# Stability in Hydrodynamics

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Let us consider an incompressible viscous fluid which fills out a spatial region $\Omega \subset \mathbb{R}^3$. We think of a bounded connected region in $\mathbb{R}^3$ with diameter $d$ or an infinite layer of depth $d$. We consider a steady flow with velocity field $\mathbf{u}_s$ and pressure $\pi_s$ under the influence of an external force $\mathbf{F}$. Then $\mathbf{u}_s$, $\pi_s$ satisfy

$$\begin{aligned} -\nu\Delta\mathbf{u}_s + \mathbf{u}_s \cdot \nabla\mathbf{u}_s + \nabla\pi_s &= \mathbf{F}, \\ \nabla \cdot \mathbf{u}_s &= 0. \end{aligned}$$

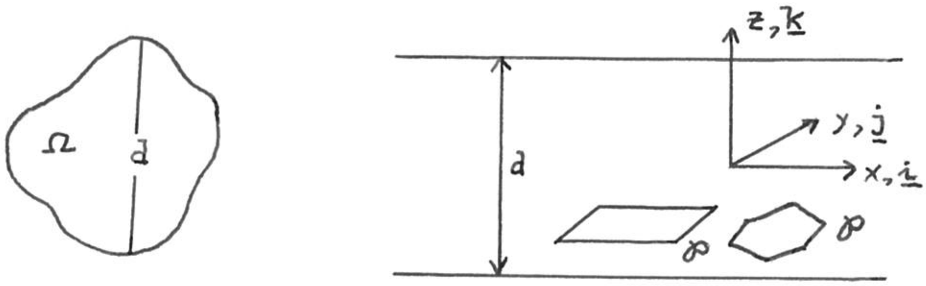


Figure 1

In the case of an infinite layer we additionally assume that $\mathbf{u}_s$, $\nabla\pi_s$, $\mathbf{F}$ are periodic with respect to the plane variables $x, y$. The plane periodicity cell $\mathcal{P}$ may be, for instance, a rectangle or a hexagon. $\nu$ is the kinematic viscosity. The density $\rho$ is assumed to be 1. On the values of $\mathbf{u}_s$ on $\partial\Omega$ or on the top $z = d/2$ and the bottom $z = -d/2$ of the layer nothing particular is assumed. We want to study the dynamic stability of our steady flow. A time-dependent disturbance under rigid boundary conditions satisfies

$$
(1) \qquad \begin{array}{l}
\partial_t \mathbf{u} - \nu \Delta \mathbf{u} + \mathbf{u}_s \cdot \nabla \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}_s + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla \pi = 0, \\
\nabla \cdot \mathbf{u} = 0, \\
\mathbf{u}|\partial\Omega = 0, \;\; \mathbf{u} = 0 \text{ at } z = \pm\frac{d}{2} \text{ resp.} \\
\mathbf{u}(0, \mathbf{x}) = \mathbf{u}_0(\mathbf{x}) \text{ with } \mathbf{u}_0 \text{ as initial value.}
\end{array}
$$

$(\mathbf{u}_s, \pi_s)$ is said to be conditionally stable with respect to $(\mathbf{u}, \pi)$ if for "small" initial values $\|\mathbf{u}_0\|_{L^2} \leq \varepsilon$ the kinetic energy $\|\mathbf{u}(t)\|_{L^2}$ stays small for $t \geq 0$ or at least becomes small for large $t$. $(\mathbf{u}_s, \pi_s)$ is said to be conditionally asymptotically stable if for $\|\mathbf{u}_0\|_{L^2} \leq \varepsilon$ the kinetic energy $\|\mathbf{u}(t)\|_{L^2}$ tends to $0$ as $t \to +\infty$. This kind of stability is therefore a stability in the sense of Ljapunov. The $L^2$-norm has to be taken either over $\Omega$ or over the layer $\Omega = \mathcal{P} \times (-d/2, d/2)$. $\varepsilon$ is called the size of the stability ball. $(\mathbf{u}_s, \pi_s)$ is said to be unconditionally asymptotically stable with respect to $(\mathbf{u}, \pi)$ if for any $\mathbf{u}_0$ the quantity $\|\mathbf{u}(t)\|_{L^2}$ tends to 0 as $t \to \infty$.

Let us mention that in the case of an infinite layer $(\mathbf{u}, \pi)$ is also periodic with respect to $x, y$ and that the periodicity of $\mathbf{u}$ is compatible with that of $\mathbf{u}_s$.

The celebrated questions of global existence and regularity are of less interest here, since one can always work with a weak solution satisfying Leray's structure theorem. If the kinetic energy is known to decay then all higher order derivatives also decay, no matter how large the initial values are, cf. [4].

To study the stability of a steady flow there are two well-known methods at our disposal. The first one is the energy-method, first applied by **Reynolds** [8] and **Orr** [7]. Its modernized version is due to **Serrin** [9]. We take the scalar product of (1) with $\mathbf{u}$. Since $\nabla \cdot \mathbf{u} = \nabla \cdot \mathbf{u}_s = 0$ and $\mathbf{u}|\partial\Omega = 0$, $\mathbf{u} = 0$ at $z = \pm d/2$ resp., we obtain

$$
\frac{1}{2}\partial_t \|\mathbf{u}\|^2 + \nu \|\nabla \mathbf{u}\|^2 \left(1 - \frac{(-\mathbf{u} \cdot \nabla \mathbf{u}_s, \mathbf{u})}{\nu \|\nabla \mathbf{u}\|^2}\right) = 0
$$

where $\|.\| = \|.\|_{L^2}$. Observe that $\mathbf{u}$ and $\mathbf{u}_s$ are real valued. If

$$
(2) \qquad \max_{\substack{\mathbf{u}, \nabla\cdot\mathbf{u}=0 \\ \mathbf{u}|\partial\Omega=0 \text{ or} \\ \mathbf{u}=0 \text{ at } z=\pm\frac{d}{2}}} \frac{(-\mathbf{u} \cdot \nabla \mathbf{u}_s, \mathbf{u})}{\nu \|\nabla \mathbf{u}\|^2} = \lambda^+ < 1,
$$

then

$$\frac{1}{2}\partial_t\|\mathbf{u}\|^2 + \nu(1-\lambda^+)\|\nabla\mathbf{u}\|^2 \leq 0$$

and $\|\mathbf{u}(t)\|^2$ decays monotonically and exponentially to 0 if $t \to +\infty$. In the marginal case $\lambda^+ = 1$ the kinetic energy is monotonically non-increasing. Observe that for $\lambda^+ < 1$ we thus have unconditional asymptotic stability. The physical interpretation of (2) is that the energy the disturbance $\mathbf{u}$ can draw from the steady flow is compared with the energy dissipation. Dynamical constraints are not considered. In (2) the maximum is in fact assumed since this variational problem can be put into a form which is accessible to Courant's method. The Euler-Lagrange system belonging to (2) is given by

$$(3) \qquad -\nu\Delta u_i + \frac{\mu}{2}\sum_{k=1}^{3}\left(u_k\frac{\partial u_{si}}{\partial x_k} + u_k\frac{\partial u_{sk}}{\partial x_i}\right) + \frac{\partial\pi}{\partial x_i} = 0,\ \nabla\cdot\mathbf{u} = 0$$

with eigenvalue parameter $\mu$ and with $\mathbf{u} = (u_1, u_2, u_3)^T$, $\mathbf{u}_s = (u_{s1}, u_{s2}, u_{s3})^T$. With $\mu^+ = \frac{1}{\lambda^+}$ and $\lambda^+$ taken from above it is readily seen that $\mu^+$ is the smallest positive eigenvalue of the Euler-Lagrange system provided $\lambda^+$ is positive. Setting $\nabla\mathbf{v} = (\partial_{x_k}v_i)_{\substack{1\leq i\leq 3\\ 1\leq k\leq 3}}$ for any vector-field $\mathbf{v}$ we can give to (3) the shorter form

$$(4) \qquad -\nu\Delta\mathbf{u} + \frac{\mu}{2}(\nabla\mathbf{u}_s + \nabla\mathbf{u}_s^T)\mathbf{u} + \nabla\pi = 0,\ \nabla\cdot\mathbf{u} = 0.$$

The second method to study stability is the method of linearized stability. We consider the spectrum of the problem

$$(5) \qquad \begin{cases} \sigma\mathbf{u} = -\nu\Delta\mathbf{u} + \mathbf{u}_s\cdot\nabla\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u}_s + \nabla\pi, \\ \nabla\cdot\mathbf{u} = 0,\ \mathbf{u}|\partial\Omega = 0 \text{ or } \mathbf{u} = 0 \text{ at } z = \pm d/2 \end{cases}$$

with $\sigma$ as eigenvalue-parameter. $\mathbf{u}$ may be complex here. This spectrum is discrete and the eigenvalues $\sigma$ can be ordered according to the size of their real parts, this is

$$\mathcal{R}e\,\sigma_1 \leq \mathcal{R}e\,\sigma_2 \leq \ldots.$$

Thus the minimum of the real parts is assumed and we set

$$(6) \qquad \xi_0 = \min\{\mathcal{R}e\,\sigma|\sigma \text{ eigenvalue in (5)}\}.$$

First we assume that $\xi_0 > 0$. Then the Green's operator belonging to (1) decays exponentially with time. By this we mean the following situation: Let $P$ be the orthogonal projection on the divergence-free part of $(L^2(\Omega))^3$. In particular $P\mathbf{u} = \mathbf{u}$ in (1) or (5). Since $P\nabla\pi = 0$ the pressure is eliminated in (1) if we apply $P$. We end up with the evolution equation

$$\begin{aligned} &\tfrac{d}{dt}\mathbf{u} + A\mathbf{u} + M(\mathbf{u}) = 0, \\ (7) \qquad &A\mathbf{u} = -\nu P\Delta\mathbf{u} + P(\mathbf{u}_s\nabla\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u}_s), \\ &M(\mathbf{u}) = P(\mathbf{u}\cdot\nabla\mathbf{u}). \end{aligned}$$

The operator $A$ turns out to be the generator of an analytic semigroup $e^{-tA}$ in $P(L^2(\Omega))^3$, $t \geq 0$. This is nothing else but the Green's operator in (1) and we have

$$\|e^{-tA}\| \leq De^{-\eta t},\ t \geq 0,$$

with positive constants $D, \eta$. In particular $\eta$ depends on $\xi_0$. Now (7) can be put into the integral form

$$(8) \qquad \mathbf{u}(t) = e^{-tA}\mathbf{u}_0 - \int_0^t e^{-(t-\sigma)A} M(\mathbf{u}(\sigma))\, d\sigma.$$

Since $M$ is quadratic in $\mathbf{u}$ it is easily seen from (8), that (7) has a unique, global (in time) strong solution provided $\mathbf{u}_0$ is small. This solution stays small for all $t$ and even exponentially decays to 0 as $t \to \infty$. Thus we have conditional asymptotic stability if $\xi_0 > 0$. Due to their smallness these perturbations are called infinitesimal. As we will see later it is just the smallness of the perturbations which causes the problems in hydrodynamic stability.

Now we turn to the case

$$\xi_0 < 0.$$

Then the steady flow is nonlinearly unstable with respect to kinetic energy. More precisely it was shown in [6] that there is an $\varepsilon_0 > 0$ with the following property: For any $\varepsilon > 0$ there is an initial value $\mathbf{u}_0$ with $\|\mathbf{u}_0\| < \varepsilon$ such that the kinetic energy of the strong solution of (7) with initial value $\mathbf{u}_0$ leaves the $\varepsilon_0$-ball during its life-time.

The marginal case for linearized stability is therefore

$$\xi_0 = 0.$$

In this case bifurcation may occur. By this we mean a branch of steady solutions of (7). Thus visible effects in experiments can be produced as the Taylor-vortices in the Taylor-Couette problem or patterns in the Bénard-problem. At the first glance one may think that linearized stability is the appropriate access to stability since it produces a necessary and sufficient condition for stability. This method however fails to match the experiments for some of the most studied simple steady flows. To give a deeper insight into the problem we introduce a control-parameter, namely the Reynolds-number

$$Re = \frac{d}{\nu} \max_{\overline{\Omega}} |\mathbf{u}_s|. \tag{9}$$

If $d, \nu, |\mathbf{u}_s|$ are endowed with dimensions, $Re$ becomes nondimensional. After appropriate scaling, (1) takes the non-dimensional form

$$\begin{aligned} \partial_t \mathbf{u} - \Delta \mathbf{u} + Re\, \mathbf{U}_s \cdot \nabla \mathbf{u} + Re\, \mathbf{u} \cdot \nabla \mathbf{U}_s + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla \pi = 0, \\ \nabla \cdot \mathbf{u} = 0 \end{aligned}$$

with the same boundary-values as in (1) and some initial value $\mathbf{u}_0$. Since $\mathbf{u}, \pi$ and $\mathbf{u}_s$ are also scaled we should in principle change the notation but did so for simplicity only for the steady flow. $Re$ is treated as a parameter we can change at will. Now we assume that there is a one-to-one correspondence between $\mu^+ = 1$ in (4) and a certain value $Re_E$ of the Reynolds-number, as well as between $\xi_0 = 0$ in (6) and another value $Re_c$ of the Reynolds-number. Of course this has to be proved in any particular case. $Re_E$ is called the energetic Reynolds-number, $Re_c$ the critical one. We also assume that for $\mu \geq 1$ in (4) the Reynolds-number belonging to $\mu$ is $\leq Re_E$, whereas for $\xi_0 < 0$ in (6) the Reynolds-number $Re$ becomes $> Re_c$. Thus we tacitly assume that all cases of interest in (4,5) can be characterized by appropriate values of $Re$. There may be also suitable choices of $Re$ different from (9) but we only want to fix the ideas here. In our situation it can be shown now that $Re_E \leq Re_c$. In general there is a large gap between $Re_E$ and $Re_c$, for instance for the simplest steady flows as plane Couette-flow $Re\, \mathbf{U}_s = Re(-z, 0, 0)^T$ or plane Poisenille-flow $Re\, \mathbf{U}_s = Re(\frac{1}{4} - z^2, 0, 0)^T$. In the case of plane Couette-flow which is driven by moving the side walls relative to each other it turns out that $Re_E$ is finite whereas $Re_c = +\infty$. Thus linearized theory predicts stability for all Reynolds-numbers. Experiments however show a subcritical transition to turbulence of which instability is only a precursor, already for finite Reynolds-numbers. The situation is similar for plane Poisenille-flow although in this case $Re_c < +\infty$. As reason it is generally assumed that for $Re$ sufficiently large the stability ball in (8) becomes so small that from the point of view of physics only the disturbance $\mathbf{u} \equiv 0$ is admitted.

The gap between $Re_E$ and $Re_c$ is simply called the stability-problem of hydrodynamics. The rare case that $\mu^+ = 1$ and $\xi_0 = 0$ occur at the same time is characterized as follows (cf. [14]):

**Theorem 1** *$\mu^+ = 1$ and $\xi_0 = 0$ in (4), (5) respectively occur at the same time if and only if the eigenspaces to eigenvalues $\sigma$ in (5) with $\mathcal{R}e\,\sigma = 0$ are all contained in the eigenspace to $\mu^+ = 1$ in (4). Otherwise $\xi_0 > 0$ if $\mu^+ = 1$.*

As for examples we refer to [1, 14]. The preceding theorem thus provides a necessary and sufficient condition that

$$Re_E = Re_c.$$

In this particular case the stability problem of hydrodynamics is solved completely since by passing the value $Re_E$ there is a transition from unconditional stability to instability. In some of the examples where $Re_E = Re_c$ holds, it is also possible to single out the most energetic perturbation. This is the perturbation which leaves the range of monotonic energy stability at earliest possibility and becomes unstable afterwards, for instance at $\min_{\alpha,\beta} Re_E$ in an infinite layer where the minimum has to be taken over all wave-numbers $\alpha, \beta$ in $x, y$-direction respectively. Cf. [1, 14] and in particular [2]. These perturbations can use the energy of the steady flow or the initial velocity in an optimal way. In general one can say that in linearized stability the cases $\xi_0 = 0$ and $\xi_0 < 0$ are of paticular importance.

There are however experiments, known since long, which show a perfect coincidence with the stability bounds given by linearized theory. This is so even if the gap between $Re_E$ and $Re_c$ may become very large. An example is the case of two rotating coaxial cylinders with a viscous incompressible fluid between them.

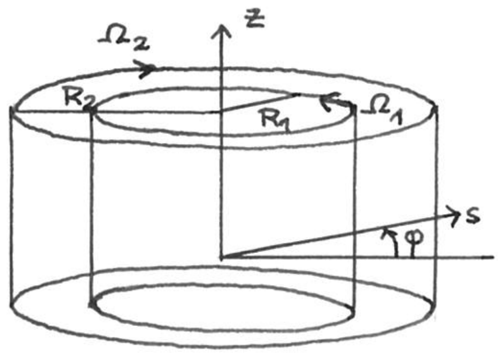


Figure 2

$\Omega_1$ is the angular velocity of the interior cylinder, $R_1$ its radius, whereas $\Omega_2$ is the angular velocity of the exterior one and $R_2$ its radius. The cylinders are considered to be sufficiently long. Therefore disturbances from the ends of the cylinders can be neglected. The quantity

$$d = R_2 - R_1$$

is called the gap between the cylinders. The basic flow $\mathbf{u}_s, \pi_s$ to be perturbed is a solution of the steady Navier-Stokes equation and is driven by the moving cylinder mantles to which it sticks. Introducing cylindrical coordinates about the axis of rotation as indicated in figure 2 we find

$$\mathbf{u}_s = v_0(s)\mathbf{e}_\varphi, \ \mathbf{F} = \text{gravity},$$
$$\pi_s = \pi_0(s) - gz + \text{const.}$$

for the basic flow with

$$v_0(s) = As + \frac{B}{s},$$
$$A = \frac{R_2^2\Omega_2 - R_1^2\Omega_1}{R_2^2 - R_1^2}, \ B = -\frac{R_1^2 R_2^2(\Omega_2 - \Omega_1)}{R_2^2 - R_1^2}.$$

The basic flow is thus axisymmetric. If it is perturbed by axisymmetric disturbances criticality is reached if the relative velocity $|\Delta v| = |\Omega_2 - \Omega_1|\frac{R_1+R_2}{2}$

between the cylinder mantles is increased sufficiently. We assume from now on that

$$(10) \qquad \frac{d}{R_2 + R_1} << 1, \; \left|\frac{\Omega_2 - \Omega_1}{\Omega_2 + \Omega_1}\right| << 1$$

which condition we call the small-gap limit. Below criticality no effects are observed whereas at criticality steady bifurcation sets in and produces the well known Taylor vortices. Pictures of this phenomenon can be found in many books on fluid dynamics.

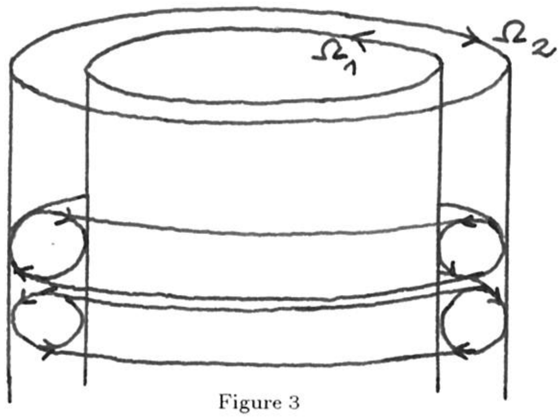

Figure 3

This coincidence of experimental results with the bounds predicted by linearized stability is explained completely in a strict mathematical sense in a forthcoming paper by **R. Kaiser** and the author (cf. [5]). In what follows we briefly sketch the mathematical part of this paper. The basic idea consists in showing that in our case unconditional stability up to criticality takes place. It was suggested by previous studies of **Busse** [3].

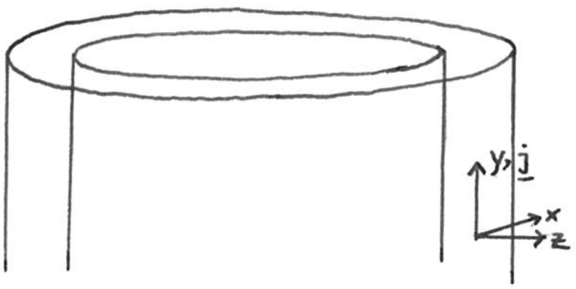


Figure 4

Since the basic flow is axisymmetric it is also a steady solution of the Navier-Stokes system with respect to a rotating cylindrical coordinate system with the $z$-axis as axis of rotation. This rotating frame is shifted to the middle between the cylinders and then replaced by a cartesian one. This turns out to be admissible at least for axisymmetric perturbations. On using (10) the basic flow is approximated and finally an appropriate scaling is introduced. The axisymmetric disturbances in this new setting are just the $x$-independent ones. On introducing the quantities

$$\begin{aligned} \Omega &= \frac{d^2}{2\nu}(\Omega_1 + \Omega_2), \ \mathbf{\Omega} = \Omega \mathbf{j} \\ Re &= \frac{d}{\nu}\,|\Omega_2 - \Omega_1|\,\frac{R_2 + R_1}{2} = \text{Reynolds-number} \end{aligned}$$

the problem in question is now mapped onto the equivalent stability problem for plane Couette-flow $Re\mathbf{U}_s = Re(-z, 0, 0)^T$ in an infinite rotating layer $-\frac{1}{2} \leq z \leq \frac{1}{2}$ with aximutal axis of rotation $\Omega\mathbf{j}$. It is governed by the system

$$\begin{aligned} &\partial_t \mathbf{u} - \Delta \mathbf{u} + Re\,\mathbf{U}_s \cdot \nabla \mathbf{u} + Re\,\mathbf{u} \cdot \nabla \mathbf{U}_s + \mathbf{u} \cdot \nabla \mathbf{u} + \\ (11) \qquad &\qquad\qquad + 2\mathbf{\Omega} \times \mathbf{u} + \nabla \pi = 0, \\ &\nabla \cdot \mathbf{u} = 0, \ \mathbf{u} = 0 \text{ at } z = \pm\tfrac{1}{2}, \ \mathbf{u}(0, x, y, z) = \mathbf{u}_0(x, y, z) \end{aligned}$$

which we have written down in its $3D$-form. $\mathbf{u}$ is periodic in the plane directions with wave-numbers $\alpha, \beta$ in $x, y$-direction respectively. The axisymmetric perturbations in the original-problem are the $x$-independent ones in (11).

For a detailed analytical study of (11) we have to make full use of the **underlying geometry** when eliminating the pressure and exploiting the structure of the nonlinearity. The projection $P$, already introduced in connection with (7), is not suitable for this purpose. Instead of using $P$ we decompose any $x, y$-periodic solenoidal vector field $\mathbf{u}$ in the layer as follows (cf. [10]):

$$\mathbf{u} = \text{curl curl } \varphi\mathbf{k} + \text{curl } \psi\mathbf{k} + \mathbf{f}.$$

$\varphi, \psi$ are periodic functions having vanishing mean value over the plane periodicity cell $\mathcal{P}$. $\mathbf{f}$ is a vector field which depends on $z$ only and has constant third component $f_3$. $f_3$ vanishes if $\mathbf{u} = 0$ at $z = \pm\frac{1}{2}$. curl curl $\varphi\mathbf{k}$ is called the poloidal part of $\mathbf{u}$, curl $\psi\mathbf{k}$ is the toroidal one and $\mathbf{f}$ is called the mean flow. We set $\Delta_2 = \partial_x^2 + \partial_y^2$ for the plane Laplacian and obtain

$$\boldsymbol{\delta}\varphi := \text{curl curl } \varphi \mathbf{k} = (\partial_{xz}\varphi, \partial_{yz}\varphi, (-\Delta_2)\varphi)^T,$$
$$\boldsymbol{\varepsilon}\psi := \text{curl } \psi \mathbf{k} = (\partial_y\psi, -\partial_x\psi, 0)^T.$$

$\boldsymbol{\delta}\cdot$, $\boldsymbol{\varepsilon}\cdot$ are used as vectorial operators too. For instance, let $\mathbf{v}$ be a vector field. Then we set $\boldsymbol{\delta} \cdot \mathbf{v} = \partial_{xz}v_1 + \partial_{yz}v_z + (-\Delta_2)v_3$ and define $\boldsymbol{\varepsilon} \cdot \mathbf{v}$ correspondingly. This decomposition was already used by **Joseph** and, mainly, by **Busse** but these authors omit $\mathbf{f}$. $\mathbf{f}$ however is needed since for $\mathbf{f} = 0$ we only obtain the solenoidal vector fields having vanishing mean value over $\mathcal{P}$. The last property however is not invariant under $Re\, \mathbf{U}_s \cdot \nabla \mathbf{u} + Re\, \mathbf{u} \cdot \nabla \mathbf{U}_s + \mathbf{u} \cdot \nabla \mathbf{u}$ and therefore these vector fields cannot serve as solutions of (11) in general. If $\mathcal{L}S$ denotes the left-hand side in (11) we form (curl curl $\mathcal{L}S, \mathbf{k}$), (curl $\mathcal{L}S, \mathbf{k}$) and end up with a system of higher order for the unknown vector $\Phi = (\varphi, \psi, f_1, f_2)^T$ where $\mathbf{f} = (f_1, f_2, 0)^T$. This system takes the form ($\tilde{\mathbf{u}} := \boldsymbol{\delta}\varphi + \boldsymbol{\varepsilon}\psi$)

$$(12) \begin{cases} (-\Delta)(-\Delta_2)\partial_t\varphi + \Delta^2(-\Delta_2)\varphi - 2(-\Delta_2)\Omega\partial_y\psi + \boldsymbol{\delta} \cdot (\mathbf{u} \cdot \nabla \mathbf{u}) = 0, \\ (-\Delta_2)\partial_t\psi + (-\Delta)(-\Delta_2)\psi - Re(-\partial_y)(-\Delta_2)\varphi + \\ \qquad 2\Omega(-\partial_y)(-\Delta_2)\varphi - \boldsymbol{\varepsilon} \cdot (\mathbf{u} \cdot \nabla \mathbf{u}) = 0, \\ \partial_t f_1 + (-\partial_z^2)f_1 + \frac{1}{|\mathcal{P}|}\int_{\mathcal{P}} (\tilde{\mathbf{u}} \cdot \nabla \tilde{\mathbf{u}})_x \, dx\, dy = 0, \\ \partial_t f_2 + (-\partial_z^2)f_2 + \frac{1}{|\mathcal{P}|}\int_{\mathcal{P}} (\tilde{\mathbf{u}} \cdot \nabla \tilde{\mathbf{u}})_y \, dx\, dy = 0 \end{cases}$$

in the axisymmetric case, this is $\partial_x. \equiv 0$. The pressure is eliminated and the system (12) is almost local, with the exception of two mean values in the subsystem for the mean flow. This is quite in contrast to (7) since $P$ there is a non-local operator. For this material we refer to [10, 5] and [14, pp. 119,120]. The system in question thus has the structure

$$(13) \qquad \partial_t\mathcal{B}\Phi + \mathcal{A}\Phi + \mathcal{C}\Phi + \mathcal{M}(\Phi_s, \Phi) + \mathcal{M}(\Phi, \Phi_s) + \mathcal{M}(\Phi, \Phi) = 0$$

for the unknown vector-field $\Phi(t) = (\varphi(t), \psi(t), f_1(t), f_2(t))^T$. $\mathcal{B}, \mathcal{A}$ are operators of higher order, $\mathcal{M}$ is a bilinear nonlinearity and $\mathcal{C}\Phi$ stands for the Coriolis-term. $\Phi_s$ is the steady flow whose stability has to be studied. The boundary values are

$$(14) \qquad \begin{aligned} &\varphi = \partial_z\varphi = 0 \text{ at } z = \pm\tfrac{1}{2}, \ \psi = 0 \text{ at } z = \pm\tfrac{1}{2}, \\ &f_1 = f_2 = 0 \text{ at } z = \pm\tfrac{1}{2}. \end{aligned}$$

In the general $3D$-case we look for a solution $\Phi$ of (13) which is periodic in the $x, y$-directions with wave numbers $\alpha, \beta$. If any $f \in L^2((-\frac{\pi}{\alpha}, \frac{\pi}{\alpha}) \times (-\frac{\pi}{\beta}, \frac{\pi}{\beta}) \times (-\frac{1}{2}, \frac{1}{2}))$ is expanded into a Fourier-series with respect to the plane variables we obtain

$$f(x, y, z) = \sum_{\kappa \in \mathbb{Z}^2} \frac{a_\kappa(z)}{\sqrt{\frac{2\pi}{\alpha} \cdot \frac{2\pi}{\beta}}} e^{i\alpha\kappa_1 x + i\beta\kappa_2 y} \tag{15}$$

for almost all $z \in (-\frac{1}{2}, \frac{1}{2})$, and Levi's theorem shows that

$$\int_\Omega |f|^2 \, dx \, dy \, dz = \sum_{\kappa \in \mathbb{Z}^2} \int_{-\frac{1}{2}}^{+\frac{1}{2}} |a_\kappa(z)|^2 \, dz,$$

$$\Omega = (-\frac{\pi}{\alpha}, \frac{\pi}{\alpha}) \times (-\frac{\pi}{\beta}, \frac{\pi}{\beta}) \times (-\frac{1}{2}, \frac{1}{2}).$$

It is now easy to see that $\mathcal{B}, \mathcal{A}$ act as positive definite selfadjoint operators on the $x, y$-periodic vector-fields. For instance, if we take $\varphi = f$ from the closed subspace $L^2_M(\Omega)$ of $L^2(\Omega)$, consisting of all functions having vanishing mean value over $(-\frac{\pi}{\alpha}, \frac{\pi}{\alpha}) \times (-\frac{\pi}{\beta}, \frac{\pi}{\beta})$, then $a_0 = 0$ in (15) and $\Delta^2(-\Delta_2)$ becomes a strictly positive definite selfadjoint operator when defined on all $f$ as in (15) with

$$a_\kappa \in H^{4,2}((-\frac{1}{2}, \frac{1}{2})), \; a_\kappa = \partial_z a_\kappa = 0 \text{ at } z = \pm\frac{1}{2}, \; \kappa \neq 0,$$

$$\sum_{\kappa \in \mathbb{Z}^2 - \{0\}} (\alpha^2\kappa_1^2 + \beta^2\kappa_2^2)^2 \int_{-\frac{1}{2}}^{+\frac{1}{2}} |(-\partial_z^2 + \alpha^2\kappa_1^2 + \beta^2\kappa_2^2)^2 a_\kappa|^2 \, dz < +\infty.$$

Also $-\Delta_2$ becomes a strictly positive definite selfadjoint operator in $L^2_M(\Omega)$ when defined in an obvious way. It is now natural to seek a solution of (13) within the class

$$\begin{aligned} \partial_t \Phi &\in L^2((0, T), \mathcal{D}(\mathcal{B})), \\ \Phi &\in L^2((0, T), \mathcal{D}(\mathcal{A})) \cap C^0([0, T], J) \end{aligned}$$

where $J$ is an appropriate interpolation space between $\mathcal{D}(\mathcal{B})$ and $\mathcal{D}(\mathcal{A})$. In our case, this is $\partial_x \Phi \equiv 0$, this solution exists for all $T > 0$. We refer to [12, 13] for this material.

On returning to our original problem (12) with $\partial_x \cdot \equiv 0$ it is readily seen that $Re_E$ and $Re_c$ are most efficiently computed when using the poloidal-, toroidal-, mean-flow language. To describe the result we introduce an auxiliary function which stems from the Bénard-problem when disturbing the motionless state under rigid boundary-conditions. In dependence of the wave-number $\beta$ in $y$-direction the function in question separates monotonic energy stability from instability (Theorem 1 here applies!). We denote this function by $R_{\min}(\beta^2)$. It marks therefore the critical Rayleigh-number against $2D$-disturbances in the Bénard-problem and looks as follows (cf. [11]):

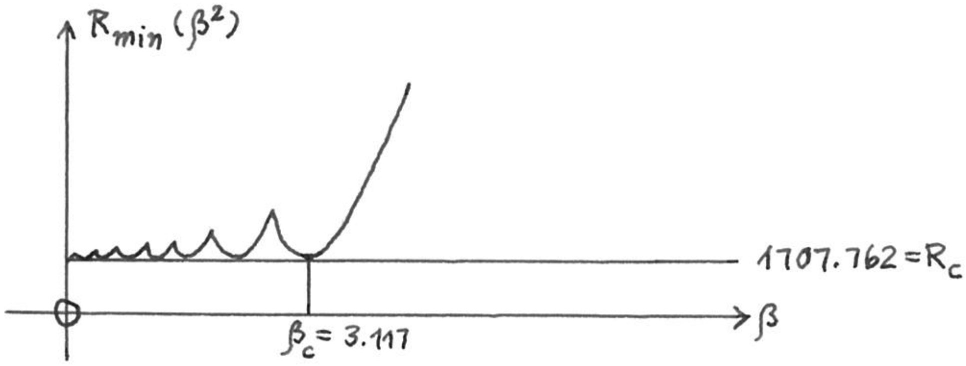


Figure 5

The minimal value $R_c = 1708$ is assumed infinitely many times. For the energetic Reynolds-number $Re_E$ in the Taylor-Couette-problem one obtains

$$Re_E = Re_E(\beta^2) = 2\sqrt{R_{\min}(\beta^2)}.$$

To fix the ideas let us assume that $\Omega = \frac{d^2}{2\nu}(\Omega_1 + \Omega_2) > 0$, $Re \neq 2\Omega$. Then we obtain

$$Re_c = 2\Omega + \frac{R_{\min}(\beta^2)}{2\Omega}$$

for the critical Reynolds-number. Now we can compare $Re_c$ and $Re_E$. If $\beta$

is fixed the situation looks as follows:

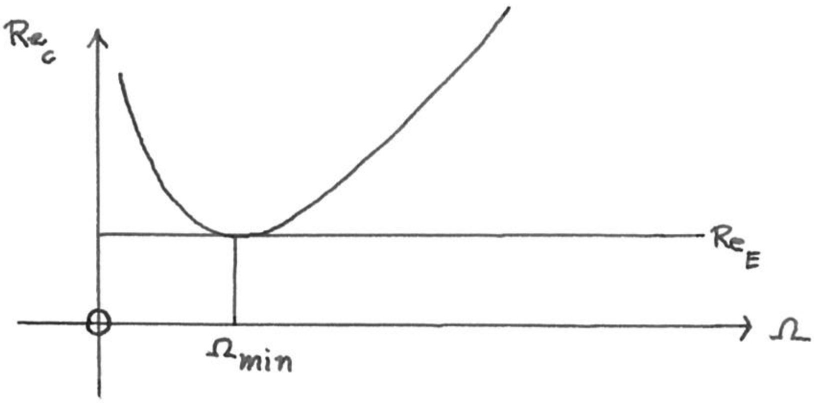


Figure 6

$Re_E$ is just the minimal value of $Re_c = Re_c(\Omega)$. It is assumed for $\Omega_{\min} = \frac{1}{2}\sqrt{R_{\min}(\beta^2)}$. Thus $Re_E = 4\Omega_{\min}$. This is an example in the spirit of Theorem 1 and was found first by **Busse** [1]. As it follows from Theorem 1 it is the only one amongst all $3D$ disturbances and all plane parallel shear flows $(f(z), 0, 0)^T$ with general profile $f$ and axis of rotation $\mathbf{\Omega} = \Omega\mathbf{j}$ (Cf. [14]). For this reason there is necessarily a gap between $Re_c$ and $Re_E$ if $\Omega \neq \Omega_{\min}$ as can be seen from the figure above. By inserting suitable testing vectors into some weak form of (12) we see that the functional

$$\begin{aligned}\mathcal{F}(t) = \|\nabla(-\partial_y^2)^{\frac{1}{2}}\varphi(t)\|^2 + \|f_2(t)\|^2 + \\ + \tfrac{4\Omega^2}{|2\Omega(Re-2\Omega)|} \cdot (\|\partial_y\psi(t)\|^2 + \|f_1(t)\|^2)\end{aligned} \tag{16}$$

exhibits unconditional stability for $Re < Re_c$. $\mathcal{F}$ is then monotonically and exponentially decaying for $t > 0$. If $Re = Re_c$, $\mathcal{F}$ is monotonically non-increasing. $\mathcal{F}$ is equivalent to the kinetic energy and equals it in the exceptional case $Re = 4\Omega$. If $\Omega < 0$ the functional in question stays the same one but if $2\Omega(Re - 2\Omega) = 0$ we have to choose a different one. We refer to [5].

Let us now consider the nonlinear steady problem which belongs to (12). We set $\vartheta = 2\Omega\partial_y\psi$, $\hat{f}_1 = 2\Omega f_1$, $\hat{f}_2 = f_2$, $\hat{\mathbf{f}} = (\hat{f}_1, \hat{f}_2)^T$ and then this problem becomes $((-\Delta_2) = -\partial_y^2)$ :

$$
(17)\qquad \begin{cases}
\Delta^2(-\Delta_2)\varphi - (-\Delta_2)\vartheta + \boldsymbol{\delta}\cdot((\boldsymbol{\delta}\varphi + \hat{\mathbf{f}})\cdot\nabla\boldsymbol{\delta}\varphi) + \\
\partial_{yz}((-\partial_y^2\varphi)\partial_z\hat{f}_2) = 0, \\
(-\Delta)\vartheta - 2\Omega(Re - 2\Omega)(-\Delta_2)\varphi + \boldsymbol{\delta}\varphi\cdot\nabla\vartheta - \\
- \int\limits_{-\pi/\beta}^{\pi/\beta} \boldsymbol{\delta}\varphi\cdot\nabla\vartheta\, dy + \hat{f}_2\partial_y\vartheta + (-\Delta_2)\varphi\partial_z\hat{f}_1 = 0, \\
(-\partial_z^2)\hat{f}_1 + \frac{1}{|\mathcal{P}|}\int\limits_{\mathcal{P}} \boldsymbol{\delta}\varphi\cdot\nabla\vartheta\, dx\, dy = 0, \\
(-\partial_z^2)\hat{f}_2 + \frac{1}{|\mathcal{P}|}\int\limits_{\mathcal{P}} \boldsymbol{\delta}\varphi\cdot\nabla\partial_{yz}\varphi\, dx\, dy = 0.
\end{cases}
$$

Thus (17) is a nonlinear eigenvalue-problem and

$$R = 2\Omega(Re - 2\Omega)$$

serves as eigenvalue parameter. Criticality precisely means that $R = R_{\min}(\beta^2)$, where $R_{\min}(\beta^2)$ is the curve from figure 5. Let us now place ourselves in the vicinity of a minimum of this curve, say around $\beta_c = 3.117$, $R_c = 1707.762$. Around $(\beta_c, R_c)$ the eigenvalues $R = R_{\min}(\beta^2)$ of the linear problem belonging to (17) turn out to be algebraically simple if one takes into consideration only vectors which are even with respect to $y$, this is: Only cos-terms appear in the Fourier-expansions (15) for $\varphi, \vartheta$. Therefore we obtain a branch $(\hat{\Phi} = (\varphi, \vartheta, \hat{f}_1, \hat{f}_2), R)$ of solutions of (17) which can be given in terms of an expansion

$$
(18)\qquad \hat{\Phi} = \sum_{\nu=1}^{\infty} \varepsilon^\nu \hat{\Phi}_\nu,\ R = \sum_{\nu=0}^{\infty} \varepsilon^\nu R_\nu,\ |\varepsilon| \leq \varepsilon_0,
$$

with Hilbert-space valued coefficients $\hat{\Phi}_\nu = \hat{\Phi}_\nu(\beta)$, $\hat{\Phi}_1 \neq 0$, and real coefficients $R_\nu = R_\nu(\beta)$, $R_0 = R_0(\beta) = R_{\min}(\beta^2)$. The unconditional asymptotic stability up to criticality, expressed by the functional $\mathcal{F}$ in (16), shows that $R_1 = 0$. Namely, if $R_1 \neq 0$ we obtain for small $|\varepsilon|$ a nonvanishing solution of (17) for $R < R_{\min}(\beta^2)$. Retransforming $\hat{\Phi}$ by setting $\psi = \frac{1}{2\Omega}(-\Delta_2)^{-1}\partial_y\vartheta$, $f_1 = \frac{1}{2\Omega}\hat{f}_1$ we arrive at a steady nonvanishing solution of (12) strictly below criticality. Taking this solution as initial value in (12) we end up with a contradiction since $\mathcal{F}(t) \to 0$, $t \to \infty$, implies that $\hat{\Phi} \equiv 0$. The stable ones amongst the solutions $(\hat{\Phi}, R)$ may be seen as Taylor-vortices as indicated in figure 3. In particular the $(\Phi, R)$ branch off to the right.

$R_2 > 0$ and

Our previous example, the Taylor-Couette-problem in the small-gap limit under axisymmetric perturbations, shows that $Re_E < Re_c$ even if the bifurcation takes place in the direction of growing Reynolds-numbers. The stability behaviour however of the basic flow below criticality may be improved by this fact. This is seen as follows: If we think of a branch like (18) with $R_1 \neq 0$ then we have a subcritical bifurcation. This in particular means that close to criticality the stability ball in (7) becomes smaller than the (small) size of the steady solutions on the branch for small $|\varepsilon|$. Otherwise one would be able to outrule $R_1 \neq 0$ by taking the steady flows on the branch as initial values in (7) as before. An analytical treatment of this question seems to be difficult.

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