\mathbf{F}$ does not contribute to the supremum of (2.13) and can be omitted altogether.

The variational class $\mathcal{V}_{\alpha\beta}$ should reflect the mean value condition, the boundary conditions, and the periodicity of the functions φ and ψ. Moreover, it should ensure that the supremum is in fact attained. A suitable choice is $\mathcal{V}_{\alpha\beta} = D(\tilde{\mathcal{A}}^{1/2}) \setminus \{(0,0)\}$, where $\tilde{\mathcal{A}}$ is that part of $\mathcal{A}$ that is operating on (φ,ψ) in the Hilbert space $\tilde{\mathcal{H}} := L_M^2(\Omega) \times L_M^2(\Omega)$.

If the class $\mathcal{V}_{\alpha\beta}$ of admissible functions is restricted to the class $\mathcal{V}_\alpha$ of functions depending only on x and z, or to the class $\mathcal{V}_\beta$ of functions depending only on y and z, the corresponding two-dimensional limits Re_E^x and Re_E^y are determined by the following simplified variational expressions:

$$\frac{1}{\mathrm{Re}_E^y} = \sup_{\beta\in\mathbb{R}_+} \sup_{(\varphi,\psi)\in\mathcal{V}_\beta} \frac{|\Re((-\Delta_2)\varphi, \partial_y\psi)|}{\|(-\Delta)\partial_y\varphi\|^2 + \|\boldsymbol{\delta}\psi\|^2}, \tag{2.15}$$

$$\frac{1}{\mathrm{Re}_E^x} = \sup_{\alpha\in\mathbb{R}_+} \sup_{(\varphi,0)\in\mathcal{V}_\alpha} \frac{|\Re((-\Delta_2)\varphi, \partial_x\partial_z\varphi)|}{\|(-\Delta)\partial_x\varphi\|^2}. \tag{2.16}$$

It is well known that $\mathrm{Re}_E = \mathrm{Re}_E^y = 82.6\ldots$ and $\mathrm{Re}_E^x = 177.2\ldots$ (cf. [Or, Jo, Bu]).

Applying the matrix notation the variational expression (2.13) takes the form

$$\frac{|(\Phi, \hat{\mathcal{C}}\Phi)|}{\|\mathcal{A}^{1/2}\Phi\|^2}$$

with the symmetric lower order operator

$$\hat{\mathcal{C}} = \frac{1}{2}\begin{pmatrix} 2(-\Delta_2)\partial_x\partial_z & (-\Delta_2)\partial_y & (-\Delta_2) & 0 \\ -(-\Delta_2)\partial_y & 0 & 0 & 0 \\ (-\Delta_2) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus, $\mathcal{A}^{-1/2}\hat{\mathcal{C}}\mathcal{A}^{-1/2}$ is a self-adjoint and compact operator in $\mathcal{H}$ and the supremum (with respect to $(\varphi,\psi) \in D(\tilde{\mathcal{A}}^{1/2})$) in (2.14) is actually a maximum. This argument applies, of course, also to the suprema in (2.15) and (2.16).

3. Generalized functional and variational problem. The usual method to proceed from the energy functional to a generalized one is to introduce additional coupling parameters and possibly additional derivatives in order to weigh the dynamic variables in an optimal way. For this purpose the generalized energy balance is considered and (analogously to the energy method) the ratio of the interaction term over the dissipative term is maximized with respect to the admissible functions. This maximum still depends on the coupling parameters and possibly discrete parameters counting the additional derivatives. Minimizing with respect to these parameters furnishes optimal (generalized) energy limits. Therefore, the first problem is to find functionals which furnish larger stability limits than those provided by the energy functional. Considering the functional (2.12) with $\mathbf{F} \equiv 0$ (as already noted, the mean flow does not contribute to the maximum in the variational problem) there is, however, not much freedom to introduce additional parameters. An obvious choice is the functional

$$\mathcal{E}_1[\varphi,\psi] := \frac{1}{2}\Big\{\|\boldsymbol{\delta}\varphi\|^2 + \lambda\,\|\boldsymbol{\varepsilon}\psi\|^2\Big\} \tag{3.1}$$

with $0 < \lambda < \infty$. Taking the scalar product of $(2.6)_{1,2}$ with (φ,ψ) in $\tilde{\mathcal{H}}$ and using the boundary conditions (2.5) one obtains the generalized energy balance

$$\partial_t \mathcal{E}_1 = -\mathcal{D}_1 + \mathrm{Re}\,\mathcal{I}_1 + \mathcal{N}_1 \tag{3.2}$$

with

$$\begin{aligned} \mathcal{D}_1[\varphi,\psi] &:= \|(-\Delta)\boldsymbol{\varepsilon}\varphi\|^2 + \lambda\|\boldsymbol{\delta}\psi\|^2, \\ \mathcal{I}_1[\varphi,\psi] &:= \Re\big((-\Delta_2)\varphi, \partial_x\partial_z\varphi\big) + \lambda\,\Re\big((-\Delta_2)\varphi, \partial_y\psi\big), \\ \mathcal{N}_1[\varphi,\psi,\mathbf{F}] &:= -\Re\big((\mathbf{u}\cdot\nabla\mathbf{u}), \boldsymbol{\delta}\varphi + \lambda\,\boldsymbol{\varepsilon}\psi\big). \end{aligned} \tag{3.3}$$

The generalized energy limit $\mathrm{Re}_{\mathcal{E}}$ is then determined by

$$\begin{aligned} {\mathrm{Re}_{\mathcal{E}}}^{-1} &= \sup_{(\alpha,\beta)\in\mathbb{R}^2_+}\ \sup_{(\varphi,\psi)\in\mathcal{V}_{\alpha\beta}} \frac{\mathcal{I}_1}{\mathcal{D}_1}[\varphi,\psi] \\ &= \sup_{(\alpha,\beta)\in\mathbb{R}^2_+}\ \sup_{(\varphi,\psi)\in\mathcal{V}_{\alpha\beta}} \frac{|\Re((-\Delta_2)\varphi, \partial_x\partial_z\varphi) + \lambda\,\Re((-\Delta_2)\varphi, \partial_y\psi)|}{\|(-\Delta)\boldsymbol{\varepsilon}\varphi\|^2 + \lambda\,\|\boldsymbol{\delta}\psi\|^2}. \end{aligned} \tag{3.4}$$

Note that in (3.4) $\mathcal{I}_1$ can always be replaced by $|\mathcal{I}_1|$; as with $(\varphi(x,y,z),\psi(x,y,z)) \in \mathcal{V}_{\alpha\beta}$, $(\varphi(-x,-y,z),\psi(-x,-y,z))$ is also admissible. Thus, $\mathcal{I}_1$ can always be chosen positive without affecting $\mathcal{D}_1$.

A comparison of (3.4) with the two-dimensional variational expressions (2.15) and (2.16) already furnishes some bounds on $\mathrm{Re}_{\mathcal{E}}$: Setting $\varphi = \varphi(x,z)$, $\psi = 0$ in (3.4) reduces the variational expression to that in (2.16), which implies the bound $\mathrm{Re}_{\mathcal{E}} \le \mathrm{Re}_E^x = 177.2\ldots$ for all $0 < \lambda < \infty$. For $\lambda \ge 1$ the substitution $\tilde{\psi} := \lambda\psi$ allows the estimate

$$\begin{aligned} &\frac{|\Re((-\Delta_2)\varphi, \partial_x\partial_z\varphi) + \lambda\,\Re((-\Delta_2)\varphi, \partial_y\psi)|}{\|(-\Delta)\boldsymbol{\varepsilon}\varphi\|^2 + \lambda\,\|\boldsymbol{\delta}\psi\|^2} = \frac{|\Re((-\Delta_2)\varphi, \partial_x\partial_z\varphi) + \Re((-\Delta_2)\varphi, \partial_y\tilde{\psi})|}{\|(-\Delta)\boldsymbol{\varepsilon}\varphi\|^2 + \frac{1}{\lambda}\|\boldsymbol{\delta}\tilde{\psi}\|^2} \\ &\ge \frac{|\Re((-\Delta_2)\varphi, \partial_x\partial_z\varphi) + \Re((-\Delta_2)\varphi, \partial_y\tilde{\psi})|}{\|(-\Delta)\boldsymbol{\varepsilon}\varphi\|^2 + \|\boldsymbol{\delta}\tilde{\psi}\|^2}, \end{aligned}$$

and restricting φ and $\tilde{\psi}$ to functions independent of x furnishes the bound $\mathrm{Re}_{\mathcal{E}} \le \mathrm{Re}_E^y = \mathrm{Re}_E = 82.6\ldots$ for $\lambda \ge 1$. Thus the question remains whether $\mathrm{Re}_{\mathcal{E}}$ does exceed Re_E for some $0 < \lambda < 1$.

A numerical computation indicates that $\mathrm{Re}_{\mathcal{E}}$ attains its upper bound Re_E^x for sufficiently small values of λ (cf. Appendix B). In order to prove this, consider the variational expression

$$\frac{\mathcal{I}_1}{\mathcal{D}_1}[\varphi,\hat{\psi}] = \frac{|\Re((-\Delta_2)\varphi, \partial_x\partial_z\varphi) + \sqrt{\lambda}\,\Re((-\Delta_2)\varphi, \partial_y\hat{\psi})|}{\|(-\Delta)\boldsymbol{\varepsilon}\varphi\|^2 + \|\boldsymbol{\delta}\hat{\psi}\|^2}, \tag{3.5}$$

where $\hat{\psi} := \sqrt{\lambda}\,\psi$. Now, inserting the mode expansions (2.8) and (2.9) for φ and $\hat{\psi}$ in (3.5) observe that the maximum for a fixed periodicity cell $\mathcal{P}$ is attained by a single mode. This can be seen as follows: Assume the maximum is attained by a (possibly infinite) linear combination of modes. By inserting this combination into the variational expression (3.5), the numerator as well as the denominator decomposes into a sum of bilinear terms each containing a single mode. Without restriction the modes can be chosen such that the expansion of the numerator contains only nonnegative terms. Applying Lemma 1 (cf. Appendix A) we can select a single mode with maximal ratio, which at most increases the value of the variational expression. Let $\boldsymbol{\kappa} \in \mathbb{Z}^2 \setminus \{0\}$ be this maximal mode. With the abbreviation $\tilde{\alpha} := \kappa_1 \alpha$, $\tilde{\beta} := \kappa_2 \beta$, $\tilde{a}(z) := a_{\boldsymbol{\kappa}}(z)$, $\tilde{b}(z) := b_{\boldsymbol{\kappa}}(z)$ we obtain

$$\begin{aligned}\frac{\mathcal{I}_1}{\mathcal{D}_1}[\tilde{a}, \tilde{b}, \tilde{\alpha}, \tilde{\beta}] &= \frac{\tilde{\alpha}\,|\Im(\tilde{a}, \tilde{a}')| + \sqrt{\lambda}\,\tilde{\beta}\,|\Im(\tilde{a}, \tilde{b})|}{\|(\tilde{\alpha}^2 + \tilde{\beta}^2)\tilde{a} - \tilde{a}''\|^2 + (\tilde{\alpha}^2 + \tilde{\beta}^2)\|\tilde{b}\|^2 + \|\tilde{b}'\|^2} \\ &\le \max\left\{ \frac{\tilde{\alpha}\,|\Im(\tilde{a}, \tilde{\alpha}')|}{\tilde{\alpha}^4\|\tilde{a}\|^2 + 2\tilde{\alpha}^2\|\tilde{a}'\|^2 + \|\tilde{a}''\|^2}, \sqrt{\lambda}\,\frac{\tilde{\beta}\,|\Im(\tilde{a}, \tilde{b})|}{\tilde{\beta}^4\|\tilde{a}\|^2 + 2\tilde{\beta}^2\|\tilde{a}'\|^2 + \tilde{\beta}^2\|\tilde{b}\|^2 + \|\tilde{b}'\|^2} \right\},\end{aligned} \tag{3.6}$$

where we used partial integration and Lemma 1 in the last line. Abbreviating the first term in (3.6) with $\mathcal{F}_1[\tilde{a}, \tilde{\alpha}]$ and the second with $\mathcal{F}_2[\tilde{a}, \tilde{b}, \tilde{\alpha}]$, it follows from (3.4)–(3.6) that

$$\mathrm{Re}_{\mathcal{E}}^{-1} = \max\left\{ \sup_{\tilde{\alpha} \in \mathbb{R}_+} \sup_{(\tilde{a}, 0) \in \mathcal{W}} \mathcal{F}_1[\tilde{a}, \tilde{\alpha}], \sqrt{\lambda} \sup_{\tilde{\beta} \in \mathbb{R}_+} \sup_{(\tilde{a}, \tilde{b}) \in \mathcal{W}} \mathcal{F}_2[\tilde{a}, \tilde{b}, \tilde{\beta}] \right\}, \tag{3.7}$$

with

$$\mathcal{W} = \left\{ (a, b) \in H^4\left(\left(-\frac{1}{2}, \frac{1}{2}\right)\right) \times H^2\left(\left(-\frac{1}{2}, \frac{1}{2}\right)\right) \;\middle|\; a = \partial_z a = b = 0 \text{ at } z = \pm\frac{1}{2} \right\} \setminus \{0\}.$$

Inserting the mode expansion (2.8) into (2.16), the first term in (3.7) turns out to be $\frac{1}{\mathrm{Re}_E^x}$, whereas $\mathcal{F}_2[\tilde{a}, \tilde{b}, \tilde{\beta}]$ is estimated with the help of inequality (A.2) as follows:

$$\begin{aligned}\mathcal{F}_2[\tilde{a}, \tilde{b}, \tilde{\beta}] &\le \frac{\tilde{\beta}\,\|\tilde{a}\|\|\tilde{b}\|}{(\tilde{\beta}^4 + 2\tilde{\beta}^2\pi^2)\|\tilde{a}\|^2 + (\tilde{\beta}^2 + \pi^2)\|\tilde{b}\|^2} \le \frac{\tilde{\beta}}{2[(\tilde{\beta}^4 + 2\tilde{\beta}^2\pi^2)(\tilde{\beta}^2 + \pi^2)]^{1/2}} \\ &\le \frac{1}{2\sqrt{2}\,\pi^2}.\end{aligned}$$

Therefore, by choosing $\sqrt{\lambda} \le \frac{2\sqrt{2}\,\pi^2}{\mathrm{Re}_E^x}$, (3.7) yields $\mathrm{Re}_{\mathcal{E}} \ge \mathrm{Re}_E^x$, hence

$$\mathrm{Re}_{\mathcal{E}} = \mathrm{Re}_E^x.$$

We formulate this result in the following proposition.

Proposition 1. *For* $0 < \lambda < \frac{8\pi^4}{\mathrm{Re}_E^x} \approx 0.025$, $0 < \mathrm{Re} < \mathrm{Re}_{\mathcal{E}}$ *with* $\mathrm{Re}_{\mathcal{E}} = \mathrm{Re}_E^x = 177.2\ldots$, $(\alpha, \beta) \in \mathbb{R}_+^2$, *and* $(\varphi, \psi) \in \mathcal{V}_{\alpha,\beta} = D(\tilde{\mathcal{A}}^{1/2}) \setminus \{(0, 0)\}$ *with* $D(\tilde{\mathcal{A}}^{1/2})$, *as explained in section* 2, *we have the bound*

$$\mathrm{Re}\,\frac{\mathcal{I}_1}{\mathcal{D}_1} \le \frac{\mathrm{Re}}{\mathrm{Re}_{\mathcal{E}}} < 1, \tag{3.8}$$

where

$$\frac{\mathcal{I}_1}{\mathcal{D}_1} = \frac{|\Re((-\Delta_2)\varphi, \partial_x\partial_z\varphi) + \lambda\,\Re((-\Delta_2)\varphi, \partial_y\psi)|}{\|(-\Delta)\boldsymbol{\varepsilon}\varphi\|^2 + \lambda\,\|\boldsymbol{\delta}\psi\|^2}.$$

Remarks. 1. The numerical computation in Appendix B indicates coincidence of $\mathrm{Re}_{\mathcal{E}}$ with Re_E^x for values of λ up to $\lambda \approx 0.042$.

2. Whether other functionals provide even larger stability limits is an open problem. Another candidate which failed to provide a larger stability limit has been discussed in [KT].

4. Nonlinear stability. For $\lambda \neq 1$ the nonlinear term $\mathcal{N}_1$ in (3.2) does not vanish. In order to dominate this term we introduce a second part $\mathcal{E}_2$ of the generalized energy functional $\mathcal{E}$,

$$\mathcal{E}_2[\mathbf{u}, \mathbf{F}] := \frac{1}{2}\{\sigma\|\boldsymbol{\varepsilon}\mathbf{u}\|^2 + \rho\,|\mathcal{P}|\|\mathbf{F}\|^2\}, \tag{4.1}$$

with yet undetermined nonnegative coupling parameters σ and ρ.

By scalar multiplication of (2.2) with $\sigma\Delta_2\mathbf{u}$ and of $(2.6)_{3,4}$ with ρF_x, ρF_y and using (2.3) and (2.5), we arrive at

$$\partial_t\mathcal{E}_2 = -\mathcal{D}_2 + \mathrm{Re}\,\mathcal{I}_2 + \mathcal{N}_2, \tag{4.2}$$

where[3]

$$\begin{aligned} &\mathcal{D}_2[\mathbf{u}, \mathbf{F}] = \sigma\|\boldsymbol{\delta}\mathbf{u}\|^2 + \rho\,|\mathcal{P}|\|\mathbf{F}'\|^2, \\ &\mathcal{I}_2[\mathbf{u}, \mathbf{F}] = \sigma\,\Re(\boldsymbol{\varepsilon}u_z, \boldsymbol{\varepsilon}u_x), \\ &\mathcal{N}_2[\mathbf{u}, \mathbf{F}] = -\sigma\,\Re(\boldsymbol{\varepsilon}\mathbf{u}\cdot\nabla\mathbf{u}, \boldsymbol{\varepsilon}\mathbf{u}) - \rho\,\Re(\tilde{\mathbf{u}}\cdot\nabla\tilde{\mathbf{u}}, \mathbf{F}). \end{aligned} \tag{4.3}$$

By defining

$$\Delta\mathrm{Re} := 1 - \frac{\mathrm{Re}}{\mathrm{Re}_{\mathcal{E}}} \tag{4.4}$$

and

$$\mathcal{D} := \Delta\mathrm{Re}\,\mathcal{D}_1 + \mathcal{D}_2, \tag{4.5}$$

the interaction term $\mathrm{Re}\,\mathcal{I}_2$ can be estimated in terms of $\mathcal{D}$:

$$\mathcal{I}_2 \le \sigma|(\boldsymbol{\varepsilon}u_z, \boldsymbol{\varepsilon}u_x)| \le \sigma^{1/2}\|\boldsymbol{\varepsilon}(-\Delta_2)\varphi\|\,\sigma^{1/2}\|\boldsymbol{\varepsilon}\mathbf{u}\| \le \sigma^{1/2}\mathcal{D}_1^{1/2}(2\mathcal{E}_2)^{1/2}.$$

Using $2\mathcal{E}_2 \le \frac{\mathcal{D}_2}{\pi^2}$, which follows with (A.2), and setting

$$\sigma := \frac{\pi^2\,\Delta\mathrm{Re}}{\mathrm{Re}^2}, \tag{4.6}$$

we obtain

$$\mathrm{Re}\,\mathcal{I}_2 \le (\Delta\mathrm{Re})^{1/2}\mathcal{D}_1^{1/2}\mathcal{D}_2^{1/2} \le \frac{1}{2}\,(\Delta\mathrm{Re}\,\mathcal{D}_1 + \mathcal{D}_2) = \frac{1}{2}\,\mathcal{D}. \tag{4.7}$$

[3]Note that no boundary terms arise in (4.2) since the terms in $\mathcal{E}_2$ differ from those in the ordinary energy at most by *horizontal* derivatives.

We estimate next the nonlinear parts in terms of $\mathcal{D}\mathcal{E}^{1/2}$, $\mathcal{E} := \mathcal{E}_1 + \mathcal{E}_2$, and begin with $\mathcal{N}_1$:

$$\mathcal{N}_1 \le |(\mathbf{u}\cdot\nabla\mathbf{u}, \boldsymbol{\delta}\varphi + \lambda\,\boldsymbol{\varepsilon}\psi)| \le \operatorname{ess\,sup}_\Omega |\mathbf{u}|\,\|\nabla\mathbf{u}\|(\|\boldsymbol{\delta}\varphi\| + \lambda\,\|\boldsymbol{\varepsilon}\psi\|).$$

The three factors are estimated separately. With (A.8) we obtain for the first factor

$$\|\mathbf{u}\|_\infty \le \frac{C}{\sqrt{2}}\,\|\boldsymbol{\delta}\tilde{\mathbf{u}}\| + \sqrt{\frac{2}{\pi}}\|\mathbf{F}'\|,$$

where $C = 8(\frac{\sqrt{2}}{m})^{3/2}$, and under the condition

$$\rho \ge \frac{4\,\sigma}{\pi\,C^2|\mathcal{P}|} \tag{4.8}$$

we get further

$$\|\mathbf{u}\|_\infty \le \frac{C}{\sqrt{2\sigma}}\Big\{\sqrt{\sigma}\,\|\boldsymbol{\delta}\tilde{\mathbf{u}}\| + (\rho\,|\mathcal{P}|)^{1/2}\|\mathbf{F}'\|\Big\} \le \frac{C}{\sqrt{\sigma}}\mathcal{D}_2^{1/2} \le \frac{C}{\sqrt{\sigma}}\mathcal{D}^{1/2}. \tag{4.9}$$

With the conditions

$$\rho \ge \lambda\,\Delta\mathrm{Re}, \qquad 0 < \lambda < 1 \tag{4.10}$$

we obtain for the second factor

$$\|\nabla\mathbf{u}\|^2 = \|\nabla\tilde{\mathbf{u}}\|^2 + |\mathcal{P}|\|\mathbf{F}'\|^2 \le \frac{\mathcal{D}_1}{\lambda} + \frac{\mathcal{D}_2}{\rho} \le \frac{1}{\lambda\,\Delta\mathrm{Re}}\,\mathcal{D};$$

thus

$$\|\nabla\mathbf{u}\| \le \frac{1}{\sqrt{\lambda\,\Delta\mathrm{Re}}}\,\mathcal{D}^{1/2}. \tag{4.11}$$

Finally, we have

$$\|\boldsymbol{\delta}\varphi\| + \lambda\,\|\boldsymbol{\varepsilon}\psi\| \le \sqrt{1+\lambda}\,(2\mathcal{E}_1)^{1/2} \le \sqrt{2}\,\sqrt{1+\lambda}\,\mathcal{E}^{1/2}. \tag{4.12}$$

The conditions (4.8) and (4.10) are satisfied for the choice

$$\rho := \Delta\mathrm{Re}\max\Big\{\lambda, \frac{\alpha\,\beta\,m^3}{\sqrt{2}\,2^7\pi\,\mathrm{Re}^2}\Big\}, \tag{4.13}$$

and by collecting the estimates (4.9), (4.11), and (4.12) we have

$$\mathcal{N}_1 \le \frac{\sqrt{2}\,C}{\sqrt{\sigma\,\Delta\mathrm{Re}}}\,\sqrt{1+1/\lambda}\,\mathcal{D}\,\mathcal{E}^{1/2}. \tag{4.14}$$

As to $\mathcal{N}_2$, we obtain by partial integration

$$\begin{aligned}\mathcal{N}_2 &= -\sigma\,\Re\sum_{i,j=1}^{3}\sum_{n=1}^{2}\int_\Omega \partial_n u_i \partial_i u_j \partial_n \overline{u}_j\,d\tau - \rho\,\Re\sum_{i,j=1}^{3}\int_\Omega \tilde{u}_i\partial_i\tilde{u}_j\overline{F}_j\,d\tau\\ &= \sigma\,\Re\sum_{i,j=1}^{3}\sum_{n=1}^{2}\int_\Omega \partial_n u_i \partial_n \partial_i \overline{u}_j u_j\,d\tau + \rho\,\Re\sum_{n=1}^{2}\int_\Omega \tilde{u}_z\tilde{u}_n\overline{F}'_n\,d\tau.\end{aligned}$$

Estimates analogous to those for $\mathcal{N}_1$, in particular (4.9), then yield

$$
\begin{aligned}
\mathcal{N}_2 &\le \sigma \|\mathbf{u}\|_\infty \|\boldsymbol{\delta}\mathbf{u}\|\,\|\varepsilon\mathbf{u}\| + \rho\,\|\tilde{\mathbf{u}}\|_\infty\,|\mathcal{P}|^{1/2}\|\mathbf{F}'\|\,\|\tilde{u}_z\| \\
&\le \frac{C}{\sqrt{\sigma}}\,\mathcal{D}^{1/2}\sqrt{\sigma}\,\|\boldsymbol{\delta}\mathbf{u}\|\,(2\mathcal{E}_2)^{1/2} + \frac{C}{\sqrt{\sigma}}\,\mathcal{D}^{1/2}\,\rho\,|\mathcal{P}|^{1/2}\|\mathbf{F}'\|\,(2\mathcal{E}_1)^{1/2} \\
&\le \frac{C}{\sqrt{\sigma}}\,\mathcal{D}^{1/2}\sqrt{1+\rho}\,\mathcal{D}_2^{1/2}(2\mathcal{E})^{1/2} \le \frac{\sqrt{2}\,C}{\sqrt{\sigma}}\,\sqrt{1+\rho}\,\mathcal{D}\,\mathcal{E}^{1/2}.
\end{aligned}
\tag{4.15}
$$

Summarizing (4.14) and (4.15) we have

$$
\mathcal{N}_1 + \mathcal{N}_2 \le \frac{1}{2}\,\mathcal{D}\left(\frac{\mathcal{E}}{\delta}\right)^{1/2}
\tag{4.16}
$$

with

$$
\delta := \frac{\sigma}{8\,C^2}\left(\sqrt{1+1/\lambda}\,\frac{1}{\sqrt{\Delta\mathrm{Re}}} + \sqrt{1+\rho}\right)^{-2}.
$$

Observe that the estimate (4.15) is based on the estimate (4.9), which increases the number of z-derivatives by only one. Previously used estimates (cf. [GP] or [St]) increase this number by two and do not work in our situation. On the other hand, functionals involving more z-derivatives do not work either, since there is not enough information about boundary values which would allow the necessary partial integrations [KT].

Finally, we add up equations (3.2) and (4.2), apply Proposition 1, and use the estimates (4.7) and (4.16). This yields the following for $\mathcal{E} = \mathcal{E}_1 + \mathcal{E}_2$:

$$
\begin{aligned}
\partial_t \mathcal{E} &= -\left[\mathcal{D}_1\left(1 - \frac{\mathrm{Re}\,\mathcal{I}_1}{\mathcal{D}_1}\right) + \mathcal{D}_2\right] + \mathrm{Re}\,\mathcal{I}_2 + \mathcal{N}_1 + \mathcal{N}_2 \\
&\le -\mathcal{D} + \frac{1}{2}\,\mathcal{D} + \frac{1}{2}\,\mathcal{D}\left(\frac{\mathcal{E}}{\delta}\right)^{1/2} \\
&\le -\frac{1}{2}\,\mathcal{D}\left[1 - \left(\frac{\mathcal{E}}{\delta}\right)^{1/2}\right].
\end{aligned}
\tag{4.17}
$$

Inequality (4.17) implies that $\mathcal{E}(t)$ is monotonically nonincreasing if $\mathcal{E}(0) < \delta$. With

$$
\frac{1}{2}\,\mathcal{D} = \frac{1}{2}\,(\Delta\mathrm{Re}\,\mathcal{D}_1 + \mathcal{D}_2) \ge \pi^2(\Delta\mathrm{Re}\,\mathcal{E}_1 + \mathcal{E}_2) \ge \pi^2\,\Delta\mathrm{Re}\,\mathcal{E},
$$

which follows from (A.2), (A.3), and $0 < \Delta\mathrm{Re} < 1$, we therefore have

$$
\partial_t \mathcal{E} \le -\frac{1}{2}\,\mathcal{D}\left[1 - \left(\frac{\mathcal{E}(0)}{\delta}\right)^{1/2}\right] \le -\pi^2\,\Delta\mathrm{Re}\,\mathcal{E}\,\left[1 - \left(\frac{\mathcal{E}(0)}{\delta}\right)^{1/2}\right],
$$

and integration yields

$$
\mathcal{E}(t) \le \mathcal{E}(0)\exp\left\{-\pi^2\,\Delta\mathrm{Re}\,\left[1 - \left(\frac{\mathcal{E}(0)}{\delta}\right)^{1/2}\right]t\right\}.
\tag{4.18}
$$

We formulate our stability result in the following theorem.

THEOREM 2. *Let us consider perturbations $\Phi = (\varphi, \psi, \mathbf{F})^{\mathrm{T}}$ of the basic flow $\mathbf{U}_0 = \mathrm{Re}(-z, 0, 0)^{\mathrm{T}}$ in the plane Couette system satisfying globally (in time) the system* (2.6) *as a strong solution (i.e., in the sense of* (2.10)*) under rigid boundary conditions* (2.5) *and being periodic in the horizontal variables x, y with wave numbers $(\alpha, \beta) \in \mathbb{R}_+^2$. Let $0 < \mathrm{Re} < \mathrm{Re}_{\mathcal{E}} = 177.2\ldots$, $\Delta\mathrm{Re} = 1 - \frac{\mathrm{Re}}{\mathrm{Re}_{\mathcal{E}}}$, $C = 8(\frac{\sqrt{2}}{m})^{3/2}$, and $m = \min(\alpha, \beta)$. Consider, furthermore, the generalized energy functional*

$$\mathcal{E}[\varphi, \psi, \mathbf{F}] = \frac{1}{2}\left\{ \|\boldsymbol{\delta}\varphi\|^2 + \lambda\, \|\boldsymbol{\varepsilon}\psi\|^2 + \sigma\, \|\boldsymbol{\varepsilon}\mathbf{u}\|^2 + \rho\, \frac{4\pi^2}{\alpha\beta}\, \|\mathbf{F}\|^2 \right\}$$

with coupling parameters $0 < \lambda < \frac{8\pi^4}{\mathrm{Re}_{\mathcal{E}}}$ and

$$\sigma = \frac{\pi^2\, \Delta\mathrm{Re}}{\mathrm{Re}^2}, \qquad \rho = \Delta\mathrm{Re}\max\left\{ \lambda,\, \frac{\alpha\, \beta\, m^3}{\sqrt{2}\, 2^7 \pi\, \mathrm{Re}^2} \right\}.$$

Then, the solution $(\varphi, \psi, \mathbf{F})$ of (2.5) *and* (2.6) *decays in the norm $\mathcal{E}^{1/2}$ exponentially to zero provided the initial value satisfies*

$$\mathcal{E}(0) < \delta = \frac{\sigma}{8\, C^2}\left(\sqrt{1 + 1/\lambda}\, \frac{1}{\sqrt{\Delta\mathrm{Re}}} + \sqrt{1+\rho} \right)^{-2}. \tag{4.19}$$

Remarks. 1. The functional $\mathcal{E}$ dominates the classical energy $E = \frac{1}{2}\, \|\mathbf{u}\|^2$. Therefore, $E(t)$ also decays to zero for $\mathrm{Re} < \mathrm{Re}_{\mathcal{E}}$. However, for $\mathrm{Re} > \mathrm{Re}_E$, $E(t)$ does not necessarily decay monotonically.

2. We did not try to obtain optimal (i.e., as large as possible) stability balls δ. Considering the restricted Reynolds number range the stability balls have not yet any importance for experiments. The emphasis of the present paper is on demonstrating that the method of generalized energy functionals also works for rigid boundary conditions.

3. The stability balls δ vanish in the limit $\Delta\mathrm{Re} \to 0$ or $m \to 0$. Asymptotically we have

$$\delta^{1/2} \sim \begin{cases} \Delta\mathrm{Re} & \text{in the limit} \quad \Delta\mathrm{Re} \to 0, \\ m^{3/2} & \text{in the limit} \quad m \to 0. \end{cases}$$

This behavior seems to be intrinsic to the functional method and it is independent of the choice of boundary conditions (cf. [RM]). The decay constant (in time) in (4.18) for a fixed value $\frac{\mathcal{E}(0)}{\delta} = \mathit{const} < 1$ decreases likewise with $\Delta\mathrm{Re}$ to zero, but it is independent of m. This is different from the case of free boundary conditions,[4] where arbitrarily slowly decaying modes always exist; e.g.,

$$\mathbf{u} = e^{-\alpha^2 t} \sin \alpha y\, \mathbf{e}_x, \qquad p \equiv 0$$

for any $\alpha = m > 0$.

4. There is another interesting approach, which is at least in parts rigorous and which aims at providing stability balls of power law type in the Reynolds number; they have the form $c\,\mathrm{Re}^{-\gamma}$, where c depends on the geometry but is independent of Re. The starting point of the method is a power law bound on the resolvent of the linearized

[4]Note that the Poincaré-type inequalities in [RM] have to be corrected; cf. [KX].

operator, which has been obtained so far only by numerical methods. In a second step the exponent γ can then rigorously be derived whereas c remains unknown. Such bounds, valid for all Reynolds numbers, have been obtained for Couette flow [KLH] and have recently been improved [LK].

5. More generally, Theorem 2 applies to (not necessarily global) strong solutions on their maximal intervals of existence. In particular, it provides an a priori bound on the horizontal derivatives of $\mathbf{u}$ in the $L^2(\Omega)$-norm under an explicit condition on its initial values. An interesting (but so far open) question is whether this condition, viz. (4.19), guarantees already global existence of the solution in time. The following is known in this respect [KW]: A strong solution which is conditionally stable in the energy norm on the maximal interval of existence exists globally in time (in the class (2.10)) provided its initial value is small in the norm of the interpolation space $\mathcal{I}$. This norm is, however, stronger than $\mathcal{E}^{1/2}$; in particular, it involves nontangential derivatives of $\mathbf{u}$, which are not controlled by $\mathcal{E}^{1/2}$. The required smallness depends on the steady flow to be perturbed and the stability behavior of the kinetic energy of the perturbation.

Appendix A. We collect in this appendix some more-or-less standard inequalities we made use of in the main text. Only Lemma 3, which presents a refined calculus inequality, is proved.

LEMMA 1. *Let $n \in \mathbb{N}$ and $a_\nu \geq 0$, $b_\nu > 0$ for $1 \leq \nu \leq n$. Then*

$$\frac{\sum_{\nu=1}^n a_\nu}{\sum_{\nu=1}^n b_\nu} \leq \max\left\{ \frac{a_\nu}{b_\nu} \;\middle|\; 1 \leq \nu \leq n \right\} =: M \tag{A.1}$$

and equality holds if and only if $a_\nu = M\, b_\nu$ for every ν.

Note that inequality (A.1) remains valid for $n \to \infty$.

Frequent use is made of the Poincaré-type inequalities

$$\|f\| \leq \frac{1}{\pi}\|\nabla f\|, \tag{A.2}$$

$$\|\nabla f\| \leq \frac{1}{\pi}\|\nabla\nabla f\| = \frac{1}{\pi}\|\Delta f\|, \tag{A.3}$$

which are valid for $\mathcal{P}$-periodic functions f decomposed according to

$$f(x,y,z) = \frac{1}{\sqrt{\mathcal{P}}} \sum_{\boldsymbol{\kappa}\in\mathbb{Z}^2} f_{\boldsymbol{\kappa}}(z) e^{i(\alpha\kappa_1 x + \beta\kappa_2 y)} \tag{A.4}$$

with (at least) $f_{\boldsymbol{\kappa}} \in H^1((-\frac{1}{2}, \frac{1}{2}))$ and (weakly) satisfying the boundary conditions $f_{\boldsymbol{\kappa}}(\pm\frac{1}{2}) = 0$, $\boldsymbol{\kappa} \in \mathbb{Z}^2$ (cf. Appendix A in [KX]). The inequalities (A.2) and (A.3) hold likewise for vector valued functions if each component satisfies such a decomposition.

The next two lemmata provide bounds on the sup-norm $\|\cdot\|_\infty = \operatorname{ess\,sup}|\cdot|$ in terms of the L_2-norm $\|\cdot\|_2 = \|\cdot\|$ in one and three dimensions.

LEMMA 2. *Let $f \in H^1((-\frac{1}{2}, \frac{1}{2}))$ with (weakly) $f(-\frac{1}{2}) = 0$. Then*

$$\|f\|_\infty^2 \leq 2\,\|f\|\,\|f'\|. \tag{A.5}$$

LEMMA 3. *Let $f : \mathbb{R}^2 \times [-\frac{1}{2}, \frac{1}{2}] \to \mathbb{R}$ be $\mathcal{P}$-periodic and decomposed according to* (A.4) *with $f_{\boldsymbol{\kappa}} \in H^1((-\frac{1}{2}, \frac{1}{2}))$ and weakly satisfying the boundary conditions $f_{\boldsymbol{\kappa}}(\pm\frac{1}{2}) = 0$ for $\boldsymbol{\kappa} \in \mathbb{Z}^2 \setminus \{0\}$, $f_0 = \frac{1}{\sqrt{|\mathcal{P}|}} \int_{\mathcal{P}} f(x,y,z)\,dxdy = 0$. Then*

$$\|f\|_\infty \leq C\,\|(-\Delta_2)^{1/2}\partial_z f\|^{1/2}\|(-\Delta_2) f\|^{1/2} \tag{A.6}$$

with $C := 8(\frac{\sqrt{2}}{m})^{3/2}$, $m := \min\{\alpha, \beta\}$.

Proof. With (A.4) and Lemma 2 one obtains

$$\begin{aligned}\operatorname{ess\,sup}_\Omega |f(x,y,z)| &\le \operatorname{ess\,sup}_{[-1/2,1/2]} \sum_{\boldsymbol{\kappa}\in\mathbb{Z}^2\setminus\{0\}} |f_{\boldsymbol{\kappa}}(z)| \\ &\le \sqrt{2} \sum_{\boldsymbol{\kappa}\in\mathbb{Z}^2\setminus\{0\}} \|f'_{\boldsymbol{\kappa}}\|^{1/2}\|f_{\boldsymbol{\kappa}}\|^{1/2}.\end{aligned}$$

Therefore, with Hölder's inequality

$$\begin{aligned}\|f\|_\infty &\le \sqrt{2} \sum_{\boldsymbol{\kappa}\in\mathbb{Z}^2\setminus\{0\}} \|f'_{\boldsymbol{\kappa}}\|^{1/2}(\alpha^2\kappa_1^2+\beta^2\kappa_2^2)^{1/4}\,\|f_{\boldsymbol{\kappa}}\|^{1/2}(\alpha^2\kappa_1^2+\beta^2\kappa_2^2)^{1/2} \\ &\qquad\times(\alpha^2\kappa_1^2+\beta^2\kappa_2^2)^{-3/4} \\ &\le \sqrt{2}\left(\sum_{\boldsymbol{\kappa}\in\mathbb{Z}^2\setminus\{0\}} \|f'_{\boldsymbol{\kappa}}\|^2(\alpha^2\kappa_1^2+\beta^2\kappa_2^2)\right)^{1/4}\left(\sum_{\boldsymbol{\kappa}\in\mathbb{Z}^2\setminus\{0\}} \|f_{\boldsymbol{\kappa}}\|^2(\alpha^2\kappa_1^2+\beta^2\kappa_2^2)^2\right)^{1/4} \\ &\qquad\times\left(\sum_{\boldsymbol{\kappa}\in\mathbb{Z}^2\setminus\{0\}}(\alpha^2\kappa_1^2+\beta^2\kappa_2^2)^{-3/2}\right)^{1/2} \\ &\le C\,\|(-\Delta_2)^{1/2}\partial_z f\|^{1/2}\|(-\Delta_2)f\|^{1/2}.\end{aligned}$$

In the last line we used the estimate

$$\left(\sum_{\boldsymbol{\kappa}\in\mathbb{Z}^2\setminus\{0\}}(\alpha^2\kappa_1^2+\beta^2\kappa_2^2)^{-3/2}\right)^{1/2} \le 2^{5/2}\left(\frac{\sqrt{2}}{m}\right)^{3/2}$$

(cf. Lemma 4.1 in [BK]). □

A more convenient form of (A.6) is

$$\|f\|_\infty \le \frac{C}{\sqrt{2}}\,\|\boldsymbol{\delta} f\|, \tag{A.7}$$

which follows from (A.6) by

$$\begin{aligned}\|f\|_\infty^2 &\le \frac{C^2}{2}\left[\|(-\Delta_2)^{1/2}\partial_z f\|^2+\|(-\Delta_2)f\|^2\right] \\ &= \frac{C^2}{2}\left[(\Delta_2 f,\partial_z^2 f)+(\Delta_2 f,\Delta_2 f)\right] \\ &= \frac{C^2}{2}\,(\Delta_2 f,\Delta f) = \frac{C^2}{2}\,\|\boldsymbol{\delta} f\|^2.\end{aligned}$$

If f has a nonzero mean value f_0 the inequalities (A.2), (A.5), and (A.7) furnish

$$\|f\|_\infty \le \|\tilde f\|_\infty + \|f_0\|_\infty \le \frac{C}{\sqrt{2}}\,\|\boldsymbol{\delta}\tilde f\| + \sqrt{\frac{2}{\pi}}\,\|f_0'\|, \tag{A.8}$$

where $\tilde f = f - f_0$.

The inequalities (A.5)–(A.8) hold likewise for vector valued functions if each component satisfies the appropriate conditions.

Appendix B. In this appendix the variational problem (3.4) with $0 \leq \lambda \leq 1$ is treated on a numerical basis. We first solve the eigenvalue problem associated with the variational problem with fixed periodicity cell $\mathcal{P}$ and subsequently perform the variation with respect to $\mathcal{P}$.

The Euler–Lagrange equations with Lagrange parameter μ read

$$\begin{aligned} \Delta^2(-\Delta_2)\varphi - \frac{\mu}{2}\Big(2\,(-\Delta_2)\partial_x\partial_z\varphi + \lambda\,(-\Delta_2)\partial_y\psi\Big) &= 0, \\ \lambda\,(-\Delta)(-\Delta_2)\psi + \frac{\mu}{2}\,\lambda\,(-\Delta_2)\partial_y\varphi &= 0. \end{aligned} \tag{B.1}$$

By inserting the mode expansions (2.8) and (2.9) the system (B.1) becomes equivalent to

$$\left.\begin{aligned} D^2_{\kappa_1\alpha,\kappa_2\beta}\,a_{\boldsymbol{\kappa}}(z) - i\frac{\mu}{2}\Big(2\alpha\kappa_1\partial_z a_{\boldsymbol{\kappa}}(z) + \lambda\,\beta\kappa_2 b_{\boldsymbol{\kappa}}(z)\Big) &= 0, \\ D_{\kappa_1\alpha,\kappa_2\beta}\,b_{\boldsymbol{\kappa}}(z) + i\frac{\mu}{2}\beta\kappa_2 a_{\boldsymbol{\kappa}}(z) &= 0, \end{aligned}\right. \qquad \boldsymbol{\kappa} \in \mathbb{Z}^2 \setminus \{0\} \tag{B.2}$$

with $D_{\tilde{\alpha},\tilde{\beta}} := \tilde{\alpha}^2 + \tilde{\beta}^2 - \partial_z^2$. The system (B.2) has to be complemented with the boundary conditions

$$a_{\boldsymbol{\kappa}} = \partial_z a_{\boldsymbol{\kappa}} = b_{\boldsymbol{\kappa}} = 0 \quad \text{at} \quad z = \pm\frac{1}{2}, \quad \boldsymbol{\kappa} \in \mathbb{Z}^2 \setminus \{0\},$$

in order to have a well-posed eigenvalue problem. As explained in section 3, the maximum is attained by a single mode. Since we are ultimately interested in the maximum with respect to all periodicity cells, it is sufficient to consider the finite dimensional system

$$\begin{aligned} D^2_{\tilde{\alpha},\tilde{\beta}}\,\tilde{a}(z) - i\frac{\mu}{2}\Big(2\tilde{\alpha}\partial_z\tilde{a}(z) + \lambda\,\tilde{\beta}\tilde{b}(z)\Big) &= 0, \\ D_{\tilde{\alpha},\tilde{\beta}}\,\tilde{b}(z) + i\frac{\mu}{2}\tilde{\beta}\tilde{a}(z) &= 0 \end{aligned} \tag{B.3}$$

together with

$$\tilde{a} = \partial_z\tilde{a} = \tilde{b} = 0 \quad \text{at} \quad z = \pm\frac{1}{2}. \tag{B.4}$$

$\mathrm{Re}_{\mathcal{E}}$ is then given by

$$\mathrm{Re}_{\mathcal{E}} = \min_{(\tilde{\alpha},\tilde{\beta})\in\mathbb{R}^2} \mu_0(\tilde{\alpha},\tilde{\beta},\lambda),$$

with μ_0 being the smallest positive eigenvalue in (B.3) and (B.4). Applying a standard shooting method based on a fourth order Runge–Kutta integration, μ_0 is determined as a function of $\tilde{\alpha}$, $\tilde{\beta}$, and λ. Subsequent minimization with respect to $\tilde{\alpha}$ and $\tilde{\beta}$ furnishes $\mathrm{Re}_{\mathcal{E}}$ as a function of λ. The result is displayed in Figure 1: With decreasing λ the stability limit $\mathrm{Re}_{\mathcal{E}}$ increases from the ordinary energy limit $\mathrm{Re}_E = 82.6\ldots$ $(\lambda = 1)$ up to the value $\mathrm{Re}_E^x = 177.2\ldots$ (Figure 1, left), and this value is, in fact, attained for finite λ ($\lambda \approx 0.042$; see Figure 1, right).

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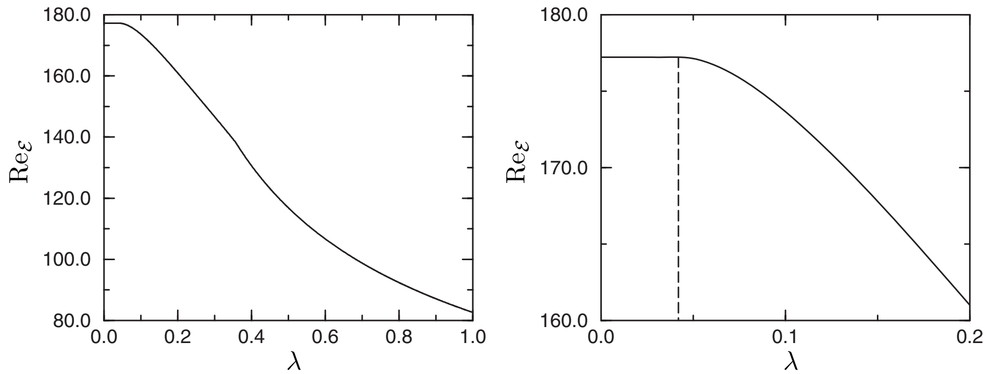

FIG. 1. *The generalized energy limit* $\mathrm{Re}_{\mathcal{E}}$ *versus coupling parameter* λ *with* $\mathcal{E}_1$ *given in* (3.1). *In the left graph,* λ *covers the range between* 0 *and* 1 *(*$\lambda = 1$ *corresponds to the ordinary energy); the right graph magnifies the region close to* $\lambda = 0$.

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