ASYMPTOTIC BEHAVIOUR OF THE DIV–CURL PROBLEM IN EXTERIOR DOMAINS

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Abstract. We study the variety of solutions of the inhomogeneous div–curl problem in exterior domains in dependence on the decay conditions on div and curl. Here we consider the Neumann as well as the Dirichlet boundary value prescription where in the first case the topological impact is decisive. In the second case the integrability conditions on div, curl and the boundary values are more difficult. Finally we present Hölder estimates for the solution of the Dirichlet or Neumann problem where it is unique.

1. Introduction

In electrostatics or magnetostatics it is a classical problem to determine a vector field v by the presciption of $\operatorname{div} v$, $\operatorname{curl} v$ and certain boundary values. If the normal component $\langle v, \nu \rangle$ is given we speak of a Neumann problem; if the tangential component $\nu \times v$ is given, the problem is called a Dirichlet problem. A basic solution theory of these problems using the fundamental theorem of vector analysis has been developed by Kress in [8] and has been extended to exterior domains in [13]. There, in an exterior domain $\hat{G}$ the condition

$$\operatorname{div} v, \operatorname{curl} v \in L^{\frac{3}{2}-\delta}(\hat{G}), \quad 0 < \delta < \delta_0, \quad \delta_0 \in (0, \tfrac{1}{2}),$$

is supposed, which means an averaged decay of $|\operatorname{div} v|$ and $|\operatorname{curl} v|$ stronger than $\frac{1}{|x|^2}$, $|x| \to \infty$. But these conditions can be weakened by modification of the integral kernels which appear in the fundamental theorem. This method goes back to a work of Otto Blumenthal [2]. There, a decomposition of vector fields defined in the entire space $\mathbb{R}^3$ into a source-free and a vorticity-free component has been proved, where the (smooth) vector field is only supposed to vanish at infinity. To obtain this, for the two components a potential and a vector potential, respectively, are constructed, similar to the

fundamental theorem, but without boundary integrals. An additional term causes a stronger decay of the integral kernels and guarantees the existence of the integrals under the conditions supposed.

In the present paper we use this method first to prove an existence theorem for the div–curl problem in the entire space, where the data for div and curl are assumed to have decay of order $O(\frac{1}{|x|^\beta})$ at infinity for some $\beta > 0$. By extension of the data and correction of the boundary values by harmonic vector fields with strong decay, the Neumann and also the Dirichlet problem for inhomogeneously harmonic vector fields can be reduced to this result. Supposing $\beta > 1$ the solution becomes unique within a certain class. In the case $0 < \beta \leq 1$ the solution may increase sublinearly. Of course, there also exist solutions with stronger asymptotic increase. Here, we study the asymptotic behaviour and the variety of solutions in dependence on $\beta > 0$.

To investigate the topological influence we consider the simple handle model (see [10, p. 224]) and ignore the abstract definition of the Betti number, but we note that this model cannot be applied to all domains in $\mathbb{R}^3$ (cf. [4]). Here, the first Betti number of a domain $G \subset \mathbb{R}^3$ is understood as the number of handles, i.e., the number of equivalence classes of simply closed curves in G which are not null homotopic in G. From Alexander's duality theorem we know that the number of handles of G is equal to the number of handles of $\hat{G} := \mathbb{R}^3 \setminus \overline{G}$. The second Betti number of G is, in our terminology, the number of bounded, connected components of the complementary domain $\hat{G} := \mathbb{R}^3 \setminus \overline{G}$.

In the last paragraph we give some Hölder estimates for the Neumann and the Dirichlet problem in the case $1 < \beta < 3$.

Solutions of the div–curl problem in exterior domains and their Hölder estimates are an important tool to study force-free magnetic fields (cf. [7]).

The Dirichlet problem for Poisson's equation $\Delta u = f$ in exterior domains is treated in [12]. This covers the case where $v = \nabla u$ is conservative.

Formulation of the problems. The problems we are going to solve in this work are the following. The notation is explained in the following subparagraph.

Problem E (div–curl problem in the entire space $\mathbb{R}^3$). *Assume $\beta > 0$,*

$$f \in C^\alpha_{\mathrm{unif}}(\mathbb{R}^3, \mathbb{R}), \quad |f(x)| = O(|x|^{-\beta}), \ |x| \to \infty,$$
$$w \in C^1(\mathbb{R}^3, \mathbb{R}^3) \cap C^\alpha_{\mathrm{unif}}(\mathbb{R}^3, \mathbb{R}^3), \quad |w(x)| = O(|x|^{-\beta}), \ |x| \to \infty,$$

with

$$\operatorname{div} w = 0 \quad in \ \mathbb{R}^3.$$

We search for a vector field $v \in C^{1+\alpha}_{\mathrm{unif}}(\mathbb{R}^3, \mathbb{R}^3)$ satisfying

$$\operatorname{div} v = f \quad and \quad \operatorname{curl} v = w \quad in \ \mathbb{R}^3$$

and certain asymptotic conditions.

Problem N (Neumann problem for inhomogeneously harmonic vector fields).
Let $G \subset \mathbb{R}^3$ be a bounded domain with smooth boundary, first Betti number $\tilde{n}$ and second Betti number zero, $\hat{G} := \mathbb{R}^3 \setminus \overline{G}$, $\beta > 0$,

$$f \in C^{\alpha}_{\mathrm{unif}}(\overline{\hat{G}}, \mathbb{R}), \quad |f(x)| \ = \ O\big(|x|^{-\beta}\big), \ |x| \to \infty,$$
$$w \in C^1(\overline{\hat{G}}, \mathbb{R}^3) \cap C^{\alpha}_{\mathrm{unif}}(\overline{\hat{G}}, \mathbb{R}^3), \ |w(x)| \ = \ O\big(|x|^{-\beta}\big), \ |x| \to \infty,$$
$$w \text{ with zero flux in } \hat{G},$$
$$g \in C^0(\partial G, \mathbb{R}).$$

Furthermore, let $\Gamma_1, \ldots, \Gamma_{\tilde{n}} \in \mathbb{R}$. We search for a vector field $v \in C^{1+\alpha}_{\mathrm{unif}}(\overline{\hat{G}}, \mathbb{R}^3)$ with the properties

$$\operatorname{div} v = f, \ \operatorname{curl} v = w \quad in \ \hat{G},$$
$$\langle v, \nu \rangle = g \quad on \ \partial G,$$
$$\int_{\partial G} \langle \nu \times v, \mathfrak{z}_j \rangle = \Gamma_j, \quad j = 1, \ldots, \tilde{n},$$

where $(\mathfrak{z}_j)_{j=1,\ldots,\tilde{n}}$ is a certain basis of the space of so-called Neumann fields in G; see 2.

The condition "w with zero flux" means that for any closed, oriented surface $\mathcal{S} \subset \hat{G}$ with outer normal ν there holds

$$\int_{\mathcal{S}} \langle w, \nu \rangle \ d\Omega \ = \ 0,$$

which implies $\operatorname{div} w = 0$. In the case of an interior domain G instead of $\hat{G}$ the additional condition $\int_G f dx = \int_{\partial G} g d\Omega$ is necessary for the solvability of the Neumann problem (N) and $\mathfrak{z}_j$ is replaced by $\hat{\mathfrak{z}}_j$.

Problem D (Dirichlet problem for inhomogeneously harmonic vector fields).
Let $G \subset \mathbb{R}^3$ be a bounded domain with smooth boundary and first Betti number $\tilde{n}$ and second Betti number zero, $\hat{G} := \mathbb{R}^3 \setminus \overline{G}$, $\beta > 0$.

Suppose

$$f \in C^\alpha_{\mathrm{unif}}(\overline{\hat{G}}, \mathbb{R}), \quad |f(x)| \,=\, O\big(|x|^{-\beta}\big), \, |x| \to \infty,$$

$$w \in C^1(\overline{\hat{G}}, \mathbb{R}^3) \cap C^\alpha_{\mathrm{unif}}(\overline{\hat{G}}, \mathbb{R}^3), \quad |w(x)| \,=\, O\big(|x|^{-\beta}\big), \, |x| \to \infty,$$

w *with zero flux in* $\hat{G}$,

$$\gamma^* \in C^{1+\alpha}(\partial G, \mathbb{R}^3) \ \mathit{with} \ \langle \gamma^*, \nu \rangle = 0 \ \mathit{on} \ \partial G.$$

Furthermore, let $E \in \mathbb{R}$. *We search for a vector field* $v \in C^{1+\alpha}_{\mathrm{unif}}(\overline{\hat{G}}, \mathbb{R}^3)$ *satisfying*

$$\mathrm{div}\, v \,=\, f, \quad \mathrm{curl}\, v \,=\, w \qquad in \ \hat{G},$$

$$\nu \times v \,=\, \gamma^* \qquad on \ \partial G,$$

$$\int_{\partial G} \langle v, \nu \rangle \, d\Omega \,=\, E.$$

The integrability conditions of the Dirichlet problem (D) are more complicated as in the case of the Neumann problem (N). They will be treated in Section 3. It will turn out that here the first Betti number $\tilde{n}$ of $\hat{G}$ is not relevant. But the variety of solutions of (D) is determined by the second Betti number of $\hat{G}$ (cf. [13]). In this work we do not consider this influence as we suppose G to be a domain.

The asymptotic behaviour of the solutions of these problems will be studied in Section 3.

General assumptions and notation. In the following, we assume $G \subset \mathbb{R}^3$ to be a bounded domain with smooth boundary; by "smooth" we mean sufficient regularity of ∂G without fixing the exact class of regularity. In all cases considered here the class C^6 will be sufficient.

Furthermore, $\hat{G}$ means the exterior domain of G; i.e., $\hat{G} := \mathbb{R}^3 \setminus \overline{G}$. ν denotes, if not explicitly defined otherwise, the outer normal with respect to G.

Furthermore, we suppose that the handle model can be applied to ∂G. The term "simply closed curve" shall be made precise here: We denote a curve to be simply closed, if it is closed, imbedded into a surface, and is the boundary of an oriented surface piece.

A closed surface $\mathcal{S}$ is here the smooth boundary $\partial\mathcal{D}$ of some bounded domain $\mathcal{D} \subset \mathbb{R}^3$, where $\mathcal{D}$ always lies locally at one side of the boundary. Then, in particular, $\mathcal{S} = \partial\mathcal{D}$ is orientable.

The Euclidean scalar product in $\mathbb{R}^3$ is denoted by $\langle \cdot, \cdot \rangle$, i.e., $\langle x, y \rangle := \sum_{j=1}^{3} x_j y_j$. For a differentiable vector field B, the Jacobian matrix will be denoted by DB. For multi-indices $\mathrm{a} = (\mathrm{a}_1, \mathrm{a}_2, \mathrm{a}_3) \in \mathbb{N}_0^3$, we set

$$D^{\mathrm{a}} f := \frac{\partial^{|\mathrm{a}|} f}{\partial x_1^{\mathrm{a}_1} \, \partial x_2^{\mathrm{a}_2} \, \partial x_3^{\mathrm{a}_3}}, \quad |\mathrm{a}| := \mathrm{a}_1 + \mathrm{a}_2 + \mathrm{a}_3, \quad \mathrm{a}! := \mathrm{a}_1! \, \mathrm{a}_2! \, \mathrm{a}_3!.$$

For a continuous function or vector field f in some (bounded or unbounded) domain $\mathcal{G}$ we set $\|f\|_{C^0(\mathcal{G})} := \sup_{x \in \mathcal{G}} |f(x)|$. Furthermore, for a function or vector field f and $k \in \mathbb{N}$

$$\|D^k f\| \quad := \quad \max_{|\mathrm{a}|=k} \|D^{\mathrm{a}} f\|.$$

Hölder spaces are the appropriate function spaces in classical potential theory. Let $\mathcal{G} \subset \mathbb{R}^3$ be a domain. For a function or a vector field $f \in C^k(\mathcal{G})$ and $0 < \alpha < 1$ we define then

$$\|f\|_{C^{k+\alpha}(\mathcal{G})} := \sum_{j=0}^{k} \|D^j f\|_{C^0(\mathcal{G})} + [D^k f]_{C^\alpha(\mathcal{G})}, \tag{1.1}$$

where

$$[D^k f]_{C^\alpha(\mathcal{G})} := \max_{|\mathrm{a}|=k} \sup_{x,y \in \mathcal{G}} \frac{|D^{\mathrm{a}} f(x) - D^{\mathrm{a}} f(y)|}{|x-y|^\alpha}.$$

Distinguishing between local and global Hölder continuity, the following notation is customary in the literature (see [5]):

In the local case the condition $\|f\|_{C^{k+\alpha}(K)} < \infty$ holds for every compact subset $K \subset \mathcal{G}$ and f belongs therefore to the space $C^{k+\alpha}(\mathcal{G}) := C_{\mathrm{loc}}^{k+\alpha}(\mathcal{G})$. If, however, the uniform condition

$$\|f\|_{C^{k+\alpha}(\overline{\mathcal{G}})} < \infty$$

holds, f is said to be $f \in C^{k+\alpha}(\overline{\mathcal{G}}) := C_{\mathrm{unif}}^{k+\alpha}(\mathcal{G})$ and $C_{\mathrm{unif}}^{k+\alpha}(\mathcal{G})$ is a Banach space with the norm defined in (1.1).

For every $f \in C^{1+\alpha}(\overline{\hat{G}})$ there exists an extension $\overline{f} \in C_{\mathrm{unif}}^{1+\alpha}(\mathbb{R}^3)$, $\overline{f}|_{\hat{G}} = f$ such that

$$\|\overline{f}\|_{C^{1+\alpha}(\mathbb{R}^3)} \leq c_1 \|f\|_{C^{1+\alpha}(\overline{\hat{G}})}, \quad \|\overline{f}\|_{C^0(\mathbb{R}^3)} \leq c_1 \|f\|_{C^0(\overline{\hat{G}})} \tag{1.2}$$

with some constant $c_1 > 0$ independent of f (see [5, Lemma 6.37]; the unboundedness of $\hat{G}$ is here not relevant.). We note that this construction is linear.

Concerning the boundary ∂G of G (or $\hat{G}$) $f \in C^{1+\alpha}(\partial G)$ means $f \circ \mu \in C^{1+\alpha}(\overline{U})$, where $\mu : U \to \partial G$ is a local chart of ∂G. The corresponding norm $\| \cdot \|_{C^{1+\alpha}(\partial G)}$ therefore depends on the chosen atlas.

In addition to the Hölder norm we need in exterior domains also a weighted norm characterizing the asymptotic behaviour to obtain global estimates for potential theoretic problems. Here we set for a function or a vector field f in $\hat{G}$ and $\beta > 0$

$$\|\|f\|\|_\beta := \sup_{x \in \hat{G}} |x|^\beta |f(x)|.$$

If $\|\|f\|\|_\beta < \infty$ we also write $|f(x)| = O(|x|^{-\beta}), |x| \to \infty$.

The differential operators ∇, div and curl are understood as usual. In the integral kernels ∇, div, curl denote the corresponding derivatives with respect to x and ∇', div$'$, curl$'$ the derivatives with respect to x'.

For tangential vector fields $\mathfrak{a}$ defined on surface pieces $\mathcal{S} \subset \mathbb{R}^3$ one can define the operator Div $\mathfrak{a}$ as follows. Let $\mathrm{x} : Q \subset \mathbb{R}^2 \to \mathcal{S}$, $(t_1, t_2) \mapsto \mathrm{x}(t_1, t_2)$ be a local chart,

$$g_{ij}(t_1, t_2) = \langle \partial_{t_i} \mathrm{x}, \partial_{t_j} \mathrm{x} \rangle (t_1, t_2) \quad (i, j = 1, 2)$$

the corresponding Gram's matrix with determinant $g = \det(g_{ij})_{ij}$,

$$\mathfrak{a}(\mathrm{x}(t_1, t_2)) = \sum_{j=1}^{2} a_j(\mathrm{x}(t_1, t_2)) \cdot \frac{\partial \mathrm{x}}{\partial t_j}(t_1, t_2)$$

a continuously differentiable tangential vector field on $\mathcal{S}$. Then we set

$$\text{Div } \mathfrak{a} := \frac{1}{\sqrt{g}} \sum_{j=1}^{2} \frac{\partial \left(\sqrt{g}\, a_j(\mathrm{x}) \right)}{\partial t_j}.$$

For C^1 vector fields v which are defined in a neighbourhood of ∂G with outer normal ν there holds the equation

$$\text{Div} (\nu \times v) = -\langle \nu, \text{curl}\, v \rangle. \tag{1.3}$$

We use the abbreviation $r := |x - x'|$, if r is not explicitly defined otherwise. For $R > 0$ and $x_0 \in \mathbb{R}^3$ we set $K_R(x_0) := \{ x \in \mathbb{R}^3 : |x - x_0| < R \}$.

2. Asymptotic behaviour of harmonic functions and vector fields. Neumann and Dirichlet fields

From classical potential theory the following theorem is well known:

Theorem 2.1. *Let $G \subset \mathbb{R}^3$ be a bounded domain with smooth boundary, $\hat{G} := \mathbb{R}^3 \setminus \overline{G}$ and $g \in C^0(\partial G, \mathbb{R})$. Then the Neumann problem or Dirichlet problem*

$$
\begin{aligned}
\Delta \varphi_\iota &= 0 && in \ \hat{G}, && \iota = N, D, \\
\langle \nabla \varphi_N, \nu \rangle &= g && on \ \partial G \ or \\
\varphi_D &= g && on \ \partial G, \ respectively, && \quad (2.1) \\
|\varphi_\iota(x)| &= O\big(|x|^{-1}\big), && |x| \to \infty, && \iota = N, D, \\
|\nabla \varphi_\iota(x)| &= O\big(|x|^{-2}\big), && |x| \to \infty, && \iota = N, D,
\end{aligned}
$$

has a unique solution $\varphi_N \in C^2(\hat{G}, \mathbb{R}) \cap C^1(\overline{\hat{G}}, \mathbb{R})$, $\varphi_D \in C^2(\hat{G}, \mathbb{R}) \cap C^0(\overline{\hat{G}}, \mathbb{R})$, respectively.

Lemmas 2.1, 2.2, 2.3, 2.4 and Theorem 2.2 to follow characterize the asymptotic behaviour of harmonic functions and can be proved by means of Green's formula, the mean value theorem, spherical harmonics or Poisson's formula. We may omit the proofs.

Lemma 2.1. *Let $G \subset \mathbb{R}^3$ be a bounded domain, $\hat{G} := \mathbb{R}^3 \setminus \overline{G}$, and $\varphi \in C^2(\hat{G}, \mathbb{R}) \cap C^0(\overline{\hat{G}}, \mathbb{R})$ with $\Delta \varphi = 0$. Suppose furthermore*

$$
|\varphi(x)| = O(|x|^\beta), \ |x| \to \infty \ for \ some \ \beta \in (0, 1).
$$

Then there exists some $u_0 \in \mathbb{R}$ with

$$
\varphi(x) \underset{unif}{\longrightarrow} u_0, \ |x| \to \infty.
$$

Lemma 2.2. *Let $G \subset \mathbb{R}^3$ be a bounded domain, $\hat{G} := \mathbb{R}^3 \setminus \overline{G}$, $\beta \in \mathbb{R}$, $\varphi \in C^2(\hat{G}, \mathbb{R})$ with $\Delta \varphi = 0$ and*

$$
|\varphi(x)| = O(|x|^\beta), \quad |x| \to \infty.
$$

Then

$$
|\nabla \varphi(x)| = O(|x|^{\beta-1}), \quad |x| \to \infty.
$$

Lemma 2.3. *Let $G, \hat{G}$ and g be as in Theorem 2.1, and let $\varphi \in C^2(\hat{G}, \mathbb{R}) \cap C^1(\overline{\hat{G}}, \mathbb{R})$ be the unique solution of the Neumann problem (2.1). Then the following equivalence holds:*

$$
\int_{\partial G} g \ d\Omega \ = 0 \quad \Longleftrightarrow \quad |\varphi(x)| = O\big(|x|^{-2}\big), \ |x| \to \infty.
$$

Harmonic vector fields (i.e., div and curl vanish) with zero normal component at the boundary are called Neumann fields. More precisely, we make the following definition.

Definition 2.1. Assume $G \subset \mathbb{R}^3$ to be a bounded domain with smooth boundary, first Betti number $\tilde{n}$ and second Betti number zero, $\hat{G} := \mathbb{R}^3 \setminus \overline{G}$, $\mu > 0$.

$$
\begin{aligned}
\mathcal{N}^\mu(\mathbb{R}^3) &:= \big\{ v \in C^1(\mathbb{R}^3, \mathbb{R}^3) : \operatorname{div} v = 0,\ \operatorname{curl} v = 0, \\
&\qquad\qquad |v(x)| = O(|x|^\mu),\ |x| \to \infty\ \big\}, \\
\mathcal{N}^\mu(\hat{G}) &:= \big\{ v \in C^1(\hat{G}, \mathbb{R}^3) \cap C^0(\overline{\hat{G}}, \mathbb{R}^3) : \operatorname{div} v = 0,\ \operatorname{curl} v = 0 \text{ in } \hat{G}, \\
&\qquad\qquad \langle v, \nu \rangle = 0 \text{ on } \partial G,\ |v(x)| = O(|x|^\mu),\ |x| \to \infty \big\}, \\
\mathcal{N}_0^\mu(\hat{G}) &:= \big\{ v \in \mathcal{N}^\mu(\hat{G}) : v \text{ without circulation } \big\}, \\
\mathfrak{Z}_\mathbb{R}(\hat{G}) &:= \big\{ v \in C^1(\hat{G}, \mathbb{R}^3) \cap C^0(\overline{\hat{G}}, \mathbb{R}^3) : \operatorname{div} v = 0,\ \operatorname{curl} v = 0 \text{ in } \hat{G}, \\
&\qquad\qquad \langle v, \nu \rangle = 0 \text{ on } \partial G,\ |v(x)| = O(|x|^{-2}),\ |x| \to \infty \big\}.
\end{aligned}
$$

Here, "v without circulation" means that for each simply closed and piecewise-continuously differentiable curve $\ell \subset \hat{G}$

$$
\int_\ell v \, d\vec{s} = \int_a^b \big\langle v(\gamma(t)), \dot{\gamma}(t) \big\rangle \, dt = 0,
$$

where $\gamma : [a, b] \to \hat{G}$ is a parametrization of ℓ. In particular, this implies that v is the gradient of some function $\Psi \in C^1(\hat{G}, \mathbb{R})$.

To construct a basis of the space $\mathfrak{Z}_\mathbb{R}(\hat{G})$ consider the $\tilde{n}$ topological independent simply closed curves $\ell_1, \dots, \ell_{\tilde{n}}$ in G which are not null homotopic in G. (See the paragraph "General assumptions and notations.")

There also exist $\tilde{n}$ closed curves $\hat{\ell}_1, \dots, \hat{\ell}_{\tilde{n}}$ in $\hat{G} := \mathbb{R}^3 \setminus \overline{G}$ having the corresponding property in $\hat{G}$.

Concerning the curves $\ell_1, \dots, \ell_{\tilde{n}}$ and $\hat{\ell}_1, \dots, \hat{\ell}_{\tilde{n}}$ we further assume that the surface piece $\mathcal{F}_j$, whose boundary is ℓ_j, has a positive orientation with respect to the orientation of ℓ_j (cf. the right-hand rule) and is penetrated by $\hat{\ell}_j$ in exactly one point. There, the orientation of $\hat{\ell}_j$ is chosen in such a way that $\hat{\ell}_j$ intersects $\mathcal{F}_j$ under an angle $< \frac{\pi}{2}$ with the normal $\tilde{\nu}_j$ on $\mathcal{F}_j$, and after interchanging ℓ_j and $\hat{\ell}_j$, etc. also, the corresponding condition is satisfied.
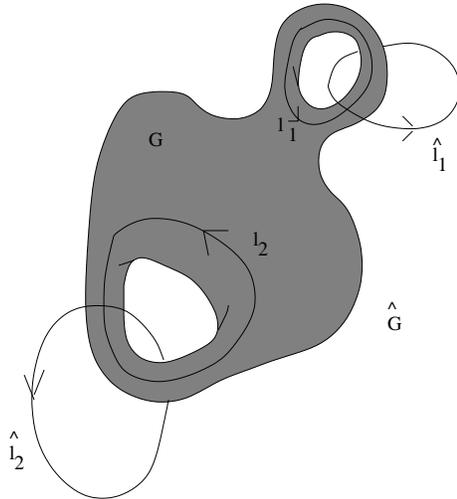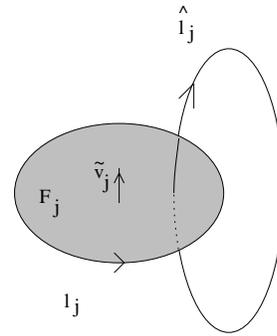
Fig. 1   Fig. 2

Now assume an electric current flowing along ℓ_i. Then the magnetic field v_i induced by this current is, up to some constant multiplier,

$$v_i(x) \quad = \quad \operatorname{curl} \frac{1}{4\pi} \int_{\ell_i} \frac{1}{r}\, d\vec{s}, \qquad x \in \mathbb{R}^3 \setminus \operatorname{trace} \ell_i,$$

where $r = |x - x'|$. This equation is the so-called Biot–Savart formula. The field v_i is harmonic for $i = 1, \ldots, \tilde{n}$ and satisfies

$$\int_{\hat{\ell}_j} v_i\, d\vec{s} \quad = \quad \delta_{ij}, \qquad i, j = 1, \ldots, \tilde{n};$$

see [13, p. 96]. Considering the restriction $v_i\big|_{\overline{\hat{G}}}$ and adding the gradient of a harmonic function φ_i in $\hat{G}$ satisfying

$$\langle \nabla \varphi_i, \nu \rangle\big|_{\partial G} \quad = \quad -\langle v_i, \nu \rangle\big|_{\partial G}$$

we obtain that $\hat{\mathfrak{z}}_i := v_i + \nabla \varphi_i$ is a Neumann field in $\hat{G}$ with

$$\int_{\hat{\ell}_j} \hat{\mathfrak{z}}_i\, d\vec{s} = \delta_{ij}, \qquad i, j = 1, \ldots, \tilde{n}. \tag{2.2}$$

This construction yields us a basis $\{\hat{\mathfrak{z}}_1, \ldots, \hat{\mathfrak{z}}_{\tilde{n}}\}$ of the space $\mathfrak{Z}_{\mathbb{R}}(\hat{G})$ of Neumann fields in $\hat{G}$. By an analogous procedure, interchanging ℓ_i and $\hat{\ell}_i$, G

and $\hat{G}$ etc., we obtain a basis $\{\mathfrak{z}_1, \ldots, \mathfrak{z}_{\tilde{n}}\}$ of $\mathfrak{Z}_{\mathbb{R}}(G)$, which is the vector space of Neumann fields in G. There holds the equation

$$\int_{\partial G} \langle \nu \times \mathfrak{z}_i, \hat{\mathfrak{z}}_j \rangle \, d\Omega = \int_{\hat{\ell}_i} \hat{\mathfrak{z}}_j \, d\vec{s} = \delta_{ij}, \quad i,j = 1, \ldots, \tilde{n} \qquad (2.3)$$

(cf. [8, (5.18)] and [13, Lemma I.3.1]).

Remark. $\dim \mathfrak{Z}(\hat{G}) = \tilde{n}$.

Lemma 2.4. *Let $G \subset \mathbb{R}^3$ be a bounded domain with smooth boundary, $\hat{G} := \mathbb{R}^3 \setminus \overline{G}$, $u_\infty \in \mathbb{R}^3$. Then the problem*

$$\begin{array}{rcll} u & = & \nabla\varphi, & \\ \Delta\varphi & = & 0 & in \ \hat{G}, \\ \langle u, \nu \rangle & = & 0 & on \ \partial G, \\ u(x) & \xrightarrow[\text{unif}]{} & u_\infty, & |x| \to \infty \end{array} \qquad (2.4)$$

has a unique solution.

Using the results above and a classification of harmonic polynomials (cf. [11]), after some technical calculations we arrive at

Theorem 2.2. *Let $G \subset \mathbb{R}^3$ be a bounded domain with smooth boundary, first Betti number $\tilde{n}$ and second Betti number zero. Let $\hat{G} := \mathbb{R}^3 \setminus \overline{G}$, $k \in \mathbb{N}_0$, $\beta \in [0,1)$. Then*

$$\begin{array}{rll} \text{(i)} \qquad \dim \mathcal{N}^k(\mathbb{R}^3) & = & k^2 + 4k + 3, \\ \text{(ii)} \qquad \dim \mathcal{N}_0^k(\hat{G}) & = & k^2 + 4k + 3, \\ \text{(iii)} \qquad \mathcal{N}_0^{k+\beta}(\hat{G}) & = & \mathcal{N}_0^k(\hat{G}), \\ \text{(iv)} \qquad \mathcal{N}^{k+\beta}(\hat{G}) & = & \mathfrak{Z}(\hat{G}) \oplus \mathcal{N}_0^k(\hat{G}), \\ \text{(v)} \qquad \dim \mathcal{N}^{k+\beta}(\hat{G}) & = & \tilde{n} + k^2 + 4k + 3. \end{array}$$

Remark. There also exist Neumann fields which increase more strongly than every polynomial. One can even prove that for every monotonically increasing function $\kappa : [0, \infty) \to [0, \infty)$ there exists a Neumann field w in $\hat{G}$ and some $R > 0$ with the property

$$|w(x_1, x_2, 0)| \quad > \quad \kappa\left(\sqrt{x_1^2 + x_2^2}\right) \quad \text{for } x_1^2 + x_2^2 > R^2.$$

In order to investigate the Dirichlet fields which form the zero space of the Dirichlet problem (D) we need some nonstandard lemmata of which we will also give proofs.

Remark. Let $G \subset \mathbb{R}^3$ be a bounded domain with (C^k-)smooth boundary and $\varphi \in C^k(\partial G, \mathbb{R})$. Then there exists an extension $\bar{\varphi} \in C_0^k(\mathbb{R}^3, \mathbb{R})$ of φ (i.e., $\bar{\varphi}|_{\partial G} = \varphi$).

Lemma 2.5. *Let $G \subset \mathbb{R}^3$ be a bounded domain with smooth boundary, $\hat{G} := \mathbb{R}^3 \setminus \overline{G}$, and $\gamma^* \in C^1(\partial G, \mathbb{R}^3)$ a tangential field; i.e., $\langle \gamma^*, \nu \rangle = 0$ on ∂G. Then the following holds:*

There exists a function $\varphi \in C^2(\partial G, \mathbb{R})$ with extension $\bar{\varphi} \in C_0^2(\mathbb{R}^3, \mathbb{R})$ and the property

$$\nu \times \nabla\bar{\varphi}\big|_{\partial G} = \gamma^*, \tag{2.5}$$

if and only if

$$\mathrm{Div}\,\gamma^* = 0 \quad and \quad \forall\, \mathfrak{z} \in \mathfrak{Z}_{\mathbb{R}}(G) \cup \mathfrak{Z}_{\mathbb{R}}(\hat{G}): \int_{\partial G} \langle \gamma^*, \mathfrak{z} \rangle \, d\Omega = 0. \tag{2.6}$$

Proof. (a) We suppose φ having the properties mentioned above. Consequently

$$\mathrm{Div}\,\gamma^* = \mathrm{Div}\,(\nu \times \nabla\bar{\varphi}) = -\langle \nu, \mathrm{curl}\,\nabla\bar{\varphi} \rangle = 0.$$

For $\mathfrak{z} \in \mathfrak{Z}_{\mathbb{R}}(G) \cup \mathfrak{Z}_{\mathbb{R}}(\hat{G})$ we have, moreover,

$$\int_{\partial G} \langle \gamma^*, \mathfrak{z} \rangle \, d\Omega = \int_{\partial G} \langle \nu \times \nabla\bar{\varphi}, \mathfrak{z} \rangle \, d\Omega = \int_{\partial G} \langle \nabla\bar{\varphi} \times \mathfrak{z}, \nu \rangle \, d\Omega.$$

In the case $\mathfrak{z} \in \mathfrak{Z}_{\mathbb{R}}(G)$ Gauss's theorem yields

$$\int_{\partial G} \langle \nabla\bar{\varphi} \times \mathfrak{z}, \nu \rangle \, d\Omega = \int_G \mathrm{div}\,\big(\nabla\bar{\varphi} \times \mathfrak{z}\big) dx = 0.$$

and in the case $\mathfrak{z} \in \mathfrak{Z}_{\mathbb{R}}(\hat{G})$ we obtain for sufficiently large $R > 0$

$$\int_{\partial G} \langle \nabla\bar{\varphi} \times \mathfrak{z}, \nu \rangle \, d\Omega = -\int_{K_R(0)\setminus\overline{G}} \mathrm{div}\,\big(\nabla\bar{\varphi} \times \mathfrak{z}\big) \, dx = 0.$$

(b) For an arbitrary vector field v we have $v\big|_{\partial G} = \langle \nu, v\big|_{\partial G}\rangle\nu + \nu \times \big(v\big|_{\partial G} \times \nu\big)$. Since γ^* is a tangential field, there holds the equivalence

$$\nu \times \nabla\bar{\varphi}\big|_{\partial G} = \gamma^* \ \Leftrightarrow\ \nu \times \big(\nabla\bar{\varphi}\big|_{\partial G} \times \nu\big) = \big(\nu \times \nabla\bar{\varphi}\big|_{\partial G}\big) \times \nu = \gamma^* \times \nu,$$

where $\nu \times \nabla\bar{\varphi}\big|_{\partial G} \times \nu$ is the tangential derivative of $\bar{\varphi}$. Thus a function $\varphi \in C^2(\partial G, \mathbb{R})$ with the properties mentioned above exists if and only if for every closed and piecewise-continuously differentiable curve $\mathcal{C}$ with trace $\mathcal{C} \subset \partial G$ the condition

$$\int_{\mathcal{C}} \big(\gamma^* \times \nu\big) \, d\vec{s} = 0$$

holds. Let $w \in C_0^2(\mathbb{R}^3, \mathbb{R})$ with $w\big|_{\partial G} = \gamma^* \times \nu$. Condition (2.6) yields on ∂G

$$\langle \operatorname{curl} w, \nu \rangle = -\operatorname{Div}\left(\nu \times w\right) = -\operatorname{Div}\left(\nu \times (\gamma^* \times \nu)\right) = -\operatorname{Div} \gamma^* = 0.$$

(1) Assume $\mathcal{C}$ to be a smooth, simply closed curve in ∂G which is null homotopic in ∂G. Then there exists some compact $\mathcal{A} \subset \partial G$ with $\partial_{\partial G}\mathcal{A} = \operatorname{trace} \mathcal{C}$, and from Stokes' theorem we can conclude

$$\int_{\mathcal{C}} \left(\gamma^* \times \nu\right) d\vec{s} = \int_{\mathcal{C}} w \, d\vec{s} = \pm \int_{\mathcal{A}} \langle \operatorname{curl} w, \nu \rangle \, d\Omega = 0.$$

(2) Now, concerning the case of non-null homotopic curves $\mathcal{C}$ in ∂G, we first investigate two special cases:

Consider the jth handle of G. On its boundary there exist, up to homotopic equivalence, two closed curves $\mathcal{C}_j$ and $\hat{\mathcal{C}}_j$, where $\mathcal{C}_j$ is null homotopic in $\overline{G}$ but not in $\overline{\hat{G}}$ and $\hat{\mathcal{C}}_j$ is null homotopic in $\overline{\hat{G}}$ but not in $\overline{G}$ (see Figure 3). Moreover, $\mathcal{C}_j$ and $\hat{\ell}_j$ are homotopic in $\overline{\hat{G}}$, $\hat{\mathcal{C}}_j$ and ℓ_j are homotopic in $\overline{G}$.

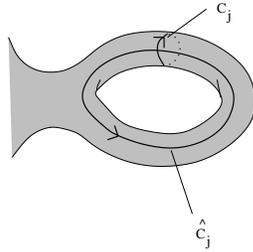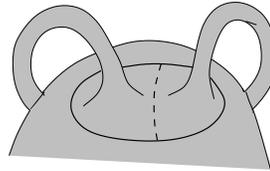
Figure 3   Figure 4

From [8, (5.18)] and [9, Satz 6.5] we know, since $\langle \operatorname{curl} w\big|_{\partial G}, \nu \rangle = 0$,

$$\int_{\hat{\mathcal{C}}_j} w \, d\vec{s} = -\int_{\partial G} \langle \nu \times w, \hat{\mathfrak{z}}_j \rangle \, d\Omega = -\int_{\partial G} \langle \gamma^*, \hat{\mathfrak{z}}_j \rangle \, d\Omega = 0. \qquad (2.7)$$

Applying this argument to $K_R(0) \backslash \overline{G}$ with $R > 0$ sufficiently large, we obtain

$$\int_{\mathcal{C}_j} w \, d\vec{s} = \int_{\partial G} \langle \nu \times w, \mathfrak{z}_j \rangle \, d\Omega = \int_{\partial G} \langle \gamma^*, \mathfrak{z}_j \rangle \, d\Omega = 0.$$

(3) By approximation of piecewise–continuously differentiable curves by smooth curves we can prove

$$\int_{\mathcal{C}} \left(\gamma^* \times \nu\right) d\vec{s} = \int_{\mathcal{C}} w \, d\vec{s} = 0$$

for every curve $\mathcal{C}$ which is homotopic in ∂G to any curve considered above.

(4) By introducing connecting paths, every simply closed curve can be decomposed into curves such that each of these is equivalent to some curve we investigated in (1), (2) and (3) (see Figure 4).

Therefore, we have

$$\int_{\mathcal{C}} \left(\gamma^* \times \nu\right) d\vec{s} = 0$$

for all closed curves $\mathcal{C}$ in ∂G, and the function φ can be defined by the curve integral. $\square$

Corollary 2.1. *Assume* $G, \hat{G}$ *and* γ^* *to be as in Lemma* 2.5. *Then the following holds:*

There exists a function $\psi \in C^2(\overline{\hat{G}}, \mathbb{R})$ *with*

$$\Delta\psi = 0 \ \ in \ \hat{G}, \ \ \nu \times \nabla\psi = \gamma^* \ on \ \partial G, \ \ |\psi(x)| = O\big(|x|^{-1}\big), \, |x| \to \infty, \quad (2.8)$$

if and only if the conditions $(2.6)_1$ *and* $(2.6)_2$ *in Lemma* 2.5 *are satisfied.*

Proof. This is an immediate consequence of Lemma 2.5 and Theorem 2.1.

Lemma 2.6. *Let* $G \subset \mathbb{R}^3$ *be a bounded domain with smooth boundary,* $\hat{G} := \mathbb{R}^3 \setminus \overline{G}$, $E \in \mathbb{R}$. *Then the problem*

$$
\begin{array}{rcll}
\Delta\varphi & = & 0 & in \ \hat{G}, \\
\nu \times \nabla\varphi & = & 0 & on \ \partial G, \\
|\varphi(x)| & \underset{\text{unif}}{\longrightarrow} & 0, & |x| \to \infty, \\
\int_{\partial G} \left\langle \nabla\varphi, \nu \right\rangle d\Omega & = & E &
\end{array}
\qquad (2.9)
$$

is uniquely solvable. The solution φ *satisfies the asymptotic condition*

$$|\varphi(x)| = O\big(|x|^{-1}\big), \quad |x| \to \infty.$$

Proof. (a) We consider the solution φ_D of the Dirichlet problem (2.1) with $g \equiv 1$, where obviously $\nu \times \nabla\varphi_D = 0$ on ∂G. The maximum principle yields

$$\left\langle \nabla\varphi_D, \nu \right\rangle \leq 0 \qquad on \ \partial G.$$

In accordance with Lemma 2.4 we have $\left\langle \nabla\varphi_D, \nu \right\rangle \not\equiv 0$ on ∂G. Thus

$$\eta_0 := \int_{\partial G} \left\langle \nabla\varphi_D, \nu \right\rangle d\Omega < 0$$

and $\varphi := \frac{E}{\eta_0} \varphi_D$ is a solution of the problem.

(b) Now let φ satisfy the conditions (2.9) with $E = 0$. Because of the second condition we have $\varphi\big|_{\partial G} \equiv c_0 \in \mathbb{R}$. The fact that $E = 0$ and the maximum principle yield $\varphi\big|_{\partial G} \equiv 0$ and finally $\varphi \equiv 0$ in $\hat{G}$. $\square$

Analogously to Lemma 2.4 we have the result

Lemma 2.7. *Let* $G \subset \mathbb{R}^3$ *be a bounded domain with smooth boundary,* $\hat{G} := \mathbb{R}^3 \setminus \overline{G}$, $u_\infty \in \mathbb{R}^3$. *Then the problem*

$$
\begin{aligned}
u &= \nabla \varphi, \\
\Delta \varphi &= 0 & \text{in } \hat{G}, \\
\nu \times u &= 0 & \text{on } \partial G, \\
\int_{\partial G} \langle \nabla \varphi, \nu \rangle \, d\Omega &= 0 \\
u(x) &\xrightarrow[\text{unif}]{} u_\infty, & |x| \to \infty
\end{aligned}
\tag{2.10}
$$

has a unique solution.

3. Solution theory for problems E, N and D

Lemma 3.1. *Assume* $\beta > 0$.
  (a) *Suppose* $f \in C^\alpha_{\text{unif}}(\mathbb{R}^3, \mathbb{R})$, $|f(x)| = O(|x|^{-\beta})$, $|x| \to \infty$ *and*

$$
u(x) := \frac{1}{4\pi} \int_{\mathbb{R}^3} \left(\nabla' \frac{1}{|x - x'|} - \nabla' \frac{1}{|x'|} \right) f(x') \, dx'
$$

for $x \in \mathbb{R}^3$. *Then* $u : \mathbb{R}^3 \to \mathbb{R}$ *is continuously differentiable satisfying*

$$
\operatorname{div} u = f, \qquad \operatorname{curl} u = 0 \qquad \text{in } \mathbb{R}^3.
$$

(b) *Suppose* $w \in C^1(\mathbb{R}^3, \mathbb{R}^3) \cap C^\alpha_{\text{unif}}(\mathbb{R}^3, \mathbb{R}^3)$, $|w(x)| = O(|x|^{-\beta})$, $|x| \to \infty$, *with* $\operatorname{div} w = 0$ *in* $\mathbb{R}^3$, *and*

$$
v(x) := \frac{1}{4\pi} \int_{\mathbb{R}^3} w(x') \times \left(\nabla' \frac{1}{|x - x'|} - \nabla' \frac{1}{|x'|} \right) dx'
$$

for $x \in \mathbb{R}^3$. *Then* $v : \mathbb{R}^3 \to \mathbb{R}^3$ *is continuously differentiable satisfying*

$$
\operatorname{div} v = 0, \qquad \operatorname{curl} v = w \qquad \text{in } \mathbb{R}^3.
$$

In the case $\beta > 1$ *the term* $\nabla' \frac{1}{|x'|}$ *in the integral kernels may be omitted.*

Proof. For $x, x' \in \mathbb{R}^3$, $x \neq x' \neq 0$ we set

$$
K(x, x') := \nabla' \frac{1}{|x - x'|} - \nabla' \frac{1}{|x'|}; \quad \text{i.e.,} \quad K_j(x, x') = \frac{x_j - x_j'}{|x - x'|^3} + \frac{x_j'}{|x'|^3}.
$$

Therefore, we have

$$
|K_j(x, x')| \leq \frac{|x|}{|x - x'| \cdot |x'|^2} + \frac{|x|}{|x - x'|^2 \cdot |x'|} + \frac{2|x|}{|x - x'|^3}.
\tag{3.1}
$$

For any bounded domain $\mathcal{G} \subset \mathbb{R}^3$ with $0, x \in \mathcal{G}$ and $\delta := \min\{\text{dist}(0, \partial\mathcal{G}),$ $\text{dist}(x, \partial\mathcal{G})\}$, $p, q > 0$ with $p + q = 3$ we obtain

$$\int_{\mathbb{R}^3\setminus\mathcal{G}} \frac{1}{|x - x'|^p \cdot |x'|^{q+\beta}} \, dx' \leq 8\pi \int_\delta^\infty \frac{r^2}{r^{3+\beta}} \, dr = \frac{8\pi}{\beta}\delta^{-\beta}. \qquad (3.2)$$

We now define

$$\Im_{11}(x) := \int_\mathcal{G} K(x, x')f(x') \, dx', \quad \Im_{12}(x) := \int_{\mathbb{R}^3\setminus\mathcal{G}} K(x, x')f(x') \, dx',$$

$$\Im_{21}(x) := \int_\mathcal{G} w(x') \times K(x, x') \, dx', \quad \Im_{22}(x) := \int_{\mathbb{R}^3\setminus\mathcal{G}} w(x') \times K(x, x') \, dx'.$$

Existence and differentiability of $\Im_{12}(x)$ and $\Im_{22}(x)$ are guaranteed by (3.1) and (3.2); existence and differentiability of $\Im_{11}(x)$ and $\Im_{21}(x)$ are well known from potential theory ([15, Satz 3.3], cf. [10, 2.4.6.]).

(a) We first consider $4\pi u(x) = \Im_{11}(x) + \Im_{12}(x)$ for $x \in \mathcal{G}$. We have

$$\Im_{11}(x) = -\nabla \int_\mathcal{G} \frac{f(x')}{|x - x'|} \, dx' - \int_\mathcal{G} f(x') \cdot \nabla' \frac{1}{|x'|} \, dx',$$

and thus ([15, Satz 3.4])

$$\text{div}\, \Im_{11}(x) = -\Delta \int_\mathcal{G} \frac{f(x')}{|x - x'|} \, dx' = 4\pi f(x), \qquad \text{curl}\, \Im_{11}(x) = 0.$$

Since $x \in \mathcal{G}$ we have, moreover,

$$\text{div}\, \Im_{12}(x) = -\int_{\mathbb{R}^3\setminus\mathcal{G}} f(x') \cdot \left(\Delta \frac{1}{|x - x'|}\right) dx' = 0,$$

and, again by differentiation under the integral,

$$\text{curl}\, \Im_{12}(x) = 0.$$

(b) We now consider $4\pi v(x) = \Im_{21}(x) + \Im_{22}(x)$ for $x \in \mathcal{G}$. A short calculation yields

$$\Im_{21}(x) = \text{curl} \int_\mathcal{G} \frac{w(x')}{|x - x'|} \, dx' - \int_\mathcal{G} w(x') \times \nabla' \frac{1}{|x'|} \, dx',$$

and therefore

$$\text{div}\, \Im_{21}(x) = 0,$$

$$\text{curl}\, \Im_{21}(x) = 4\pi w(x) \quad + \quad \nabla \text{div} \int_\mathcal{G} \frac{w(x')}{|x - x'|} \, dx' \qquad (3.3)$$

$$= 4\pi w(x) - \nabla \int_{\partial\mathcal{G}} \frac{1}{|x - \xi'|} \cdot \langle w(\xi'), \nu(\xi') \rangle \, d\Omega',$$

where we used $\operatorname{div} w = 0$. Now we investigate the integral $\Im_{22}(x)$. There arises, again after some calculations and using $\operatorname{div} w = 0$,

$$\operatorname{div} \Im_{22}(x) = \int_{\mathbb{R}^3 \setminus \mathcal{G}} \left\langle w(x'), \operatorname{curl} \nabla \frac{1}{|x - x'|} \right\rangle dx' = 0$$

and

$$\operatorname{curl} \Im_{22}(x) = \int_{\mathbb{R}^3 \setminus \mathcal{G}} \operatorname{curl} \left(w(x') \times \nabla' \frac{1}{|x - x'|} \right) dx'$$

$$= \nabla \int_{\partial \mathcal{G}} \frac{1}{|x - \xi'|} \cdot \langle w(\xi'), \nu(\xi') \rangle \, d\Omega' \tag{3.4}$$

From (3.3) and (3.4) we can conclude $\operatorname{curl} v = w$. $\qquad \square$

Lemma 3.2. *Assume* $\beta > 0$, $f \in C^0(\mathbb{R}^3, \mathbb{R})$ *and* $|f(x)| = O(|x|^{-\beta})$, $|x| \to \infty$. *Let*

$$u(x) \quad := \begin{cases} \dfrac{1}{4\pi} \displaystyle\int_{\mathbb{R}^3} f(x') \cdot \left(\nabla' \dfrac{1}{|x - x'|} - \nabla' \dfrac{1}{|x'|} \right) dx' & \text{for } 0 < \beta \leq 1, \\ \dfrac{1}{4\pi} \displaystyle\int_{\mathbb{R}^3} f(x') \cdot \nabla' \dfrac{1}{|x - x'|} \, dx' & \text{for } 1 < \beta \leq \infty. \end{cases}$$

Then

$$|u(x)| = \begin{cases} O(|x|^{1-\beta}), & |x| \to \infty, & \text{for } 0 < \beta < 1, \\ O(\ln|x|), & |x| \to \infty, & \text{for } \beta = 1. \\ O(\dfrac{1}{|x|^{\beta-1}}), & |x| \to \infty, & \text{for } 1 < \beta < 3, \\ O(\dfrac{\ln|x|}{|x|^2}), & |x| \to \infty, & \text{for } \beta = 3, \\ O(\dfrac{1}{|x|^2}), & |x| \to \infty, & \text{for } 3 < \beta \leq \infty. \end{cases}$$

The corresponding estimates also hold for v *as in Lemma* 3.1.

Proof. (a) Suppose $0 < \beta < 3$, $0 < \delta < \frac{1}{4}$, $x \in \mathbb{R}^3 \setminus \{0\}$. We obtain

$$\left| \int_{K_{\delta|x|}(0)} f(x') \cdot \nabla' \frac{1}{|x - x'|} \, dx' \right| \leq \||f\||_\beta \int_{K_{\delta|x|}(0)} \frac{1}{|x - x'|^2 \cdot |x'|^\beta} \, dx'$$

$$\leq \frac{\||f\||_\beta}{(1 - \delta)^2 \, |x|^2} \int_{K_{\delta|x|}(0)} \frac{1}{|x'|^\beta} \, dx' \leq \frac{4\pi \delta^{3-\beta} \||f\||_\beta}{(1 - \delta)^2 (3 - \beta)} |x|^{1-\beta} \tag{3.5}$$

since $|x'| < \delta|x|$ and $|x - x'| \geq (1-\delta)|x|$ for $x' \in K_{\delta|x|}(0)$. For $0 < \beta < 1$, we have furthermore

$$\Big| \int_{K_{\delta|x|}(0)} f(x') \cdot \nabla' \frac{1}{|x'|} \, dx' \Big| \leq \frac{4\pi\delta^{1-\beta}}{1-\beta} \cdot \||f\||_\beta \cdot |x|^{1-\beta}; \qquad (3.6)$$

in the case $\beta = 1$, however, for arbitrary $\delta_0 > 0$ and $|x| > \frac{\delta_0}{\delta}$,

$$\Big| \int_{K_{\delta|x|}(0)} f(x') \cdot \nabla' \frac{1}{|x'|} \, dx' \Big| \leq \int_{K_{\delta_0}(0)} \frac{|f(x')|}{|x'|^2} \, dx' + \int_{K_{\delta|x|}(0) \setminus K_{\delta_0}(0)} \frac{|f(x')|}{|x'|^2} \, dx'$$

$$\leq 4\pi \Big(\delta_0 \cdot \|f\|_\infty + \||f\||_1 \cdot \ln \frac{\delta|x|}{\delta_0} \Big). \qquad (3.7)$$

For $x' \in K_{\delta|x|}(x)$ we have $|x'| \geq (1-\delta)|x|$. Thus, for $|x| > 0$ and arbitrary $\beta > 0$ we arrive at

$$\Big| \int_{K_{\delta|x|}(x)} f(x') \cdot \nabla' \frac{1}{|x - x'|} \, dx' \Big| \leq \frac{4\pi\delta}{(1-\delta)^\beta} \cdot \||f\||_\beta \cdot |x|^{1-\beta}. \qquad (3.8)$$

Furthermore, for $0 < \beta \leq 1$, there holds

$$\Big| \int_{K_{\delta|x|}(x)} f(x') \cdot \nabla' \frac{1}{|x'|} \, dx' \Big| \leq \||f\||_\beta \cdot \int_{(1-\delta)|x| \leq |x'| \leq (1+\delta)|x|} |x'|^{-(2+\beta)} \, dx'$$

$$= \begin{cases} \frac{4\pi}{1-\beta} \cdot \big((1+\delta)^{1-\beta} - (1-\delta)^{1-\beta} \big) \cdot \||f\||_\beta \cdot |x|^{1-\beta}, & \text{for } 0 < \beta < 1, \\ 4\pi \ln \frac{1+\delta}{1-\delta} \cdot \||f\||_\beta, & \text{for } \beta = 1. \end{cases} \qquad (3.9)$$

Now we define $\Gamma := \mathbb{R}^3 \setminus \big(K_{\delta|x|}(0) \cup K_{\delta|x|}(x) \big)$,

$$\Gamma_1 := \big\{ x' \in \Gamma : |x - x'| < |x'| \big\}, \quad \Gamma_2 := \big\{ x' \in \Gamma : |x - x'| \geq |x'| \big\},$$

and obtain for $p, q \geq 0$, $p + q = 3$ and $\beta > 0$ (cf. (3.1), (3.2))

$$\int_\Gamma \frac{|x|}{|x - x'|^p \cdot |x'|^{q+\beta}} \, dx' \leq \int_{\Gamma_1} \frac{|x|}{|x - x'|^{3+\beta}} \, dx' + \int_{\Gamma_2} \frac{|x|}{|x'|^{3+\beta}} \, dx' \leq \frac{8\pi}{\beta\delta^\beta} \cdot |x|^{1-\beta}. \qquad (3.10)$$

Finally, the statement follows from (3.5), (3.6), (3.7), (3.8), (3.9), (3.1), and (3.10).

(b) Suppose again $0 < \delta < \frac{1}{4}$, $\beta > 1$, and define Γ as in the proof of (a). We now have

$$\Big| \int_\Gamma f(x') \cdot \nabla' \frac{1}{|x - x'|} \, dx' \Big| \leq \int_\Gamma \frac{\||f\||_\beta}{|x - x'|^2 \cdot |x'|^\beta} \, dx' \leq \frac{8\pi}{(\beta-1)\delta^{\beta-1}} \cdot \frac{\||f\||_\beta}{|x|^{\beta-1}}. \qquad (3.11)$$

From (3.5), (3.8) and (3.11) we obtain the desired estimate for the case $1 < \beta < 3$.

In the case $\beta = 3$ we have for arbitrary $\delta_0 > 0$ and $|x| < \frac{\delta_0}{\delta}$

$$\Big| \int_{K_{\delta|x|}(0)} f(x') \cdot \nabla' \frac{1}{|x - x'|} dx' \Big| \leq \int_{K_{\delta_0}(0)} \frac{|f(x')|}{|x - x'|^2} dx' + \int_{K_{\delta|x|}(0) \setminus K_{\delta_0}(0)} \frac{|f(x')|}{|x - x'|^2} dx'$$

$$\leq \frac{4\pi \delta_0^3}{3(1-\delta)^2} \cdot \frac{\|f\|_\infty}{|x|^2} + \frac{4\pi}{(1-\delta)^2} \cdot \frac{\||f\||_3}{|x|^2} \cdot \ln \frac{\delta|x|}{\delta_0}. \tag{3.12}$$

In the case $\beta = 3$ the desired estimate follows from (3.8), (3.11) and (3.12).

Now assume $\beta > 3$, let $R_0 > 0$ and $0 < \sigma < \frac{R_0}{4}$. Then $x' \in K_\sigma(0)$ satisfies $|x - x'| \geq |x| - |x'| > |x| - \sigma$, and for $|x| \geq R_0$ and thus $|x| - \sigma \geq \frac{3}{4}|x|$ we have

$$\Big| \int_{K_\sigma(0)} f(x') \cdot \nabla' \frac{1}{|x - x'|} dx' \Big| \leq \frac{1}{(|x| - \sigma)^2} \cdot \Big| \int_{K_\sigma(0)} f(x') \, dx' \Big| \leq \frac{(4\sigma)^3 \pi \|f\|_\infty}{3^3 |x|^2}. \tag{3.13}$$

We define

$$\tilde{\Gamma} \quad := \quad \mathbb{R}^3 \setminus \big(K_\sigma(0) \cup K_{\delta|x|}(x) \big),$$

$$\tilde{\Gamma}_1 := \big\{ x' \in \tilde{\Gamma} : |x - x'| < |x'| \big\}, \quad \tilde{\Gamma}_2 := \big\{ x' \in \tilde{\Gamma} : |x - x'| \geq |x'| \big\}$$

and obtain

$$\Big| \int_{\tilde{\Gamma}_1} f(x') \cdot \nabla' \frac{1}{|x - x'|} \, dx' \Big| \leq \int_{\tilde{\Gamma}_1} \frac{\||f\||_\beta}{|x - x'|^{2+\beta}} \, dx' \leq \frac{4\pi}{(\beta-1)\delta^{\beta-1}} \cdot \frac{\||f\||_\beta}{|x|^{\beta-1}},$$

$$\Big| \int_{\tilde{\Gamma}_2} f(x') \cdot \nabla' \frac{1}{|x - x'|} \, dx' \Big| \leq \frac{4\||f\||_\beta}{|x|^2} \cdot \int_{\tilde{\Gamma}_2} \frac{1}{|x'|^\beta} \, dx' \leq \frac{16\pi}{(\beta-3)\sigma^{\beta-3}} \cdot \frac{\||f\||_\beta}{|x|^2}. \tag{3.14}$$

From (3.8), (3.13) and (3.14) we can conclude the result for the case $\beta > 3$.

Theorem 3.1. *Assume $\beta > 0$ and the assumptions of Problem* E *to be satisfied.*

(a) If $0 < \beta \leq 1$ there exist solutions $v \in C^{1+\alpha}(\mathbb{R}^3, \mathbb{R}^3)$ of Problem E *satisfying*

$$|v(x)| = \left\{ \begin{array}{ll} O\big(|x|^{1-\beta}\big), & |x| \to \infty \quad\quad for\ \beta < 1, \\ O\big(\ln|x|\big), & |x| \to \infty \quad\quad for\ \beta = 1. \end{array} \right.$$

The solutions of the homogeneous problem are constant vectors.

(b) *If $\beta > 1$ there exists a unique solution $v \in C^{1+\alpha}(\mathbb{R}^3, \mathbb{R}^3)$ of Problem E satisfying*

$$|v(x)| = \begin{cases} O\big(\dfrac{1}{|x|^{\beta-1}}\big), & |x| \to \infty & for\ 1 < \beta < 3, \\[2mm] O\big(\dfrac{\ln|x|}{|x|^2}\big), & |x| \to \infty & for\ \beta = 3, \\[2mm] O\big(\dfrac{1}{|x|^2}\big), & |x| \to \infty & for\ \beta > 3. \end{cases}$$

Proof. The existence of the solutions follows from the Lemmas 3.1 and 3.2. The uniqueness can be proved by Green's formula or by means of Theorem 2.2. $\qquad\square$

These preparations lead us to the solvability result of Problem N.

Theorem 3.2. *Assume $\beta > 0$ and the assumptions of Problem N to be satisfied.*

(a) *If $0 < \beta \leq 1$ there exist solutions v of Problem N satisfying*

$$|v(x)| = \begin{cases} O\big(|x|^{1-\beta}\big), & |x| \to \infty & for\ \beta < 1, \\ O\big(\ln|x|\big), & |x| \to \infty & for\ \beta = 1. \end{cases}$$

The general solution of the corresponding homogeneous problem ($f = 0$, $w = 0$, $g = 0$, $\Gamma_j = 0$) is a three-dimensional real vector space. Two solutions v_1, v_2 of the same inhomogeneous problem are equal, if

$$v_1(x) - v_2(x) \xrightarrow[\text{unif}]{} 0, \quad |x| \to \infty.$$

(b) *If $1 < \beta \leq \infty$ there exists a unique solution v of Problem N satisfying*

$$|v(x)| = \begin{cases} O\big(\dfrac{1}{|x|^{\beta-1}}\big), & |x| \to \infty & for\ 1 < \beta < 3, \\[2mm] O\big(\dfrac{\ln|x|}{|x|^2}\big), & |x| \to \infty & for\ \beta = 3, \\[2mm] O\big(\dfrac{1}{|x|^2}\big), & |x| \to \infty & for\ \beta > 3. \end{cases}$$

(c) *If $f = 0$, $w = 0$, $\Gamma_1 = \cdots = \Gamma_{\tilde{n}} = 0$ and $\displaystyle\int_{\partial G} g\, d\Omega = 0$, then even*

$$|v(x)| = O\Big(\dfrac{1}{|x|^3}\Big), \quad |x| \to \infty.$$

Proof. (1) We choose $R_0 > 0$ such that $\overline{G} \subset K_{R_0}(0)$, and set $\mathcal{G} := K_{R_0}(0) \setminus \overline{G}$. Let $\bar{f} \in C^\alpha(\overline{K_{R_0}(0)}, \mathbb{R})$ be an extension of $f|_{\overline{\mathcal{G}}}$ to $\overline{K_{R_0}(0)}$ and $A \in$

$C^2(\overline{\mathcal{G}}, \mathbb{R}^3)$ a vector potential of w with extension $\bar{A} \in C^2(\overline{K_{R_0}(0)}, \mathbb{R}^3)$ to $\overline{K_{R_0}(0)}$. Then

$$\bar{w} \quad := \quad \begin{cases} w & \text{in } \hat{G}, \\ \operatorname{curl} \bar{A} & \text{in } \overline{G} \end{cases}$$

is a continuously differentiable extension of w to $\mathbb{R}^3$ satisfying $\operatorname{div} \bar{w} = 0$ in $\mathbb{R}^3$. Now let $\tilde{u}$ be a solution of Problem E with prescriptions $\bar{f}$ and $\bar{w}$ according to Theorem 3.1. Furthermore let $\varphi = \varphi_N \in C^2(\hat{G}, \mathbb{R}) \cap C^1(\overline{\hat{G}}, \mathbb{R})$ be the unique solution of the Neumann problem (2.1) according to Theorem 2.1 with the boundary value prescription $\langle \nabla \varphi, \nu \rangle = g - \langle \tilde{u}|_{\partial G}, \nu \rangle$. Thus, $u := \tilde{u} + \nabla \varphi$ satisfies $\operatorname{div} u = f$, $\operatorname{curl} u = w$ in $\hat{G}$, $\langle u, \nu \rangle = g$ on ∂G and the asymptotic conditions for $|x| \to \infty$.

Let $\Gamma'_1, \dots, \Gamma'_{\tilde{n}}$ be the generalized circulations of u; i.e., $\int_{\partial G} \langle \nu \times u, \mathfrak{z}_j \rangle \, d\Omega = \Gamma'_j$. We set

$$v(x) \quad := \quad u(x) - \sum_{j=1}^{\tilde{n}} (\Gamma_j - \Gamma'_j) \, \hat{\mathfrak{z}}_j(x)$$

for $x \in \hat{G}$. From equation (2.3) and $|\hat{\mathfrak{z}}_j(x)| = O(|x|^{-2})$, $|x| \to \infty$, we conclude that v is a solution of the Neumann problem (N).

(2) Now let u be a solution of the homogeneous Problem N. Since u is without circulation in $\hat{G}$, there exists a potential $\psi \in C^2(\hat{G}, \mathbb{R})$ with $u = \nabla \psi$ in $\hat{G}$. Because of $\operatorname{div} u = 0$ we have $\Delta \psi = 0$ in $\hat{G}$, and therefore also $\Delta \nabla \psi = 0$ in $\hat{G}$.

In the case $|u(x)| \to 0$, $|x| \to \infty$ we have $u = 0$ in $\hat{G}$ according to Lemma 2.4. If, however, we have $|\nabla \psi(x)| = O(|x|^{\beta_0})$, $|x| \to \infty$ for some $\beta_0 \in (0, 1)$ there exists, according to Lemma 2.1, $u_\infty \in \mathbb{R}^3$ such that

$$\nabla \psi(x) \quad \underset{\text{unif}}{\longrightarrow} \quad u_\infty, \quad |x| \to \infty.$$

Now Lemma 2.4 yields the rest.

The statement of (c) follows from Lemma 2.3. $\qquad \square$

Now we are going to investigate the integrability conditions of Problem D.

Remark 3.1. The condition

$$\operatorname{Div} \gamma^* \quad = \quad -\langle \nu, w \rangle \qquad \text{on } \partial G$$

is necessary for the solvability of Problem D; cf. equation (1.3).

Lemma 3.3. *For $\beta > 1$ the condition*

$$\forall \hat{\mathfrak{z}} \in \mathfrak{Z}_{\mathbb{R}}(\hat{G}): \quad \int_{\hat{G}} \langle w, \hat{\mathfrak{z}} \rangle \, dx \, + \, \int_{\partial G} \langle \gamma^*, \hat{\mathfrak{z}} \rangle \, d\Omega \quad = \quad 0$$

is necessary for the solvability of Problem D.

Proof. Without restriction assume $1 < \beta < 3$. Let $R > 0$ be such that $\overline{G} \subset K_R(0)$. Let $\hat{\mathfrak{z}} \in \mathfrak{Z}_{\mathbb{R}}(\hat{G})$, $\operatorname{curl} v = w$ in $\hat{G}$ and $\nu \times v = \gamma^*$ on ∂G. Then we have

$$\int_{\hat{G} \cap K_R(0)} \langle \operatorname{curl} v, \hat{\mathfrak{z}} \rangle \, dx \, = \, \int_{\hat{G} \cap K_R(0)} \operatorname{div} \left(v \times \hat{\mathfrak{z}} \right) \, dx \, + \, \int_{\hat{G} \cap K_R(0)} \langle v(x), \underbrace{\operatorname{curl} \hat{\mathfrak{z}}(x)}_{=0} \rangle \, dx$$

$$= \, - \int_{\partial G} \langle v \times \hat{\mathfrak{z}}, \nu \rangle \, d\Omega' \, + \int_{\partial K_R(0)} \langle v(\xi') \times \hat{\mathfrak{z}}(\xi'), \frac{\xi'}{R} \rangle \, d\Omega'.$$

According to the assumption there holds

$$|\langle w(x), \hat{\mathfrak{z}}(x) \rangle| = O\big(|x|^{-(2+\beta)}\big), \quad |v(x) \times \hat{\mathfrak{z}}(x)| = O\big(|x|^{-(1+\beta)}\big), \quad |x| \to \infty,$$

and thus

$$\lim_{R \to \infty} \int_{\hat{G} \cap K_R(0)} \langle w, \hat{\mathfrak{z}} \rangle \, dx = \int_{\hat{G}} \langle w, \hat{\mathfrak{z}} \rangle \, dx, \quad \lim_{R \to \infty} \int_{\partial K_R(0)} \langle (v \times \hat{\mathfrak{z}})(\xi'), \frac{\xi'}{R} \rangle \, d\Omega' = 0.$$

This implies the statement of the lemma. $\qquad \square$

Obviously the condition in Lemma 3.3 can not be extended to the case $\beta \in (0, 1)$. In what follows, we look for a suitable generalization which also holds in that case.

Lemma 3.4. *Suppose G and $\hat{G}$ as in Problem* D*; assume $\mathcal{G} \subset \mathbb{R}^3$ to be a bounded, simply connected domain with smooth boundary and $\overline{G} \subset \mathcal{G}$.*

Let $u \in C^1(\hat{G}, \mathbb{R}^3)$ be a vector field with

$$\forall x \in \hat{G} \cap \mathcal{G}: \quad \operatorname{curl} u(x) \quad = \quad 0.$$

Then

$$\text{(i)} \quad \operatorname{Div} (\nu \times u) \quad = \quad 0 \quad \text{on } \partial G,$$

$$\text{(ii)} \quad \forall \hat{\mathfrak{z}} \in \mathfrak{Z}_{\mathbb{R}}(\hat{G}): \quad \int_{\partial G} \langle \nu \times u, \hat{\mathfrak{z}} \rangle \, d\Omega \, = \, 0.$$

Proof. Equation (i) is clear. We determine $\Gamma_1, \ldots, \Gamma_{\tilde{n}}$ such that $\tilde{u} := u + \sum_{j=1}^{\tilde{n}} \Gamma_j \, \hat{\mathfrak{z}}_j$ is without circulation in $\hat{G} \cap \mathcal{G}$ (cf. (2.3), (2.7)). Let $\psi \in C^2(\overline{\hat{G} \cap \mathcal{G}}, \mathbb{R})$ be a potential of $\tilde{u}$. Assume $\mathcal{G}' \subset \mathbb{R}^3$ to be a bounded, simply connected domain with smooth boundary satisfying $\overline{G} \subset \mathcal{G}' \subset \overline{\mathcal{G}'} \subset \mathcal{G}$, and

let $\tilde{\psi} \in C_0^2(\overline{\hat{G}}, \mathbb{R})$ be an extension of $\psi|_{\hat{G} \cap \mathcal{G}'}$. Then for sufficiently large $R > 0$ and arbitrary $\hat{\mathfrak{z}} \in \mathfrak{Z}_{\mathbb{R}}(\hat{G})$, we have

$$\int_{\partial G} \langle \nu \times \tilde{u}, \hat{\mathfrak{z}} \rangle \, d\Omega = \int_{\partial G} \langle \nu \times \nabla \tilde{\psi}, \hat{\mathfrak{z}} \rangle \, d\Omega = \int_{\partial G} \langle \nabla \tilde{\psi} \times \hat{\mathfrak{z}}, \nu \rangle \, d\Omega$$

$$= \int_{\partial K_R(0)} \langle (\nabla \tilde{\psi} \times \hat{\mathfrak{z}})(\xi'), \frac{\xi'}{R} \rangle \, d\Omega' - \int_{K_R(0) \cap \hat{G}} \operatorname{div}(\nabla \tilde{\psi} \times \hat{\mathfrak{z}}) \, dx = 0.$$

Since $|\hat{\mathfrak{z}}(x)| = O(|x|^{-2})$, $|x| \to \infty$ for $\hat{\mathfrak{z}} \in \mathfrak{Z}_{\mathbb{R}}(\hat{G})$ we obtain

$$\forall \hat{\mathfrak{z}} \in \mathfrak{Z}_{\mathbb{R}}(\hat{G}): \quad \int_{\partial G} \langle \nu \times \hat{\mathfrak{z}}_j, \hat{\mathfrak{z}} \rangle \, d\Omega \quad = \quad 0, \qquad j = 1, \ldots, \tilde{n}, \qquad (3.15)$$

and we can conclude equation (ii). $\qquad \square$

Lemma 3.5. *Let the assumptions of Problem* D *be satisfied and* $\mathcal{G}$ *be a bounded, simply connected domain with smooth boundary and* $\overline{G} \subset \mathcal{G}$. *Furthermore, let* $w_0 \in C_0^1(\overline{\hat{G}}, \mathbb{R}^3)$ *be a vector field with zero flux and*

$$\forall \, x \in \mathcal{G} \cap \hat{G}: \qquad w(x) = w_0(x).$$

Then the condition

$$\forall \hat{\mathfrak{z}} \in \mathfrak{Z}_{\mathbb{R}}(\hat{G}): \quad \int_{\hat{G}} \langle w_0, \hat{\mathfrak{z}} \rangle \, dx + \int_{\partial G} \langle \gamma^*, \hat{\mathfrak{z}} \rangle \, d\Omega \quad = \quad 0$$

is necessary for the solvability of Problem D.

Proof. Let $v_0 \in C^2(\hat{G}, \mathbb{R}^3)$ with $\operatorname{curl} v_0 = w_0$. Then for arbitrary $\hat{\mathfrak{z}} \in \mathfrak{Z}_{\mathbb{R}}(\hat{G})$ Lemmas 3.4 and 3.3 yield

$$\int_{\partial G} \langle \gamma^*, \hat{\mathfrak{z}} \rangle \, d\Omega = \int_{\partial G} \langle \nu \times v, \hat{\mathfrak{z}} \rangle \, d\Omega = \int_{\partial G} \langle \nu \times v_0, \hat{\mathfrak{z}} \rangle \, d\Omega = - \int_{\hat{G}} \langle w_0, \hat{\mathfrak{z}} \rangle \, dx. \quad \square$$

Remarks. (1) For any w as in Problem D such a corresponding w_0 can be obtained by construction of a C_0^2 extension of a vector potential of $w|_{\hat{G} \cap \mathcal{G}}$.

(2) The condition $w_0 \in C_0^1(\overline{\hat{G}}, \mathbb{R}^3)$ can be replaced by

$$w_0 \in C^1(\overline{\hat{G}}, \mathbb{R}^3), \quad |w_0(x)| = O(|x|^{-\beta}), \; |x| \to \infty, \quad \beta > 1.$$

(3) According to Lemma 3.4 the value of $\int_{\hat{G}} \langle w_0, \hat{\mathfrak{z}} \rangle \, dx$ does not depend on the special choice of w_0.

Theorem 3.3. *Suppose the assumptions of Problem* D *to be satisfied. Let*

$$\operatorname{Div} \gamma^* = -\langle \nu, w \rangle \qquad on \; \partial G.$$

(a) *Let $0 < \beta \leq 1$. Assume w_0 as in Lemma* 3.5. *If*

$$\forall \hat{\mathfrak{z}} \in \mathfrak{Z}_{\mathbb{R}}(\hat{G}) : \quad \int_{\hat{G}} \langle w_0, \hat{\mathfrak{z}} \rangle \, dx + \int_{\partial G} \langle \gamma^*, \hat{\mathfrak{z}} \rangle \, d\Omega = 0,$$

then Problem D *has a solution satisfying the asymptotic conditions as in Theorem* 3.1. *The general solution of the corresponding homogeneous problem* (*i.e.,* $f = 0$, $w = 0$, $g = 0$, $E = 0$) *is a three-dimensional real vector space.*
  (b) *Let $\beta > 1$. If*

$$\forall \hat{\mathfrak{z}} \in \mathfrak{Z}_{\mathbb{R}}(\hat{G}) : \quad \int_{\hat{G}} \langle w, \hat{\mathfrak{z}} \rangle \, dx + \int_{\partial G} \langle \gamma^*, \hat{\mathfrak{z}} \rangle \, d\Omega = 0,$$

then Problem D *has a unique solution satisfying the asymptotic conditions as in Theorem* 3.1.

Proof. As in the proof of Theorem 3.2 we extend f and w by $\bar{f}$ and $\bar{w}$ to the entire space $\mathbb{R}^3$. Then let $\tilde{v}$ be any solution of Problem E with the corresponding asymptotic behaviour according to Theorem 3.1.
  Then we have $\mathrm{Div}(\nu \times \tilde{v}) = -\langle \nu, w \rangle$ on ∂G and

$$\forall \, x \in \mathcal{G} \cap \hat{G} : \quad \mathrm{curl}\, \tilde{v}(x) = \overline{w}(x) = w(x) = w_0(x). \qquad (3.16)$$

Now we set

$$\Gamma_j \quad := \quad \int_{\partial G} \langle \gamma^* - \nu \times \tilde{v}, \, \mathfrak{z}_j \rangle \, d\Omega, \quad j = 1, \dots, \tilde{n},$$

and for $\xi \in \partial G$

$$\gamma^{**}(\xi) := \gamma^*(\xi) - \nu(\xi) \times \tilde{v}(\xi) + \sum_{j=1}^{\tilde{n}} \Gamma_j \nu(\xi) \times \hat{\mathfrak{z}}_j(\xi).$$

Then, from (2.3), (3.16), (3.15) and Lemma 3.5 we can conclude

$$\mathrm{Div}\, \gamma^{**} = 0, \quad \int_{\partial G} \langle \gamma^{**}, \mathfrak{z}_j \rangle \, d\Omega = \int_{\partial G} \langle \gamma^{**}, \hat{\mathfrak{z}}_j \rangle \, d\Omega = 0, \quad j = 1, ..., \tilde{n}.$$

Let ψ be any solution of Problem (2.8) in Corollary 2.1 and φ the appropriate solution of Problem (2.9) in Lemma 2.6. Then

$$v \quad := \quad \tilde{v}\big|_{\hat{G}} - \sum_{j=1}^{\tilde{n}} \Gamma_j \, \hat{\mathfrak{z}}_j + \nabla \psi + \nabla \varphi$$

is a solution of Problem D with the corresponding asymptotic behaviour.
  Now let $u \in C^1(\overline{\hat{G}}, \mathbb{R}^3)$ be any Dirichlet field; i.e.,

$$\mathrm{div}\, u = 0, \ \mathrm{curl}\, u = 0 \ \text{in}\ \hat{G}, \ \nu \times u = 0 \ \text{on}\ \partial G.$$

Since u is without circulation in $\hat{G}$ there exists a potential $\psi \in C^2(\overline{\hat{G}}, \mathbb{R})$ satisfying $\Delta\psi = 0$ and $u = \nabla\psi$ in $\hat{G}$, and $\psi|_{\partial G}$ is constant. In the case (b) Lemma 2.6 yields the uniqueness.

In the case (a) the solutions of the homogeneous problem are those of Problem (2.10) in Lemma 2.7. $\qquad\qquad\square$

4. Hölder estimates

The aim of this paragraph is to transfer the Hölder estimates for the div–curl problem for inhomogeneously harmonic vector fields in interior domains which are already known (cf. [3]) to the exterior case. To realize this, an additional term characterizing the asymptotic behaviour of the data must be introduced. Of course, the uniqueness of the solution is necessary for the existence of such estimates. Here, we treat only the case $1 < \beta < 3$.

Lemma 4.1. *Let $\beta > 1$, $f \in C_{\mathrm{unif}}^{\alpha}(\mathbb{R}^3, \mathbb{R})$, $\|\|f\|\|_{\beta} < \infty$,*

$$\Phi : \mathbb{R}^3 \to \mathbb{R}^3, \quad \Phi(x) := \frac{1}{4\pi} \int_{\mathbb{R}^3} \nabla\frac{1}{r}(x, x') f(x') \, dx'.$$

Then $\Phi \in C_{\mathrm{unif}}^{1+\alpha}(\mathbb{R}^3, \mathbb{R})$ and there exists some $c(\alpha, \beta) > 0$ independent of f satisfying

$$\|\Phi\|_{C^{1+\alpha}(\mathbb{R}^3)} \leq c(\alpha, \beta) \cdot \big(\|f\|_{C^{\alpha}(\mathbb{R}^3)} + \|\|f\|\|_{\beta}\big).$$

This lemma can be proved by methods which are similar to those of [5, Lemma 4.4]. We may omit the proof.

Lemma 4.2. *Let $G \subset \mathbb{R}^3$ be a bounded domain with smooth boundary, $\hat{G} := \mathbb{R}^3 \setminus \overline{G}$, $f \in C^{\alpha}(\overline{G}, \mathbb{R})$. Let furthermore*

$$\varphi : \mathbb{R}^3 \to \mathbb{R}, \quad \varphi(x) := \frac{1}{4\pi} \int_G \frac{1}{|x - x'|} \cdot f(x') \, dx'.$$

Then the following estimate holds:

$$\|\varphi\|_{C^{2+\alpha}(\overline{\hat{G}})} \leq c(\alpha, G) \cdot \|f\|_{C^{\alpha}(\overline{G})}.$$

This is a classical potential-theoretic result and can be proved by the aid of the Hölder–Korn–Lichtenstein–Giraud inequality, the jump relations and estimates up to the boundary for second-order elliptic differential operators (cf. [1, part II, chapter 5]).

Lemma 4.3. *Let $G \subset \mathbb{R}^3$ be a bounded domain with smooth boundary, $\hat{G} := \mathbb{R}^3 \setminus \overline{G}$, $\beta > 1$, $f \in C^\alpha_{\mathrm{unif}}(\overline{\hat{G}}, \mathbb{R})$ with $\| |f| \|_\beta < \infty$,*

$$\Phi : \hat{G} \to \mathbb{R}^3, \quad \Phi(x) := \frac{1}{4\pi} \int_{\hat{G}} \nabla \frac{1}{r}(x, x') f(x') \, dx'.$$

Then there holds the inequality

$$\|\Phi\|_{C^{1+\alpha}(\overline{\hat{G}})} \quad \leq \quad c(\alpha, \beta, G) \cdot \big(\|f\|_{C^\alpha(\overline{\hat{G}})} + \| |f| \|_\beta \big).$$

This is a consequence of the two lemmata above and the extendability result of [5, Lemma 6.37].

Lemma 4.4. *Let the assumptions of Problem* N *be satisfied, $\beta > 1$, $g \in C^{1+\alpha}(\partial G, \mathbb{R})$. Let v be the unique solution of Problem* N *according to Theorem* 3.2 (b)*. Then $\gamma^* := \nu \times v$ satisfies the following integral equation:*

$$
\begin{aligned}
\gamma^*(\xi) + \Re\gamma^*(\xi) = -\frac{1}{2\pi} \, \nu(\xi) \, \times \, \Big(& \int_{\hat{G}} \nabla \frac{1}{r}(\xi, x') \, f(x') \, dx' \\
& - \int_{\hat{G}} \nabla \frac{1}{r}(\xi, x') \times w(x') \, dx' + \int_{\partial G} \nabla \frac{1}{r}(\xi, \xi') \, g(\xi') \, d\Omega' \Big),
\end{aligned}
\tag{4.1}
$$

where $\xi \in \partial G$, $r = |\xi - x'|$, $r = |\xi - \xi'|$, respectively, and

$$\Re\gamma^*(\xi) := -\frac{1}{2\pi} \int_{\partial G} \nu(\xi) \, \times \, \big(\nabla \frac{1}{r}(\xi, \xi') \, \times \, \gamma^*(\xi') \big) \, d\Omega'.$$

At this occasion we remark that the boundary integral in equation (4.1) only exists in the sense of Cauchy's principal value, whereas the integral in the definition of $\Re\gamma^*$ exists in the L^1 sense (cf. [15, Satz 4.4] and [13, Definition I.3.2]).

Proof. Suppose $R > 0$ such that $\overline{G} \subset K_R(0)$ and set $\hat{G}^{(R)} := K_R(0) \setminus \overline{G}$. In [13, Satz I.3.6] the corresponding integral equation has been proved for bounded domains. We apply this result to $\hat{G}^{(R)}$ and let R tend to ∞.   □

Theorem 4.1. *Let $G \subset \mathbb{R}^3$ be a bounded domain with smooth boundary, $1 < \beta < 3$, $v \in C^1(\overline{\hat{G}}, \mathbb{R}^3)$ with*

$$|v(x)| = O\big(|x|^{-(\beta-1)}\big), \quad |\mathrm{div}\, v(x)|, |\mathrm{curl}\, v(x)| = O\big(|x|^{-\beta}\big), \quad |x| \to \infty.$$

Then for $x \in \hat{G}$ we have

$$v(x) = -\tfrac{1}{4\pi} \int_{\hat{G}} \nabla \tfrac{1}{r}(x, x') \operatorname{div} v(x')\, dx' + \tfrac{1}{4\pi} \int_{\hat{G}} \nabla \tfrac{1}{r}(x, x') \times \operatorname{curl} v(x')\, dx'$$
$$- \tfrac{1}{4\pi} \int_{\partial G} \nabla \tfrac{1}{r}(x, \xi') \cdot \big\langle \nu(\xi'), v(\xi') \big\rangle\, d\Omega'$$
$$+ \tfrac{1}{4\pi} \int_{\partial G} \nabla \tfrac{1}{r}(x, \xi') \times \big(\nu(\xi') \times v(\xi') \big)\, d\Omega'.$$

Proof. The corresponding statement for $\hat{G}^{(R)}$ follows immediately from the fundamental theorem. The limit $R \to \infty$ then yields the formula above. $\square$

Theorem 4.2. *Let the assumptions of Problem* N *be satisfied,* $1 < \beta < 3$, $g \in C^{1+\alpha}(\partial G, \mathbb{R})$. *Let* v *be the unique solution of Problem* N *according to Theorem* 3.2 (b). *Then there exists some constant* $c = c(\alpha, \beta, G) > 0$, *independent of* v *and the given data, such that*

$$\|v\|_{C^{1+\alpha}(\overline{\hat{G}})} + \|\|v\|\|_{\beta-1} \le c \cdot \big(\|f\|_{C^{\alpha}(\overline{\hat{G}})} + \|\|f\|\|_{\beta} + \|w\|_{C^{\alpha}(\overline{\hat{G}})} + \|\|w\|\|_{\beta}$$
$$+ \|g\|_{C^{1+\alpha}(\partial G)} + \sum_{j=1}^{\tilde{n}} |\Gamma_j| \big).$$

Proof. (a) Let $\gamma^* := \nu \times v|_{\partial G}$. Obviously $\gamma^* \in C^1(\partial G, \mathbb{R}^3)$. From the boundedness of $\|\gamma^*\|_{C^{1+\alpha}(\partial G)}$ we later conclude $\gamma^* \in C^{1+\alpha}(\partial G, \mathbb{R}^3)$. The operator $\mathfrak{R}$ is a compact operator in $\mathcal{T}_{C^0}(\partial G)$, where

$$\mathcal{T}_{C^0}(\partial G) := \big\{ \gamma^* \in C^0(\partial G, \mathbb{R}^3) : \langle \nu, \gamma^* \rangle = 0 \big\}$$

is the space of continuous tangential vector fields on ∂G; cf. [13, Satz I.3.7]. There it has been shown that $\mathcal{N}(I + \mathfrak{R}) = \{\nu \times v : v \in \mathfrak{Z}_{\mathbb{R}}(\hat{G})\}$ and the Riesz number of $\mathfrak{R}$ is 1. Here, I denotes the identity, $\mathcal{N}(I + \mathfrak{R})$ the zero space, $\mathcal{R}(I + \mathfrak{R})$ the range of the operator $I + \mathfrak{R}$ in $\mathcal{T}_{C^0}(\partial G)$. If the regularity of the boundary is sufficient, we have $\mathcal{N}(I + \mathfrak{R}) \subset C^{1+\alpha}(\partial G, \mathbb{R}^3)$. Riesz's decomposition theorem then yields

$$\mathcal{T}_{C^0}(\partial G) = \mathcal{N}(I + \mathfrak{R}) \oplus \mathcal{R}(I + \mathfrak{R})$$

with the closed spaces $\mathcal{N}(I + \mathfrak{R})$ and $\mathcal{R}(I + \mathfrak{R})$, which are invariant under the operator $I + \mathfrak{R}$. Thus, for any tangential vector field $\gamma^* \in C^{1+\alpha}(\partial G, \mathbb{R}^3)$ there exists a unique decomposition

$$\gamma^* = \gamma_1^* + \gamma_2^*, \quad \gamma_1^* \in \mathcal{R}(I + \mathfrak{R}), \quad \gamma_2^* \in \mathcal{N}(I + \mathfrak{R}).$$

One can prove that (cf. [6, Satz 6.2])

$$\mathfrak{R}\big(\mathcal{T}_{C^0}(\partial G) \cap C^{1+\alpha}(\partial G, \mathbb{R}^3)\big) \subset \mathcal{T}_{C^0}(\partial G) \cap C^{2+\alpha}(\partial G, \mathbb{R}^3)$$

and

$$\|\mathfrak{R}\gamma^*\|_{C^{2+\alpha}(\partial G)} \leq c(\alpha, G) \|\gamma^*\|_{C^{1+\alpha}(\partial G)}$$

for any $\gamma^* \in \mathcal{T}_{C^0}(\partial G) \cap C^{1+\alpha}(\partial G, \mathbb{R}^3)$. The imbedding

$$C^{2+\alpha}(\partial G, \mathbb{R}^3) \hookrightarrow C^{1+\alpha}(\partial G, \mathbb{R}^3)$$

being compact, the operator $\mathfrak{R}$ is compact with respect to the norm $\|\cdot\|_{C^{1+\alpha}(\partial G)}$. Therefore, there exists some constant $c = c(\alpha, G) > 0$ such that

$$\|(I + \mathfrak{R})\gamma_1^*\|_{C^{1+\alpha}(\partial G)} \geq c \|\gamma_1^*\|_{C^{1+\alpha}(\partial G)}$$

for $\gamma_1^* \in \mathcal{R}(I + \mathfrak{R}) \cap C^{1+\alpha}(\partial G, \mathbb{R}^3)$. For the generalized circulations of γ_1^*

$$\tilde{\Gamma}_j := \int_{\partial G} \langle \gamma_1^*, \mathfrak{z}_j \rangle d\Omega, \quad j = 1, \ldots, \tilde{n},$$

there holds an estimate

$$|\tilde{\Gamma}_j| \leq c(G) \|\gamma_1^*\|_{C^0(\partial G)} \leq c(G) \|\gamma_1^*\|_{C^{1+\alpha}(\partial G)}.$$

Hence, we obtain

$$\int_{\partial G} \langle \gamma_2^*, \mathfrak{z}_j \rangle d\Omega = \Gamma_j - \tilde{\Gamma}_j; \quad \text{thus} \quad \gamma_2^* = \nu \times \Big(\sum_{j=1}^{\tilde{n}} (\tilde{\Gamma}_j - \Gamma_j) \hat{\mathfrak{z}}_j \Big).$$

As a consequence of this we have

$$\|\gamma_2^*\|_{C^{1+\alpha}(\partial G)} \leq \max_{j=1,\ldots,\tilde{n}} \|\nu \times \hat{\mathfrak{z}}_j\|_{C^{1+\alpha}(\partial G)} \cdot \sum_{j=1}^{\tilde{n}} |\Gamma_j - \tilde{\Gamma}_j|$$

$$\leq c(\alpha, G) \cdot \Big(\sum_{j=1}^{\tilde{n}} |\Gamma_j| + \|\gamma_1^*\|_{C^{1+\alpha}(\partial G)} \Big),$$

and therefore

$$\|\gamma^*\|_{C^{1+\alpha}(\partial G)} \leq c(\alpha, G) \cdot \Big(\|(I + \mathfrak{R})\gamma^*\|_{C^{1+\alpha}(\partial G)} + \sum_{j=1}^{\tilde{n}} |\Gamma_j| \Big).$$

(b) Applying the Schauder estimates for boundary-layer potentials (cf. [16]) we obtain

$$\Big\| \int_{\partial G} \nabla \frac{1}{r}(\cdot, \xi') g(\xi') \, d\Omega' \Big\|_{C^{1+\alpha}(\overline{\hat{G}})} \leq c(\alpha, G) \|g\|_{C^{1+\alpha}(\partial G)},$$

$$\Big\| \int_{\partial G} \nabla \frac{1}{r}(\cdot, \xi') \times \gamma^*(\xi') \, d\Omega' \Big\|_{C^{1+\alpha}(\overline{\hat{G}})} \leq c(\alpha, G) \|\gamma^*\|_{C^{1+\alpha}(\partial G)}.$$

Because of the quadratic asymptotic decay of the integrals there also hold estimates of the form

$$\||\int_{\partial G} \nabla \frac{1}{r}(\cdot, \xi') g(\xi') \, d\Omega' \||_{\beta-1} \quad \leq \quad c(\beta, G) \|g\|_{C^{1+\alpha}(\partial G)},$$

$$\||\int_{\partial G} \nabla \frac{1}{r}(\cdot, \xi') \times \gamma^*(\xi') \, d\Omega' \||_{\beta-1} \quad \leq \quad c(\beta, G) \|\gamma^*\|_{C^{1+\alpha}(\partial G)}.$$

Lemma 4.3 yields

$$\| \int_{\hat{G}} \nabla \frac{1}{r}(\cdot, x') f(x') \, dx' \|_{C^{1+\alpha}(\overline{\hat{G}})} \leq c(\alpha, \beta, G) \cdot \big(\|f\|_{C^{\alpha}(\overline{\hat{G}})} + \||f\||_{\beta} \big),$$

$$\| \int_{\hat{G}} \nabla \frac{1}{r}(\cdot, x') \times w(x') \, dx' \|_{C^{1+\alpha}(\overline{\hat{G}})} \leq c(\alpha, \beta, G) \cdot \big(\|w\|_{C^{\alpha}(\overline{\hat{G}})} + \||w\||_{\beta} \big).$$

The estimates in the proof of Lemma 3.2 yield

$$\||\int_{\hat{G}} \nabla \frac{1}{r}(\cdot, x') f(x') \, dx' \||_{\beta-1} \quad \leq \quad c(\beta) \cdot \||f\||_{\beta},$$

$$\||\int_{\hat{G}} \nabla \frac{1}{r}(\cdot, x') \times w(x') \, dx' \||_{\beta-1} \quad \leq \quad c(\beta) \cdot \||w\||_{\beta}.$$

(c) From Lemma 4.4 and part (a) of the proof we arrive at

$$\|\gamma^*\|_{C^{1+\alpha}(\partial G)} \leq c(\alpha, \beta, G) \cdot \big(\|f\|_{C^{\alpha}(\overline{\hat{G}})} + \||f\||_{\beta} + \|w\|_{C^{\alpha}(\overline{\hat{G}})} + \||w\||_{\beta}$$
$$+ \|g\|_{C^{1+\alpha}(\partial G)} + \sum_{j=1}^{\tilde{n}} |\Gamma_j| \big).$$

Finally, using Theorem 4.1 and the estimates in part (b) of the proof, we conclude

$$\|v\|_{C^{1+\alpha}(\overline{\hat{G}})} + \||v\||_{\beta-1} \leq c(\alpha, \beta, G) \cdot \big(\|f\|_{C^{\alpha}(\overline{\hat{G}})} + \||f\||_{\beta} + \|w\|_{C^{\alpha}(\overline{\hat{G}})} + \||w\||_{\beta}$$
$$+ \|g\|_{C^{1+\alpha}(\partial G)} + \sum_{j=1}^{\tilde{n}} |\Gamma_j| \big). \quad \square$$

Remark 4.1. Concerning Problem D we obtain instead of Lemma 4.4 the integral equation

$$g(\xi) - \mathfrak{K}g(\xi) = \frac{1}{2\pi} \left\langle \nu(\xi), \int_{\hat{G}} \nabla \frac{1}{r} \left(\xi, x'\right) f(x') \, dx' \right.$$
$$\left. - \int_{\hat{G}} \nabla \frac{1}{r} \left(\xi, x'\right) \times w(x') \, dx' - \operatorname{curl} \int_{\partial G} \frac{1}{r} \left(\xi, \xi'\right) \gamma^*(\xi') \, d\Omega' \right\rangle,$$

for $\xi \in \partial G$, where $g(\xi) = \langle \nu(\xi), v(\xi) \rangle$ is the normal component of the vector field v in $\xi \in \partial G$ and

$$\mathfrak{K}g(\xi) := -\frac{1}{2\pi} \int_{\partial G} \left\langle \nu(\xi), \nabla \frac{1}{r} \left(\xi, \xi'\right) g(\xi') \right\rangle d\Omega'.$$

The estimate corresponding to Theorem 4.2 is

$$\|v\|_{C^{1+\alpha}(\overline{\hat{G}})} + \||v\||_{\beta-1} \leq c(\alpha, \beta, G) \cdot \big(\|f\|_{C^\alpha(\overline{\hat{G}})} + \||f\||_\beta +$$
$$+ \|w\|_{C^\alpha(\overline{\hat{G}})} + \||w\||_\beta + \|\gamma^*\|_{C^{1+\alpha}(\partial G)} + |E| \big).$$

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