Decomposition of Solenoidal Fields into Poloidal Fields, Toroidal Fields and the Mean Flow. Applications to the Boussinesq-Equations

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0. Introduction and Notations

Let there be given a solenoidal vector field $\underline{u}$ over an infinite layer $(x,y,z) \in \mathbf{R}^2 \times (a,b)$. $\underline{u}$ is assumed to be periodic in x, y with respect to a square or a rectangle. We prove here that $\underline{u}$ can be decomposed in a unique way into $(\underline{k} = (0,0,1))$

$$
\begin{aligned}
(0.1) \qquad \underline{u}(x,y,z) &= \operatorname{curl}\operatorname{curl}\varphi(x,y,z)\underline{k} + \operatorname{curl}\psi(x,y,z)\underline{k} + F(z) \\
&= P(x,y,z) + T(x,y,z) + F(z).
\end{aligned}
$$

$P = \operatorname{curl}\operatorname{curl}\varphi\underline{k}$ is called the poloidal part of $\underline{u}$, $T = \operatorname{curl}\psi\underline{k}$ is the toroidal part and the field F, which depends on z only and has constant third component, is the mean flow. φ, ψ are functions which are determined uniquely if we require them to have vanishing mean value over a periodicity cell $\mathcal{P}$. $P+T$ is then nothing else but that part of $\underline{u}$ which has vanishing mean value over $\mathcal{P}$. A corresponding decomposition was already derived in [1,2] in the case of a spherical layer. In this situation it turns out that the mean flow is not needed, i.e. $F \equiv 0$. In [5, p. 236] an attempt was made to decompose $\underline{u}$ into $P + T$ in our situation, on $\underline{u} = (u_x, u_y, u_z)^{\mathrm{T}}$ the condition $\int_{\mathcal{P}} u_z(x,y,a)\,dx\,dy = 0$ was imposed. This is however not sufficient as it is exhibited by the mean flow (cf. the Remark in section 1). In fact $\int_{\mathcal{P}} u_x\,dx\,dy = \int_{\mathcal{P}} u_y\,dx\,dy = \int_{\mathcal{P}} u_z\,dx\,dy = 0$ is needed. The exact result concerning the decomposition (0.1) can be seen from Theorem 1.4 to follow. P, T, F can be understood as orthogonal projections from the L^2-space of periodic solenoidal fields (in a sense which is made precise in Theorem 1.4) onto three pairwise orthogonal subspaces. The

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regularity properties of these projections in terms of L^2-Sobolev spaces are studied also in Theorem 1.4.

The decomposition 0.1 is now applied to the Boussinesq-equations

$$(0.2) \quad \begin{cases} \underline{u}' - \Delta\underline{u} + \underline{u}\cdot\nabla\underline{u} - \sqrt{R}\,\vartheta\underline{k} + \nabla\pi = 0, \quad \nabla\cdot\underline{u} = 0, \\ \mathrm{Pr}\,\vartheta' - \Delta\vartheta + \mathrm{Pr}\,\underline{u}\cdot\nabla\vartheta - \sqrt{R}\,u_z = 0 \end{cases}$$

over the infinite layer $\mathbf{R}^2 \times (-\frac{1}{2}, \frac{1}{2})$. $\mathrm{Pr} > 0$ is the Prandtl-number, $R > 0$ is the Rayleigh-number, $\underline{u}$, ϑ have the usual meaning, π is the pressure. The boundary-conditions at $z = \pm\frac{1}{2}$ are the usual ones: Stress-free boundaries or rigid boundaries. They are explained in section 2. $.'$ refers to the derivative with respect to time, and we have also to prescribe the initial values $\underline{u}_0$, ϑ_0 at time $t = 0$. $\underline{u}$, ϑ and π are required to be periodic in (x, y) with respect to a square $\mathcal{P} = (-\frac{\pi}{\alpha}, \frac{\pi}{\alpha})^2$ with wave-number α in both directions; the generalization to $\mathcal{P} = (-\frac{\pi}{\alpha}, \frac{\pi}{\alpha}) \times (-\frac{\pi}{\beta}, \frac{\pi}{\beta})$ is at hand and does not need further consideration. When applying (0.1) to $\underline{u}$ as above the boundary conditions on $\underline{u}$ go over into equivalent ones on φ, ψ and F. Moreover, $F_3 \equiv 0$. The system (0.2) itself is transformed into an equivalent one for $\Phi = (\varphi, \psi, \vartheta, F_1, F_2)^{\mathrm{T}}$. It has the form

$$(0.3) \quad \mathcal{B}\Phi' + \mathcal{A}\Phi - \sqrt{R}\,\mathcal{C}\Phi + \mathcal{M}(\Phi) = 0$$

with matrix operators $\mathcal{B}$, $\mathcal{A}$, $\mathcal{C}$ and a nonlinear term $\mathcal{M}$. $\mathcal{A}$, $\mathcal{B}$ turn out to be diagonal and strictly positive definite selfadjoint operators in an appropriate Hilbert space H. H is simply the product $L^2_M(\Omega) \times L^2_M(\Omega) \times L^2(\Omega) \times (L^2_M((-\frac{1}{2}, \frac{1}{2})))^2$ or $L^2_M(\Omega) \times L^2_M(\Omega) \times L^2(\Omega) \times (L^2((-\frac{1}{2}, \frac{1}{2})))^2$ for stress-free boundaries or rigid boundaries with $\Omega = \mathcal{P} \times (-\frac{1}{2}, \frac{1}{2})$. The subscript $._M$ indicates that φ, ψ have vanishing mean value over $\mathcal{P}$, whereas F is required to have vanishing mean value over $(-\frac{1}{2}, \frac{1}{2})$ in the case of stress-free boundaries. The pressure is eliminated. Whereas the necessity of (0.3) is easy to show, the proof of the sufficiency throws some light on the mean flow. While it's possible to solve (0.3) in a reduced form, i.e. without mean flow (this is even easier), it's not possible to obtain back (0.2) from (0.3) in this case unless

$$(0.4) \quad \int_{\mathcal{P}} (P + T)_x\, P_z\, dx\, dy = \int_{\mathcal{P}} (P + T)_y\, P_z\, dx\, dy = 0.$$

(0.4) holds if φ, ψ, ϑ exhibit certain symmetries. We refer for this to [3, pp. 347, 357].

Of course one can try to eliminate $\nabla\pi$ in (0.2) by using the classical tool of projecting $L^2(\Omega)$ on its divergence-free part. This has been done by Iooss in [4] for rigid boundaries. Therefore some remarks on the advantages of

(0.3) are in order. While the projection Q just mentioned is a nonlocal operator and therefore yields a nonlocal nonlinearity $Q(\underline{u} \cdot \nabla \underline{u})$ when applied, the main part of $\mathcal{M}(\Phi)$ is purely local. There is only one nonlocal part in $\mathcal{M}(\Phi)$. It occurs within the subsystem for the mean flow and consists of $(\int_{\mathcal{P}}(P + T) \cdot \nabla (P + T)_x, \, dx \, dy, \int_{\mathcal{P}}(P + T) \cdot \nabla (P + T)_y, \, dx \, dy)$. In particular it's possible to study (0.3) within various subspaces which are invariant under the nonlinearity and to subject (0.3) to a numerical analysis. This was almost exclusively done by physicists. We refer for this e.g. to the paper [3, sections 2, 4]. The mathematical background for (0.3) is treated in detail in [6]. In the present paper we review in section 2 some results of [6]. (0.3) may also be used to study the regularity behaviour of Φ near $t = 0$ by imposing suitable compatibility conditions. While this question may be of more mathematical interest, the problem of energy-stability is not. Due to the fact that the highest order derivatives of $u_z = (-\Delta_2)\varphi$, $\Delta_2 := \partial_x^2 + \partial_y^2$, are isolated in the first row of (0.3) and the pressure is eliminated at the same time, a calculus estimate already yields the precise bounds in the case of stress-free boundaries.

We introduce some notation. A vector field $\underline{u}$ or F is usually written as a column, i.e. $\underline{u} = (u_1, u_2, u_3)^{\mathrm{T}} = (u_x, u_y, u_z)^{\mathrm{T}}$, $F = (F_1, F_2, F_3)^{\mathrm{T}} = (F_x, F_y, F_z)^{\mathrm{T}}$ with the symbol $.^{\mathrm{T}}$ for transposition. $H^l(\Omega) = H^{l,2}(\Omega)$ for any open set Ω of $\mathbf{R}^n$ are the usual Sobolev spaces of integer order $l \geq 0$. In section 1 we will also introduce the Sobolev spaces $H^l_{\mathcal{P}}$ of $\mathcal{P}$-periodic functions in the plane with exponent of integration 2. If (a, b) is an open interval on the z-axis, then

$$W^k((a, b), H^l_{\mathcal{P}})$$

consists of the mappings $f: (a, b) \to H^l_{\mathcal{P}}$ with $\partial_z^p \mathrm{D}_{xy}^q f \in L^2((a, b), H^0_{\mathcal{P}}) = L^2((a, b), L^2(\mathcal{P}))$ for any integers $p, q \geq 0$ with $p \leq k$, $q \leq l$ and $p + q \leq \max\{k, l\}$. D_{xy}^q stands for any derivative of order q in the periodic variables x, y. $W^k((a, b), H^l_{\mathcal{P}})$ becomes a Hilbert space in the usual way. A selfadjoint operator A in a Hilbert space $\mathcal{H}$ is called strictly positive definite iff $(Au, u) \geq \gamma \|u\|^2$, $u \in \mathcal{D}(A)$, for some $\gamma > 0$. $C^k([a, b], \mathcal{H})$ is the usual space of k-times continuously differentiable functions on $[a, b]$ with values in the space $\mathcal{H}$.

1. A poloidal-toroidal representation for periodic solenoidal fields in an infinite layer

We want to explain how a solenoidal vector field $\underline{u}$ defined in the three-dimensional layer $L = \mathbf{R}^2 \times (a, b) \subset \mathbf{R}^3$ can be represented in terms of

poloidal and toroidal fields

$$P(x, y, z) = \operatorname{curl} \operatorname{curl} \varphi(x, y, z)\underline{k}, \qquad T(x, y, z) = \operatorname{curl} \psi(x, y, z)\underline{k}$$

if $\underline{u}$ and the flux functions φ, ψ are assumed to be periodic with respect to the first two arguments.

For simplicity we restrict ourselves to the case where the lengths of the periods in x and y are equal. Moreover we will deal only with the exponent of integration 2. Thus let us set $\mathcal{P} = (-\pi, \pi)^2$ and consider the Hilbert space $L_{\mathcal{P}}^2$ consisting of all quadratically integrable (complex valued) functions on $\mathcal{P}$ which we will regard to be extended into the whole plane $\mathbb{R}^2$ periodically. Given two functions f, $f_\beta \in L_{\mathcal{P}}^2$ we will call f_β the weak β-th derivative of f *in the sense of periodic distributions in* $\mathbb{R}^2$ iff

$$(1.1) \quad \int_{\mathcal{P}} f_\beta(x, y)\phi(x, y)\, dx\, dy = (-1)^{|\beta|} \int_{\mathcal{P}} f(x, y) \mathrm{D}^\beta \phi(x, y)\, dx\, dy \quad \forall \phi \in C_{\mathcal{P}}^\infty,$$

where

$$C_{\mathcal{P}}^\infty := \left\{ \phi \in C^\infty(\mathbb{R}^2, \mathbb{C}) \mid \phi \text{ periodic in } x \text{ and } y \text{ with respect to } \mathcal{P} \right\}$$

denotes the space of the $\mathcal{P}$-periodic testing functions. We will then write $\mathrm{D}^\beta f$ instead of f_β. Further we define the following Sobolev spaces of $\mathcal{P}$-periodic functions:

$$H_{\mathcal{P}}^m := \left\{ f \in L_{\mathcal{P}}^2 \mid \mathrm{D}^\beta f \in L_{\mathcal{P}}^2 \text{ in the sense of } (1.1)\ \forall |\beta| \le m \right\}$$

endowed with the norms

$$\|f\|_{\mathcal{P},m} := \left(\sum_{|\beta| \le m} \left\| \mathrm{D}^\beta f \right\|_{L_{\mathcal{P}}^2}^2 \right)^{1/2}, \qquad m \in \mathbb{N}_0.$$

Consider now our main device in such spaces, i.e. the Fourier expansion. Assume

$$f(x, y) = (2\pi)^{-1} \sum_{\kappa \in \mathbf{Z}^2} a_\kappa e^{i\kappa \cdot (x,y)} \quad \text{in } L_{\mathcal{P}}^2.$$

If there exists $\mathrm{D}^\beta f \in L_{\mathcal{P}}^2$ in the sense of periodic distributions in $\mathbb{R}^2$ we will infer from (1.1) by using the testing functions $\exp(-i\kappa \cdot (x, y))$ that $a_\kappa (i\kappa)^\beta$ are the Fourier coefficients of $\mathrm{D}^\beta f$. Therefore it is easy to see that $\|f\|_{\mathcal{P},m}$ and $\left(|a_0|^2 + \sum_{\kappa \ne 0} |a_\kappa|^2 |\kappa|^{2m} \right)^{1/2}$ define equivalent norms in $H_{\mathcal{P}}^m$. Especially it will suffice to show the convergence of this series in order to prove $f \in H_{\mathcal{P}}^m$ for some $f \in L_{\mathcal{P}}^2$.

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Let $\Omega = \mathcal{P} \times (a, b)$ denote the three dimensional box built over the period rectangle $\mathcal{P}$. For $f \in L^2((a, b), L_{\mathcal{P}}^2)$ let

$$\langle f \rangle_z := \fint_{\mathcal{P}} f(x, y, z) \, dx \, dy$$

be the normalized mean value over $\mathcal{P}$. Finally let us introduce $\Delta_2 := \partial_x^2 + \partial_y^2$, the Laplacian in two dimensions in each hyperplane $\mathbf{R}^2 \times \{z\}$ of the layer L. This differential operator will arise as $-\langle \operatorname{curl} \operatorname{curl}(\,.\,\underline{k}), \underline{k} \rangle$ when multiplying a poloidal field and the vector $\underline{k}$.

Thus we start with some considerations on this operator.

Proposition 1.1:

a) *Let $f \in L_{\mathcal{P}}^2$. The problem $\Delta_2 u = f$ (in the sense of periodic distributions in $\mathbf{R}^2$) admits a solution $u \in L_{\mathcal{P}}^2$ iff $\langle f \rangle := \fint_{\mathcal{P}} f(x, y) \, dx \, dy = 0$. u is uniquely determined by its mean value $\langle u \rangle$. If $\langle u \rangle = 0$ then the estimate $\|u\|_{\mathcal{P},0} \leq \|f\|_{\mathcal{P},0}$ is valid.*

b) *If $f \in H_{\mathcal{P}}^m$, then $u \in H_{\mathcal{P}}^{m+2}$, and u can be estimated via the inequality $\|u\|_{\mathcal{P},m+2} \leq c(m)(\|u\|_{\mathcal{P},0} + \|f\|_{\mathcal{P},m})$.*

c) *Let G be a domain in $\mathbf{R}^n$, and $\lambda \mapsto f(\cdot, \lambda) \in C^k(G, H_{\mathcal{P}}^m)$ with $\langle f(\cdot, \lambda) \rangle = 0$ $\forall \lambda \in G$. Then the zero mean valued solution $u(\cdot, \lambda)$ of $\Delta_2 u(\cdot, \lambda) = f(\cdot, \lambda)$ $\forall \lambda$ lies in the space $C^k(G, H_{\mathcal{P}}^{m+2})$, and for all $\lambda \in G$, $|\beta| \leq k$, the pointwise estimate $\left\| D_\lambda^\beta u(\cdot, \lambda) \right\|_{\mathcal{P},m+2} \leq c(m) \left\| D_\lambda^\beta f(\cdot, \lambda) \right\|_{\mathcal{P},m}$ holds true.*

PROOF:

a) Let $u, f \in L_{\mathcal{P}}^2$ satisfy $\Delta_2 u = f$ in the sense mentioned above. Let $(a_\kappa)_{\kappa \in \mathbf{Z}^2}$ be the Fourier coefficients of u. As explained in the preliminary examination $(-a_\kappa |\kappa|^2)_{\kappa \in \mathbf{Z}^2}$ turn out to be necessarily the Fourier coefficients of f. Thus, $\langle f \rangle = (-(2\pi)^{-1} a_\kappa |\kappa|^2)_{\kappa=0} = 0$. Further, a_κ, $\kappa \neq 0$ are uniquely determined by f, whence u will be also unique if f and $a_0 = 2\pi \langle u \rangle$ are prescribed.

On the other hand, if f is given by Fourier coefficients $(b_\kappa)_{\kappa \in \mathbf{Z}^2}$ and $b_0 = 0$, the function

$$u(x, y) = (2\pi)^{-1} \sum_{\kappa \neq 0} \frac{-b_\kappa}{|\kappa|^2} e^{i\kappa \cdot (x,y)} + (2\pi)^{-1} a_0$$

obviously defines the required solution of $\Delta_2 u = f$ with mean value $(2\pi)^{-1} a_0$.
b) As $f \in H_{\mathcal{P}}^m$ means that $\sum_{\kappa \neq 0} |b_\kappa|^2 |\kappa|^{2m} < \infty$, the fact $a_\kappa = -b_\kappa / |\kappa|^2$ for $\kappa \neq 0$ implies

$$|a_0|^2 + \sum_{\kappa \neq 0} |a_\kappa|^2 |\kappa|^{2m+4} = |a_0|^2 + \sum_{\kappa \neq 0} |b_\kappa|^2 |\kappa|^{2m} < \infty,$$

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i.e. $u \in H_{\mathcal{P}}^{m+2}$ and the asserted estimate.

c) If $\langle u \rangle = 0$ we get $a_0 = 0$, thus b) changes to $\|u\|_{\mathcal{P},m+2} \le c(m)\|f\|_{\mathcal{P},m}$. Apply this to the difference $u(\cdot,\lambda) - u(\cdot,\lambda')$ in order to obtain $(\lambda \mapsto u(\cdot,\lambda)) \in C(G, H_{\mathcal{P}}^{m+2})$. Now apply the same estimate to

$$\frac{1}{h}\Big(u(\cdot,\lambda_0 + he_i) - u(\cdot,\lambda_0)\Big) - u_{\lambda_0,v},$$

where $u_{\lambda_0,v}$ shall be the solution to $\mathrm{D}_\lambda^{e_i} f(\cdot,\lambda_0)$ with mean value zero. It follows that $\lambda \mapsto u(\cdot,\lambda)$ admits partial derivatives at the point λ_0. The assertion will then result inductively. $\qquad\qquad\Box$

Remark. In c) we may replace $C^k(G, H_{\mathcal{P}}^m)$ by its completion $H^k(G, H_{\mathcal{P}}^m)$ with respect to the norm $\|u\| = (\sum_{|\beta|\le k} \int_G \big\|D_\lambda^\beta u(\cdot,\lambda)\big\|_{\mathcal{P},m}^2 \, d\lambda)^{1/2}$, or by the space $W^k(G, H_{\mathcal{P}}^m)$ mentioned in the introduction. In the latter case the pointwise estimate reads

$$\big\|\mathrm{D}_\lambda^\beta u(\cdot,\lambda)\big\|_{\mathcal{P},j+2} \le c(j)\big\|\mathrm{D}_\lambda^\beta f(\cdot,\lambda)\big\|_{\mathcal{P},j}$$

for all $j \le m$ and $|\beta| \le k$ with $|\beta| + j + 2 \le \max\{k, m+2\}$. In the rest of this section we will use W^k instead of the more natural H^k because W^k is needed in section 2.

If in a) the right hand side f is real valued, then the solution u will also be real valued, provided its prescribed mean value belongs to $\mathbf{R}$, too. In fact, we might as well make use of the Fourier expansion in terms of $\cos(\kappa\cdot(x,y))$ and $\sin(\kappa\cdot(x,y))$, thus we would stay properly in real function spaces. However, this would not be as handy as the present notation.

For shortness we introduce the abbreviations

$$\underline{\varepsilon} := \mathrm{curl}\,(\,.\,\underline{k}) = \begin{pmatrix} \partial_y \\ -\partial_x \\ 0 \end{pmatrix}, \quad \underline{\delta} := \mathrm{curl}\,\mathrm{curl}\,(\,.\,\underline{k}) = \begin{pmatrix} \partial_x\partial_z \\ \partial_y\partial_z \\ -\Delta_2 \end{pmatrix},$$

these operators are intended to act on functions defined in L.

Using Proposition 1.1 we obtain at once a result for the problem $\Delta_2 u = f$ in the layer L.

Corollary 1.2: *Let $f \in W^k((\,a,b\,), H_{\mathcal{P}}^m)$ with $\langle f \rangle_z = 0 \; \forall' z \in (\,a,b\,)$. Then there exists exactly one $u \in L_{\mathrm{loc}}^1((\,a,b\,), L_{\mathcal{P}}^2)$ satisfying $\langle u \rangle_z = 0 \; \forall' z \in (\,a,b\,)$ and $\Delta_2 u = f$ in the sense of periodic distributions in L, i.e. when using testing functions $\zeta \in C^\infty(L)$, that are periodic in x and y with respect to $\mathcal{P}$*

and vanish near the boundary $(\mathbf{R}^2 \times \{a\}) \cup (\mathbf{R}^2 \times \{b\})$. *The solution u belongs to* $W^k((\,a,b\,), H_P^{m+2})$, *and the following inequality holds true:*

$$(1.2) \qquad \|\partial_z^\nu u\|_{\mathcal{P},j+2} \leq c(j)\|\partial_z^\nu f\|_{\mathcal{P},j}$$

$\forall z,\ j \leq m$ *and* $\nu \leq k$ *with* $\nu + j + 2 \leq \max\{k, m+2\}$. *In particular we get* $\underline{\varepsilon}u \in W^k((\,a,b\,), H_P^{m+1})$. *If* $k \geq 1$, *then* $(\underline{\delta}u)_i \in W^{k-1}((\,a,b\,), H_P^{m+1})$ *for* $i = 1, 2$, *and* $(\underline{\delta}u)_3 \in W^k((\,a,b\,), H_P^m)$. *All corresponding norms can be estimated with the aid of (1.2). Moreover* $\underline{\varepsilon}u$ *and* $\underline{\delta}u$ *have zero mean values.*

PROOF: As $f \in W^k((\,a,b\,), H_P^m)$ implies that the Fourier coefficients $b_\kappa(z)$ belonging to f lie in $H^k((\,a,b\,))$ and that the z-derivatives of f may be calculated termwise, the maintained existence and uniqueness as well as regularity result readily from Proposition 1.1. Expanding $\underline{\varepsilon}u$ and $\underline{\delta}u$ into their Fourier series we see that the coefficients of order zero vanish whence so do their mean values. $\qquad\square$

The next lemma reveals the sufficiency of some necessary conditions to the flux functions φ and ψ, which will be derived from an assumed poloidal-toroidal decomposition in the main theorem named after the lemma.

Lemma 1.3: *Let* $V = (V_1, V_2, V_3)^{\mathrm{T}} \in L^2((\,a,b\,), L_P^2)^3$ *with* $\langle V \rangle_z = 0\ \forall z \in (\,a,b\,)$ *satisfy* $\langle \underline{k}, V \rangle = 0$, $\mathrm{div}\,V = 0$, $\underline{\varepsilon} \cdot V = 0$ *in the sense of periodic distributions in* L. *Then* $V = 0$ *in* $L^2((\,a,b\,), L_P^2)^3$.

PROOF: Put $V_3 = 0$ into the presumed condition

$$\int_L \langle V, \nabla\zeta \rangle\, dx\, dy\, dz = 0 = \int_L \langle V, \underline{\varepsilon}\zeta \rangle\, dx\, dy\, dz,$$

$\zeta(x, y, z) = \gamma(z)\phi(x, y)$ with $\gamma \in C_0^\infty(\,a,b\,)$, $\phi \in C_P^\infty$, in order to get the equations

$$(1.3) \qquad \int_{\mathbf{R}^2} \left(\frac{\partial\phi}{\partial x}(x, y) V_1(x, y, z) + \frac{\partial\phi}{\partial y}(x, y) V_2(x, y, z) \right) dx\, dy = 0$$

$$(1.4) \qquad \int_{\mathbf{R}^2} \left(\frac{\partial\phi}{\partial y}(x, y) V_1(x, y, z) - \frac{\partial\phi}{\partial x}(x, y) V_2(x, y, z) \right) dx\, dy = 0$$

for all $\phi \in C_P^\infty$ and almost all $z \in (\,a,b\,)$. Choosing in particular $\phi = \frac{\partial\xi}{\partial x}$ in (1.3), $\phi = \frac{\partial\xi}{\partial y}$ in (1.4) and adding (1.3), (1.4) we obtain

$$(1.5) \qquad \int_{\mathbf{R}^2} \Delta_2\xi(x, y) V_1(x, y, z)\, dx\, dy = 0 \quad \forall \xi \in C_P^\infty.$$

Hence, by means of Proposition 1.1, $V_1(x, y, z) = c_1(z)$ does not depend on x and y. Analogously take $\phi = \frac{\partial\xi}{\partial y}$ in (1.3), $\phi = \frac{\partial\xi}{\partial x}$ in (1.4) and subtract to get (1.5) with V_2 replaced by V_1 whence as above $V_2(x, y, z) = c_2(z)$. Recalling that V should have mean values zero we are set. $\qquad\square$

Remark. The zero mean value assumption must not be removed, because otherwise it would be impossible to exclude the case

$$V(x,y,z) = \begin{pmatrix} V_1(z) \\ V_2(z) \\ 0 \end{pmatrix}.$$

In fact, every such V (being smooth enough) serves as an example for a solenoidal field with vanishing third component and $\underline{\varepsilon} \cdot V = 0$, but which is not identical to zero.

The following theorem describes the correct representation of a solenoidal field $\underline{u}$ in the layer $L = \mathbf{R}^2 \times (a,b)$.

Theorem 1.4: *Let $\underline{u} = (u_1, u_2, u_3)^{\mathrm{T}}$ be a vector field defined in the layer L, the components of the field $\underline{u}$ shall satisfy u_1, $u_2 \in W^k((a,b), H_P^{m+1})$ and $u_3 \in W^{l+1}((a,b), H_P^n)$, k, l, m, $n \in \mathbf{N}_0$. Assume $\mathrm{div}\,\underline{u} = 0$ in L. Then there exist fields P, T, $F \in L^2((a,b), L_P^2)^3$ that are determined uniquely by their following properties:*

$$P = \underline{\delta}\varphi, \qquad T = \underline{\varepsilon}\psi$$

(in the sense of periodic distributions in L) with zero mean valued functions φ, $\psi \in L^2((a,b), L_P^2)$,

$$F(x,y,z) = (F_1(z), F_2(z), F_3)$$

independent of x, y, and

(1.6) $$\underline{u} = P + T + F.$$

Especially, $F(z) = \langle \underline{u} \rangle_z \in H^k((a,b))^3$. Moreover $\varphi \in W^{l+1}((a,b), H_P^{n+2})$, $\psi \in W^k((a,b), H_P^{m+2})$, and the estimates

$$\|\partial_z^\nu \varphi(\cdot, z)\|_{P, j+2} \leq c(j)\|\partial_z^\nu u_3(\cdot, z)\|_{P, j}$$

for $j \leq n$, $0 \leq \nu \leq l+1$ with $\nu + j + 2 \leq \max\{l+1, n+2\}$ and

$$\|\partial_z^\nu \psi(\cdot, z)\|_{P, j+2} \leq c(j)\|\partial_z^\nu \underline{\varepsilon} \cdot \underline{u}(\cdot, z)\|_{P, j}$$

for $j \leq m$, $0 \leq \nu \leq k$ with $\nu + j + 2 \leq \max\{k, m+2\}$ are valid. The regularity of P and T is inferred by that of φ and ψ.

PROOF: Because $\underline{\delta}\varphi$ and $\underline{\varepsilon}\psi$ have zero mean values (cf. Corollary 1.2), we get $\langle \underline{u} \rangle_z = F(z)$ necessarily. Then $F_3 \equiv \mathrm{const}$ follows from $\mathrm{div}\,\underline{u} = 0$. Thus let us consider $\underline{u} - F$ instead of $\underline{u}$ and assume now that $F \equiv 0$, i.e. $\underline{u}$ has mean value zero.

It is evident that for any toroidal field $T = \underline{\varepsilon}\psi$ the conditions $\mathrm{div}T = 0$, $\underline{\varepsilon} \cdot T = \Delta_2\psi$, $\langle \underline{k}, T \rangle = 0$ must hold true. Analogously $\mathrm{div}P = 0$, $\underline{\varepsilon} \cdot P = 0$, $\langle \underline{k}, P \rangle = -\Delta_2\varphi$ for any poloidal field $P = \underline{\delta}\varphi$. Therefore the claim $\underline{u} = P + T$ implies that $\underline{\varepsilon} \cdot T = \underline{\varepsilon} \cdot \underline{u}$ and $\langle \underline{k}, P \rangle = \langle \underline{k}, \underline{u} \rangle$. Thus by means of Lemma 1.3, P and T have to be unique. Moreover we see that necessarily

$$(1.7) \qquad -\Delta_2\varphi = \langle \underline{k}, \underline{u} \rangle, \quad -\Delta_2\psi = -\underline{\varepsilon} \cdot \underline{u}$$

for the flux functions φ and ψ. These equations can be solved in a unique way, as indicated by Corollary 1.2. Once that φ and ψ have been determined according to (1.7), we may apply Lemma 1.3 to $\underline{u} - P - T$. (Note that now $\underline{u}$ is assumed to be zero mean valued!) Thus we obtain $\underline{u} - P - T = 0$, i.e. the condition (1.7) is already sufficient for (1.6).

The regularity assertions stated in the theorem are an immediate consequence of Corollary 1.2. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Remark. It should be emphasized that (1.6) actually defines an orthogonal decomposition where orthogonality is meant to be taken with respect to the inner product

$$(f, g)_z = \int_{\mathcal{P}} f(x, y, z)\overline{g(x, y, z)}\, dx\, dy,$$

for almost all $z \in (a, b)$. For, if $a_\kappa(z)$ denote the Fourier coefficients of φ and $b_\kappa(z)$ those of ψ, then the coefficient vectors of P and T are

$$c_\kappa(z) = \begin{pmatrix} \frac{\partial}{\partial z}a_\kappa(z)i\kappa_1 \\ \frac{\partial}{\partial z}a_\kappa(z)i\kappa_2 \\ -a_\kappa(z)|\kappa|^2 \end{pmatrix} \quad \text{and} \quad d_\kappa(z) = \begin{pmatrix} b_\kappa(z)i\kappa_2 \\ -b_\kappa(z)i\kappa_1 \\ 0 \end{pmatrix},$$

respectively. Thus,

$$\int_{\mathcal{P}} \left\langle P(x, y, z), \overline{T(x, y, z)} \right\rangle dx\, dy = \sum_{\kappa \neq 0} c_\kappa(z)^{\mathrm{T}}\overline{d_\kappa(z)} = 0.$$

2. Applications to the Boussinesq-Equations. Remarks on Energy Stability

In what follows we set $a = -\frac{1}{2}$, $b = \frac{1}{2}$, $\mathcal{P} = (-\frac{\pi}{\alpha}, \frac{\pi}{\alpha})^2$ for some $\alpha > 0$ (the wave-number). This setting turns out to be useful in what follows. It's clear that the results of the first section apply to this situation. The Boussinesq-equations (0.2) over the infinite layer $\mathbf{R}^2 \times (-\frac{1}{2}, \frac{1}{2})$ are usually connected with two types of boundary-conditions, namely:

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1st *Case: Rigid boundaries:* $\underline{u} = 0$ at $z = \pm\frac{1}{2}$, $\vartheta = 0$ at $z = \pm\frac{1}{2}$.

2nd *Case: Stress-free boundaries:* $\partial_z u_x = \partial_z u_y = u_z = 0$ at $z = \pm\frac{1}{2}$, $\vartheta = 0$ at $z = \pm\frac{1}{2}$.

Now let $\underline{u} \in W^2((-\frac{1}{2}, \frac{1}{2}), H_p^2)^3$ be solenoidal. We decompose it according to Theorem 1.4 into P, T, F. If $\underline{u}$ satisfies one of the boundary-conditions above we obtain

$$F_3 \equiv 0$$

(observe that $\int_P \Delta_2 \varphi(x, y, \pm\frac{1}{2})\, dx\, dy = 0$). Exploiting that $\underline{u}$ is solenoidal we arrive in the first case at

$$\Delta_2 \varphi = 0 \quad \text{at} \quad z = \pm\tfrac{1}{2},$$
$$\partial_z \Delta_2 \varphi = 0 \quad \text{at} \quad z = \pm\tfrac{1}{2}.$$

This implies $\varphi = \partial_z \varphi = 0$ at $z = \pm\frac{1}{2}$. Thus $\partial_y(\partial_y \psi + F_1) = \partial_y^2 \psi = 0$, $\partial_x(-\partial_x \psi + F_1) = -\partial_x^2 \psi = 0$ at $z = \pm\frac{1}{2}$ and consequently $\psi = 0$, $F = 0$ at $z = \pm\frac{1}{2}$. In an analogous fashion we obtain in the second case $\varphi = \partial_z^2 \varphi = 0$, $\partial_z \psi = 0$, $\partial_z F = 0$ at $z = \pm\frac{1}{2}$. Evidently our conditions on φ, ψ, ϑ and F imply the corresponding ones on $\underline{u}$, ϑ.

In the next step we express the Boussinesq-equations in terms of φ, ψ, ϑ and F. To this end we are needing certain regularity assumptions on $\underline{u}$, ϑ (which implies some regularity for $\nabla \pi$ or π). It's not necessary to state these assumptions here. We will express them in terms of φ, ψ, ϑ, F. In [6] it's shown, amongst other things, that the initial-boundary value problem in φ, ψ, ϑ, F is well posed within appropriate spaces. We will indicate here, however, how one obtains from the system for φ, ψ, ϑ and F the original system (0.1). To be sure of this step is of course an absolute necessity.

/2

From (0.2) it follows that

$$(2.1) \quad \begin{cases} \operatorname{curl}(\underline{u}' - \Delta\underline{u} + \underline{u} \cdot \nabla\underline{u} - \sqrt{R}\,\vartheta\underline{k}) &= 0, \\ \operatorname{div}\underline{u} &= 0, \\ \operatorname{Pr}\vartheta' - \Delta\vartheta + \operatorname{Pr}\underline{u} \cdot \nabla\vartheta - \sqrt{R}\,u_z &= 0. \end{cases}$$

By Lemma 1.3 this is equivalent to

$$(2.2) \quad \begin{cases} \left\langle \operatorname{curl}(\underline{u}' - \Delta\underline{u} + \underline{u} \cdot \nabla\underline{u} - \sqrt{R}\,\vartheta\underline{k}), \underline{k} \right\rangle &= 0, \\ \left\langle \operatorname{curl}\operatorname{curl}(\underline{u}' - \Delta\underline{u} + \underline{u} \cdot \nabla\underline{u} - \sqrt{R}\,\vartheta\underline{k}), \underline{k} \right\rangle &= 0, \\ \int_P \operatorname{curl}(\underline{u}' - \Delta\underline{u} + \underline{u} \cdot \nabla\underline{u} - \sqrt{R}\,\vartheta\underline{k})\, dx\, dy &= 0, \\ \operatorname{div}\underline{u} &= 0, \\ \operatorname{Pr}\vartheta' - \Delta\vartheta + \operatorname{Pr}\underline{u} \cdot \nabla\vartheta - \sqrt{R}\,u_z &= 0. \end{cases}$$

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Taking the decomposition

$$\underline{u} = \operatorname{curl}\operatorname{curl}\varphi\underline{k} + \operatorname{curl}\psi\underline{k} + F = \underline{\delta}\varphi + \underline{\varepsilon}\psi + F$$

from Theorem 1.4 we infer from the first two equations in (2.2) that

(2.3)
$$(-\Delta)(-\Delta_2)\varphi' + \Delta^2(-\Delta_2)\varphi - \sqrt{R}(-\Delta_2)\vartheta +$$
$$+\underline{\delta} \cdot ((\underline{\delta}\varphi + \underline{\varepsilon}\psi + F) \cdot \nabla(\underline{\delta}\varphi + \underline{\varepsilon}\psi + F)) = 0,$$

(2.4)
$$(-\Delta_2)\psi' + (-\Delta)(-\Delta_2)\psi -$$
$$-\underline{\varepsilon} \cdot ((\underline{\delta}\varphi + \underline{\varepsilon}\psi + F) \cdot \nabla(\underline{\delta}\varphi + \underline{\varepsilon}\psi + F)) = 0,$$

whereas the last one reads

(2.5)
$$\operatorname{Pr}\vartheta' - \Delta\vartheta + \operatorname{Pr}(\underline{\delta}\varphi + \underline{\varepsilon}\psi + F) \cdot \nabla\vartheta - \sqrt{R}(-\Delta_2)\varphi = 0.$$

The third equation in (2.2) reads

(2.6)
$$\partial_z(F_1') - \partial_z^3 F_1 + (\frac{\alpha}{2\pi})^2\partial_z^2\int_{\mathcal{P}}(-\Delta_2\varphi)(\partial_x\partial_z\varphi + \partial_y\psi)\,dx\,dy = 0,$$

(2.7)
$$\partial_z(F_2') - \partial_z^3 F_2 + (\frac{\alpha}{2\pi})^2\partial_z^2\int_{\mathcal{P}}(-\Delta_2\varphi)(\partial_y\partial_z\varphi - \partial_x\psi)\,dx\,dy = 0.$$

Integrating the first and second row in (0.2) over $\mathcal{P}$ we obtain

(2.8)
$$F_1' - \partial_z^2 F_1 + (\frac{\alpha}{2\pi})^2\int_{\mathcal{P}}\underline{u} \cdot \nabla u_x\,dx\,dy = -(\frac{\alpha}{2\pi})^2\int_{\mathcal{P}}\partial_x\pi\,dx\,dy,$$

(2.9)
$$F_2' - \partial_z^2 F_2 + (\frac{\alpha}{2\pi})^2\int_{\mathcal{P}}\underline{u} \cdot \nabla u_y\,dx\,dy = -(\frac{\alpha}{2\pi})^2\int_{\mathcal{P}}\partial_y\pi\,dx\,dy.$$

Since

$$\int_{\mathcal{P}}\underline{u} \cdot \nabla u_x\,dx\,dy = \partial_z\int_{\mathcal{P}}(-\Delta_2\varphi)(\partial_x\partial_z\varphi + \partial_y\psi)\,dx\,dy$$

$$\int_{\mathcal{P}}\underline{u} \cdot \nabla u_y\,dx\,dy = \partial_z\int_{\mathcal{P}}(-\Delta_2\varphi)(\partial_y\partial_z\varphi - \partial_x\psi)\,dx\,dy$$

the equations (2.6), (2.7) imply that $\int_{\mathcal{P}}\partial_x\pi\,dx\,dy$, $\int_{\mathcal{P}}\partial_y\pi\,dx\,dy$ depend on t only. Thus $\pi = \tilde{\pi} + c(t)^{\mathrm{T}} \cdot \binom{x}{y} + d(t,z)$ where the two-vector c is arbitrary and depends on t only. d is subject to the condition

(2.10)
$$\partial_z\Big(d + (\frac{\alpha}{2\pi})^2\int_{\mathcal{P}}u_z^2\,dx\,dy\Big) = \sqrt{R}(\frac{\alpha}{2\pi})^2\int_{\mathcal{P}}\vartheta\,dx\,dy$$

which is implied by integration of the third row in (0.2) over $\mathcal{P}$. $\tilde{\pi}$ is periodic in x, y with respect to $\mathcal{P}$ and fulfills $\int_{\mathcal{P}}\tilde{\pi}\,dx\,dy = 0$.

The system we want to work with is now given by (2.3), (2.4), (2.5) and

$$(2.11) \qquad F' - \partial_z^2 F + (\frac{\alpha}{2\pi})^2 \begin{pmatrix} \int_{\mathcal{P}} \underset{\sim}{u} \cdot \nabla \tilde{u}_x \, dx \, dy \\ \int_{\mathcal{P}} \underset{\sim}{u} \cdot \nabla \tilde{u}_y \, dx \, dy \end{pmatrix} = 0.$$

We have set $c_1 = c_2 \equiv 0$, and $\tilde{u} = \underline{\delta}\varphi + \underline{\varepsilon}\psi$ in the decomposition $\underline{u} = \underline{\delta}\varphi + \underline{\varepsilon}\psi + F$. Observe that $\int_{\mathcal{P}} \underline{u} \cdot \nabla\underline{u} \, dx \, dy = \int_{\mathcal{P}} \tilde{u} \cdot \nabla\tilde{u} \, dx \, dy$ and that $\tilde{u}$ is simply that part of $\underline{u}$ which has mean value 0 over $\mathcal{P}$. The system in question is written now in matrix-form. Set

$$\mathcal{B} = \begin{pmatrix} (-\Delta)(-\Delta_2) & 0 & 0 & 0 & 0 \\ 0 & (-\Delta_2) & 0 & 0 & 0 \\ 0 & 0 & \Pr I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{pmatrix},$$

$$\mathcal{A} = \begin{pmatrix} \Delta^2(-\Delta_2) & 0 & 0 & 0 & 0 \\ 0 & (-\Delta)(-\Delta_2) & 0 & 0 & 0 \\ 0 & 0 & (-\Delta) & 0 & 0 \\ 0 & 0 & 0 & (-\partial_z^2) & 0 \\ 0 & 0 & 0 & 0 & (-\partial_z^2) \end{pmatrix},$$

$$\mathcal{C} = \begin{pmatrix} 0 & 0 & (-\Delta_2) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ (-\Delta_2) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\Phi = \begin{pmatrix} \varphi \\ \psi \\ \vartheta \\ F_1 \\ F_2 \end{pmatrix}.$$

The nonlinear part is denoted by $\mathcal{M}(\Phi)$. Then the system we are going to consider is simply

$$(2.12) \qquad \begin{cases} \mathcal{B}\Phi' + \mathcal{A}\Phi - \sqrt{R}\,\mathcal{C}\Phi + \mathcal{M}(\Phi) & = 0 \\ \Phi(0) & = \Phi_0 \end{cases}$$

under boundary conditions as indicated in the beginning of this section. To obtain (0.1) from this system we set $\underline{u} = \underline{\delta}\varphi + \underline{\varepsilon}\psi + F$. From the solution which is constructed in [6] it's easily seen that F', $\partial_z^2 F$ possess a further z-derivative for $t > 0$. Then (2.3), (2.4), (2.5), (2.6), (2.7) are at our disposal

which are a reformulation of (2.2). (2.2) however yields (2.1), i.e. we obtain $\underline{u}' - \Delta\underline{u} + \underline{u} \cdot \nabla\underline{u} - \sqrt{R}\,\vartheta\underline{k} = -\nabla\pi$ with a periodic pressure gradient. π is decomposed as before into $\pi = \tilde{\pi} + d$, where d is subject to (2.10).

The system (2.12) is most easily treated within an appropriate Hilbert space H, where $\mathcal{A}$, $\mathcal{B}$ become strictly positive definite selfadjoint operators and $\mathcal{C}$ is hermitian. As Hilbert space H we take

$$H = \mathcal{H}_M \times \mathcal{H}_M \times \mathcal{H} \times \mathcal{H}^1 \times \mathcal{H}^1$$

with $\varphi \in \mathcal{H}_M$, $\psi \in \mathcal{H}_M$, $\vartheta \in \mathcal{H}$, $F_1 \in \mathcal{H}^1$, $F_2 \in \mathcal{H}^1$ in the case of rigid boundaries. Here

$$
\begin{aligned}
\mathcal{H}_M &= \left\{ \tilde{\varphi} \;\middle|\; \tilde{\varphi} \in W^0((-\tfrac{1}{2},\tfrac{1}{2}), L_P^2), \int_\mathcal{P} \tilde{\varphi}\, dx\, dy = 0 \right\}, \\
\mathcal{H} &= W^0((-\tfrac{1}{2},\tfrac{1}{2}), L_P^2), \\
\mathcal{H}^1 &= \left\{ f \;\middle|\; f \in L^2((-\tfrac{1}{2},\tfrac{1}{2})) \right\}.
\end{aligned}
$$

$\mathcal{H}_M$, $\mathcal{H}$ are made Hilbert spaces in the usual way. For $\mathcal{H}^1$ we choose the inner product

$$(f,g) = (\frac{2\pi}{\alpha})^2 \int_{-\frac{1}{2}}^{\frac{1}{2}} f \cdot \overline{g}\, dz.$$

In the case of stress-free boundaries we take

$$H = \mathcal{H}_M \times \mathcal{H}_M \times \mathcal{H} \times \mathcal{H}_M^1 \times \mathcal{H}_M^1$$

with $\mathcal{H}_M^1$ being the closed subspace of $\mathcal{H}^1$ which consists of the f having vanishing mean value over $(-\tfrac{1}{2},\tfrac{1}{2})$. Now we can define $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ by defining $A = \Delta^2(-\Delta_2)$, $\tilde{B} = (-\Delta)(-\Delta_2)$ for φ, $B = (-\Delta)(-\Delta_2)$ for ψ and $-\Delta$ for ϑ, $-\partial_z^2$ for F_1, F_2, $-\Delta_2$ for φ, ψ and ϑ as well. Observe that we have two different kinds of operators $(-\Delta)(-\Delta_2)$ in the case of stress-free boundaries.

Definition 2.1: *We expand φ, ψ, ϑ in series*

$$(2.13) \qquad \varphi(x,y,z) = \frac{\alpha}{2\pi} \sum_{\kappa \in \mathbb{Z}^2 \setminus \{0\}} a_\kappa(z) e^{i\alpha\kappa \cdot (x,y)},$$

$$(2.14) \qquad \psi(x,y,z) = \frac{\alpha}{2\pi} \sum_{\kappa \in \mathbb{Z}^2 \setminus \{0\}} b_\kappa(z) e^{i\alpha\kappa \cdot (x,y)},$$

$$(2.15) \qquad \vartheta(x,y,z) = \frac{\alpha}{2\pi} \sum_{\kappa \in \mathbb{Z}^2} c_\kappa(z) e^{i\alpha\kappa \cdot (x,y)},$$

the series being convergent in $W^0((-\tfrac{1}{2},\tfrac{1}{2}), L_P^2)$. Set

$$
\begin{aligned}
A_\kappa &= (\alpha^2|\kappa|^2 - \partial_z^2)^2 \\
&= \alpha^4|\kappa|^4 - 2\alpha^2|\kappa|^2\partial_z^2 + \partial_z^4, \quad \kappa \in \mathbb{Z}^2 \setminus \{0\}, \\
\mathcal{D}(A_\kappa) &= \left\{ f \mid f \in H^4((-\tfrac{1}{2}, \tfrac{1}{2})) \text{ with either } f = \partial_z f = 0 \right.\\
&\qquad\qquad \left. \text{ at } z = \pm\tfrac{1}{2} \text{ or } f = \partial_z^2 f = 0 \text{ at } z = \pm\tfrac{1}{2} \right\}.
\end{aligned}
$$

Then A_κ is a strictly positive definite selfadjoint operator in $L^2((-\tfrac{1}{2}, \tfrac{1}{2}))$. We define $A = \Delta^2(-\Delta_2)$ on

$$
\begin{aligned}
\mathcal{D}(A) = \Big\{ \varphi \mid \varphi &\in \mathcal{H}_M, \quad \varphi \text{ is expanded as in (2.13)}, \\
&a_\kappa \in \mathcal{D}(A_\kappa), \quad \kappa \in \mathbb{Z}^2 \setminus \{0\}, \\
&\sum_{\kappa \in \mathbb{Z}^2 \setminus \{0\}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \alpha^4|\kappa|^4 |A_\kappa a_\kappa|^2 \, dz < +\infty \Big\}
\end{aligned}
$$

by

$$
A\varphi = \frac{\alpha}{2\pi} \sum_{\kappa \in \mathbb{Z}^2 \setminus \{0\}} \alpha^2|\kappa|^2 A_\kappa a_\kappa e^{i\alpha\kappa \cdot}.
$$

Set

$$
\begin{aligned}
\tilde{B}_\kappa &= \alpha^2|\kappa|^2 - \partial_z^2, \quad \kappa \in \mathbb{Z}^2 \setminus \{0\}, \\
\mathcal{D}(\tilde{B}_\kappa) &= \left\{ f \mid f \in H^2((-\tfrac{1}{2}, \tfrac{1}{2})) \text{ with } f = 0 \text{ at } z = \pm\tfrac{1}{2} \right\}.
\end{aligned}
$$

Then $\tilde{B}_\kappa$ is a strictly positive definite selfadjoint operator in $L^2((-\tfrac{1}{2}, \tfrac{1}{2}))$. We define $\tilde{B} = (-\Delta)(-\Delta_2)$ on

$$
\begin{aligned}
\mathcal{D}(\tilde{B}) = \Big\{ \varphi \mid \varphi &\in \mathcal{H}_M, \quad \varphi \text{ is expanded as in (2.13)}, \\
&a_\kappa \in \mathcal{D}(\tilde{B}_\kappa), \quad \kappa \in \mathbb{Z}^2 \setminus \{0\}, \\
&\sum_{\kappa \in \mathbb{Z}^2 \setminus \{0\}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \alpha^4|\kappa|^4 \big|\tilde{B}_\kappa a_\kappa\big|^2 \, dz < +\infty \Big\}
\end{aligned}
$$

by

$$
\tilde{B}\varphi = \frac{\alpha}{2\pi} \sum_{\kappa \in \mathbb{Z}^2 \setminus \{0\}} \alpha^2|\kappa|^2 \tilde{B}_\kappa a_\kappa e^{i\alpha\kappa \cdot}.
$$

Let

$$
\begin{aligned}
B_\kappa &= \alpha^2|\kappa|^2 - \partial_z^2, \quad \kappa \in \mathbb{Z}^2 \setminus \{0\}, \\
\mathcal{D}(B_\kappa) &= \left\{ f \mid f \in H^2((-\tfrac{1}{2}, \tfrac{1}{2})) \text{ with either } f = 0 \right.\\
&\qquad\qquad \left. \text{ at } z = \pm\tfrac{1}{2} \text{ or } \partial_z f = 0 \text{ at } z = \pm\tfrac{1}{2} \right\}.
\end{aligned}
$$

Then B_κ is a strictly positive definite selfadjoint operator in $L^2((-\frac{1}{2}, \frac{1}{2}))$ ($|\kappa| \geq 1!$). We define $B = (-\Delta)(-\Delta_2)$ on

$$\mathcal{D}(B) = \Big\{ \psi \mid \psi \in \mathcal{H}_M, \quad \psi \text{ is expanded as in (2.14)},$$
$$b_\kappa \in \mathcal{D}(B_\kappa), \quad \kappa \in \mathbb{Z}^2 \setminus \{0\},$$
$$\sum_{\kappa \in \mathbb{Z}^2 \setminus \{0\}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \alpha^4 |\kappa|^4 |B_\kappa b_\kappa|^2 \, dz < +\infty \Big\}$$

by

$$B\psi = \frac{\alpha}{2\pi} \sum_{\kappa \in \mathbb{Z}^2 \setminus \{0\}} \alpha^2 |\kappa|^2 B_\kappa b_\kappa e^{i\alpha\kappa \cdot \cdot}.$$

It is now obvious how $-\Delta$ is defined for ϑ, i.e. in $\mathcal{H}$, $-\Delta_2$ in $\mathcal{H}_M$ or $\mathcal{H}$, $-\partial_z^2$ in $\mathcal{H}^1$ or $\mathcal{H}_M^1$.

Next we prove

Theorem 2.2: *A, B are strictly positive definite selfadjoint operators in H.*

PROOF: The assertion is proved by showing that A, B, $\tilde{B}$, $-\Delta$, $-\partial_z^2$ are strictly positive definite selfadjoint operators in the corresponding Hilbert spaces. It's sufficient to do this for A: Either the proofs are very similar to each other or the assertion is well known as in the case of $-\partial_z^2$. As for A it's clear that A is densely defined and hermitian. Now we have to show that

$$(A \pm i)\varphi = f$$

is uniquely solvable in $\mathcal{D}(A)$ for any given $f \in \mathcal{H}_M$. To this end we take the expansion

$$f = \frac{\alpha}{2\pi} \sum_{\kappa \in \mathbb{Z}^2 \setminus \{0\}} f_\kappa e^{i\alpha\kappa \cdot \cdot}$$

and set

$$a_\kappa = \frac{1}{\alpha^2 |\kappa|^2} \Big(A_\kappa \pm \frac{i}{\alpha^2 |\kappa|^2} \Big)^{-1} f_\kappa.$$

It's clear that φ with expansion coefficients a_κ is the required solution. The strict positivity follows from Parseval's equation. $\qquad\qquad\square$

The choice of the various Hilbert spaces of functions with vanishing mean values corresponds to the invariance properties of the nonlinear terms. For these and other invariance properties see [6, sect. IV]. The norm $\|A \cdot \|$ is equivalent with the norm of $W^4((-\frac{1}{2}, \frac{1}{2}), H_P^6)$. Corresponding equivalences

hold for the other operators. See [6, sect. III]. The spaces within which we solve (2.12) are now at hand. We are looking for solutions Φ with

$$(2.16) \qquad\qquad \Phi \quad \in \quad L^2((0,T), \mathcal{D}(\mathcal{A})),$$

$$(2.17) \qquad\qquad \Phi' \quad \in \quad L^2((0,T), \mathcal{D}(\mathcal{B})),$$

and, as a result of interpolation,

$$(2.18) \qquad\qquad \nabla \mathcal{B} \Phi \quad \in \quad C^0([0,T], H)$$

where ∇ refers to each component of $\mathcal{B}\Phi$. In particular Φ_0 is required to fulfill $\|\nabla\mathcal{B}\Phi_0\| < +\infty$. This construction is carried through in [6, sect. IV]. In general T is not allowed to exceed a maximal finite value, unless $\mathrm{Pr} = +\infty$ or the solution represents a convection roll. Imposing compatibility conditions the regularity behaviour near $t = 0$ can be studied.

When constructing the solution with properties (2.16), (2.17), (2.18) one is faced with a characteristic difficulty in the case of rigid boundaries. Δ^2 in the first row of (2.12) is no longer the square in the operator-theoretical sense of $(-\Delta)$ in front of $(-\Delta_2)\varphi'$ as it is for stress-free boundaries. Therefore the range $\mathrm{Pr} \in [\frac{1}{2}, 2]$ has to be excluded. This difficulty can be removed if one considers a somewhat weaker form of solutions as is done in [6, sect. V].

Now we want to consider the energy-equality for solutions with properties (2.16), (2.17), (2.18). It reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|\mathcal{B}^{\frac{1}{2}}\Phi(t)\right\|^2 + 2\left\|\mathcal{A}^{\frac{1}{2}}\Phi(t)\right\|^2 - 2\sqrt{R}(\mathcal{C}\Phi(t), \Phi(t)) = 0$$

with $\left\|\mathcal{B}^{\frac{1}{2}}\Phi(t)\right\|$ as (kinetic) energy at time t. It's therefore interesting to ask for

$$\sqrt{R_{\min}(\alpha^2)} = \inf_{\Phi \in \mathcal{D}(\mathcal{A})\backslash\{0\}} \frac{(\mathcal{A}\Phi, \Phi)}{|(\mathcal{C}\Phi, \Phi)|}$$

in dependence on α^2. If $0 < R \le R_{\min}(\alpha^2)$ then the energy can be estimated a-priori independently of R and is monotonically non increasing or even decays exponentially (if $R < R_{\min}(\alpha^2)$). This variational problem can be attacked in the way that one wants to find

$$\inf \frac{\|\nabla \underline{u}\|^2 + \|\nabla\vartheta\|^2}{2|(u_z, \vartheta)|}$$

where the infimum has to be taken over $(\underline{u}, \vartheta)$ with $\underline{u} \neq 0$, $\vartheta \neq 0$, $\nabla \cdot \underline{u} = 0$, $\underline{u}$, ϑ periodic with respect to any rectangle. The condition $\nabla \cdot \underline{u} = 0$ is covered by introducing a Lagrange-multiplier. Then the Euler-Lagrange equations are considered. After solving them one is confronted with the problem to show that the solution is a minimizer of the functional under consideration

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on suitable subclasses of the admissible $\underline{u}$, ϑ. Thus it seems to be easier to start with Courant's classical method of finding the eigenvalues of a compact selfadjoint operator. Here we only prove an estimate from below for $(\mathcal{A}\Phi, \Phi)$ which turns out to be sharp in the case of stress-free boundaries. We have

Proposition 2.3: *For $\Phi \in \mathcal{D}(\mathcal{A})$ the following estimate holds:*

$$(2.19) \qquad (\mathcal{A}\Phi, \Phi) \geq \min_{\kappa \in \mathbf{Z}^2 \backslash \{0\}} \sqrt{R_i(\alpha^2 |\kappa|^2)} \cdot |(\mathcal{C}\Phi, \Phi)|, \quad i = 0, 1,$$

with

$$R_0(\alpha^2) = \frac{(\alpha^2 + \pi^2)^3}{\alpha^2}, \quad \alpha > 0,$$

in the case of stress-free boundaries, and

$$R_1(\alpha^2) = \frac{(\alpha^2 + \pi^2)}{\alpha^2}(\alpha^4 + \lambda(\alpha^2))$$

in the case of rigid boundaries. Here $\lambda(\alpha^2)$ is the smallest eigenvalue of $\partial_z^4 - 2\alpha^2 \partial_z^2$ in $L^2((-\frac{1}{2}, \frac{1}{2}))$ under boundary conditions $f = \partial_z f = 0$ at $z = \pm\frac{1}{2}$. In the case of stress-free boundaries we have

$$R_{\min}(\alpha^2) = \min_{\kappa \in \mathbf{Z}^2 \backslash \{0\}} R_0(\alpha^2 |\kappa|^2).$$

PROOF: If we take Parseval's equation for $(\mathcal{A}\Phi, \Phi)$ and $(\mathcal{C}\Phi, \Phi)$ together with the expansions (2.13), (2.15) the estimate (2.19) is easily shown. One only has to use the extremal property of the smallest eigenvalue of $-\partial_z^2$, $(-\partial_z^2)^2$ under Dirichlet-0-conditions (which are π^2, π^4) and of $\partial_z^4 - 2\alpha^2 \partial_z^2$ under boundary conditions $f = \partial_z f = 0$ at $z = \pm\frac{1}{2}$. It's easily seen that $\min_{\kappa \in \mathbf{Z}^2 \backslash \{0\}} R_i(\alpha^2 |\kappa|^2)$ is assumed for some κ_i, $i = 0, 1$. In the case of stress-free boundaries R_0 assumes its minimal value $R_c = 27\pi^4/4$ in $\alpha_c = \pi/\sqrt{2}$. Thus $\min_{\kappa \in \mathbf{Z}^2 \backslash \{0\}} R_0(\alpha^2 |\kappa|^2) = R_0(\alpha^2)$ for $\alpha \geq \alpha_c$. The functional $(\mathcal{A}\Phi, \Phi)/(\mathcal{C}\Phi, \Phi)$ assumes the value $\sqrt{R_0(\alpha^2)}$ in $\Phi = (\varphi, 0, \vartheta, 0, 0)^\mathsf{T}$ with $\varphi(x, y, z) = \cos \alpha x \cos \pi z$, $\vartheta = (\alpha^2 + \pi^2)^{\frac{1}{2}} \alpha \varphi$. The situation is different for $\alpha \in (0, \alpha_c)$, since R_0 is monotonically decreasing on $(0, \alpha_c]$. If $\kappa_0 = (\kappa_{01}, \kappa_{02})$ and if $R_0(\alpha^2 |\kappa_0|^2)$ is minimal, then $(\mathcal{A}\Phi, \Phi)/(\mathcal{C}\Phi, \Phi)$ assumes $\sqrt{R_0(\alpha^2 |\kappa_0|^2)}$ in $\Phi = (\varphi, 0, \vartheta, 0, 0)^\mathsf{T}$ with $\varphi(x, y, z) = \cos \alpha \kappa_{01} x \cos \alpha \kappa_{02} y \cos \pi z$ and $\vartheta = (\alpha^2 |\kappa_0|^2 + \pi^2)^{\frac{1}{2}} \alpha |\kappa_0| \varphi$. The assertion is proved. $\square$

Anything what was said before remains true if the periodicity cell $(-\frac{\pi}{\alpha}, \frac{\pi}{\alpha})^2$ is replaced by a rectangle $(-\frac{\pi}{\alpha}, \frac{\pi}{\alpha}) \times (-\frac{\pi}{\beta}, \frac{\pi}{\beta})$, with the single modification that $\alpha^2 |\kappa|^2$ has to be replaced by $\alpha^2 \kappa_1^2 + \beta^2 \kappa_2^2$. Thus, if one wants that the energy is monotonically non increasing for any Φ being periodic in x, y with respect to a rectangle, one needs $R \leq R_c$ in the case of stress-free boundaries.

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