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L^p-Decay Rates for Homogeneous Wave-Equations

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Abstract. This paper deals with the question of the asymptotic behaviour of the solution of Cauchy's problem for

$$\frac{\partial^2 u}{\partial t^2} - \Delta u + m u = 0, \quad m \geqq 0,$$

if t tends to infinity. This question is of particular interest for the behaviour in the large of the solutions of nonlinear wave-equations.

0. Preliminary Remarks and Notations

In this paper we calculate the decay-rates of the $L^p(\mathbb{R}^n)$-norms, $\infty \geqq p \geqq 2$, of the solutions of Cauchy's problem for

$$\frac{\partial^2 u}{\partial t^2} - \Delta u + m u = 0, \quad m \geqq 0,$$

if t tends to infinity. The space dimension n is supposed to be greater than or equal 3. Since the papers of Strauss [4] and Segal [3] appeared it was clear that such estimates are of some importance for the question of the behaviour in the large of the solutions of Cauchy's problem for

$$\frac{\partial^2 u}{\partial t^2} - \Delta u + m u + F\left(u, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}\right) = 0$$

with a nonlinear term F. A great part of our estimates deals with expressions

$$\int_0^t \cos(-\Delta + m)^{\frac{1}{2}}(t-s) f(s)\, ds$$

$$\int_0^t (-\Delta + m)^{-\frac{1}{2}} \sin(-\Delta + m)^{\frac{1}{2}}(t-s) f(s)\, ds,$$

which are occurring in the treatment of nonlinear problems.

The method employed is similar to that outlined in [7], where we treated a perturbative problem for the equation

$$\frac{\partial^2 u}{\partial t^2} - \Delta u + m u + F\left(u, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}\right) = 0$$

in higher space dimensions. It essentially uses the well-known formulas for the classical solution of the homogeneous wave-equation which can be found in [2].

We introduce some notations: $\mathbb{N}$ is the set of all positive integers, $\mathbb{R}^+$ is the set of all nonnegative reals. Let

$$D_j = \frac{1}{i} \frac{\partial}{\partial x_j}, \quad 1 \leq j \leq n.$$

For n-tuples $\alpha = (\alpha_1, \ldots, \alpha_n)$ of nonnegative integers we set

$$|\alpha| = \sum_{\nu=1}^{n} \alpha_\nu, \quad D^\alpha = \prod_{j=1}^{n} D_j^{\alpha_j}, \quad x^\alpha = \prod_{j=1}^{n} x_j^{\alpha_j}, \quad x \in \mathbb{R}^n.$$

$H^{k,p}(\mathbb{R}^n)$, $1 \leq p \leq \infty$, $k \in \mathbb{N} \cup \{0\}$, is the Banach-space of all functions u with distributional derivatives up to the order k lying in $L^p(\mathbb{R}^n)$ (continuous and bounded for $p = \infty$) with norms

$$\|u\|_{H^{k,p}(\mathbb{R}^n)} = \Big\{ \sum_{|\alpha| \leq k} \|D^\alpha u\|^p_{L^p(\mathbb{R}^n)} \Big\}^{1/p}$$

and

$$\|u\|_{H^{k,\infty}(\mathbb{R}^n)} = \sum_{|\alpha| \leq k} \sup_{x \in \mathbb{R}^n} |D^\alpha u(x)|$$

respectively. For $p = 2$ we write $\|.\|_k$ instead of $\|u\|_{H^{k,2}(\mathbb{R}^n)}$. The $L^2(\mathbb{R}^n)$-norm is denoted by $\|.\|$. Let X be a Banach-space. $C^\nu([a,b], X)$, $-\infty < a < b < +\infty$, is the Banach-space of all ν-times continuously differentiable mappings $u: [a,b] \to X$ with norm

$$\sum_{\lambda=0}^{\nu} \sup_{t \in [a,b]} \left\| \left(\frac{d^\lambda}{dt^\lambda} u \right)(t) \right\|$$

$C_{\mathrm{loc}}(\mathbb{R}^+, X)$ is the set of all ν-times continuously differentiable mappings $u: \mathbb{R}^+ \to X$. For a subset $G \subset \mathbb{R}^n$ we denote with $C^\nu(G)$ the Banach-space of all functions having continous and bounded derivatives up to the order ν.

At last we set

$$K_\rho(y) = \{x \mid x \in \mathbb{R}^n, |x - y| < \rho\}, \quad \rho > 0, \ y \in \mathbb{R}^n.$$

$c_1, c_2, \ldots$ are positive constants which are independent of the occurring functions.

I. The Case of Odd Space Dimensions and Vanishing Mass

Let $\varphi \in C_0^{(n-1)/2+2}(\mathbb{R}^n)$, $\psi \in C_0^{(n-1)/2+1}(\mathbb{R}^n)$. The solution of the homogeneous wave-equation

$$\frac{\partial^2}{\partial t^2} u - \Delta u = 0$$

with initial-data $u(0,x)=\varphi(x)$, $\left(\frac{\partial}{\partial t}u\right)(0,x)=\psi(x)$ is given by

$$\Phi(x,t)=\sum_{\nu=0}^{(n-3)/2}(\nu+1)\,a_\nu t^\nu\left(\frac{\partial^\nu}{\partial t^\nu}Q_1\right)(x,t)$$
$$+t\sum_{\nu=0}^{(n-3)/2}a_\nu t^\nu\left(\left(\frac{\partial^{\nu+1}}{\partial t^{\nu+1}}Q_1\right)(x,t)+\left(\frac{\partial^\nu}{\partial t^\nu}Q_2\right)(x,t)\right)$$

(see [2], p. 394) where the a_ν's are constants and where

$$Q_1(x,t)=\frac{1}{\omega_n}\int_{\Omega_n}\varphi(x+t\xi)\,d\xi,$$
$$Q_2(x,t)=\frac{1}{\omega_n}\int_{\Omega_n}\psi(x+t\xi)\,d\xi.$$

Ω_n denotes the unit-sphere in the n-dimensional Euclidean space $\mathbb{R}^n$, ω_n is its surface, $d\xi$ is its surface-element. Carrying out the differentiation in the formula for Φ and using the theorem of Gauss in the form

$$\int_{\Omega_n}f(t\xi)\,\xi_\nu\,d\xi=t^{-(n-1)}\int_{|x|\leqq t}\frac{\partial}{\partial x_\nu}f(x)\,dx,\qquad f\in C^1(|x|\leqq t),\ t>0,$$

one gets the estimate

$$|\Phi(x,t)|\leqq c_1\Bigg\{|Q_1(x,t)|+t\,|Q_2(x,t)|$$
$$+\sum_{\nu=0}^{(n-3)/2}t^{\nu+1-(n-1)}\sum_{\substack{2\leqq|\alpha|\leqq(n-3)/2+2\\0\leqq|\gamma|=|\alpha|-2}}t^{-|\gamma|}\Big|\int_{K_t(0)}(D^\alpha\varphi)(x+y)\,y^\gamma\,dy\Big|$$
$$+\sum_{\nu=1}^{(n-3)/2+1}t^{\nu-(n-1)}\sum_{\substack{2\leqq|\alpha|\leqq(n-3)/2+1\\0\leqq|\gamma|=|\alpha|-2}}t^{-|\gamma|}\Big(\Big|\int_{K_t(0)}(D^\alpha\varphi)(x+y)\,y^\gamma\,dy\Big|$$
$$+\Big|\int_{K_t(0)}(D^\alpha\psi)(x+y)\,y^\gamma\,dy\Big|\Big)\Bigg\}.$$

Moreover, we have

$$|Q_1(x,t)|=\frac{1}{t\,\omega_n}\left|\sum_{\nu=1}^{n}\int_{\Omega_n}\varphi(x+t\xi)\,t\,\xi_\nu\,\xi_\nu\,d\xi\right|,$$
$$\leqq\frac{1}{t^n\,\omega_n}\sum_{\substack{0\leqq|\alpha|\leqq 1\\0\leqq|\gamma|\leqq 1}}\Big|\int_{K_t(0)}(D^\alpha\varphi)(x+y)\,y^\gamma\,dy\Big|.$$

Let us regard a term $t^{-|\gamma|}\int\limits_{K_t(0)}(D^\alpha\varphi)(x+y)\,y^\gamma\,dy$. Let $\lambda,\mu\geqq 0$, $1>\mu$, $1\geqq\lambda+\mu$. The Hausdorff-Young inequality immediately furnishes the relation

$$\|t^{-|\gamma|}\int\limits_{K_t(0)}(D^\alpha\varphi)(x+y)\,y^\gamma\,dy\|_{L^{1/(1-\lambda-\mu)}(\mathbb{R}^n)}\leqq\|D^\alpha\varphi\|_{L^{1/(1-\mu)}(\mathbb{R}^n)}\,t^{-|\gamma|}\,\|\,|y^\gamma|\,\|_{L^{1/(1-\mu)}(K_t(0))}$$
$$\leqq c_2\,t^{n(1-\lambda)}\,\|D^\alpha\varphi\|_{L^{1/(1-\mu)}(\mathbb{R}^n)}.$$

From this we obtain the inequality

$$\begin{aligned}\|\Phi(t)\|_{L^{1/(1-\lambda-\mu)}(\mathbb{R}^n)}\leqq c_3\,t^{n(1-\lambda)-(n-1)/2}\Big\{&t^{-(n-1)/2}\,\|\varphi\|_{L^{1/(1-\mu)}(\mathbb{R}^n)}\\&+t^{-(n-3)/2}\,\|\psi\|_{L^{1/(1-\mu)}(\mathbb{R}^n)}+\sum_{1\leqq|\alpha|\leqq(n-3)/2+2}\|D^\alpha\varphi\|_{L^{1/(1-\mu)}(\mathbb{R}^n)}\\&+\sum_{1\leqq|\alpha|\leqq(n-3)/2+1}\|D^\alpha\psi\|_{L^{1/(1-\mu)}(\mathbb{R}^n)}\Big\},\quad t\geqq 1.\end{aligned}\tag{1}$$

We remark that $\mu=0$, $\lambda=1$, $1/(1-\lambda-\mu)=+\infty$ is admitted as the preceding calculations show. So we get the results of Chapter I.A. of [6] as a special case. We will state our results in a form which is especially suitable for non-linear situations: Let a mapping

$$v\colon\ \mathbb{R}^+\to C_0^{(n-3)/2+1}(\mathbb{R}^n)$$

be given with supp $v(t)\subset K_{\rho(t)}(0)$, $\rho(t)$ a positive continuous function. Moreover, let

$$v(.)\in C^0_{\mathrm{loc}}(\mathbb{R}^+,H^{(n-3)/2+2+k,\infty}(\mathbb{R}^n)),\quad k\in\mathbb{N}.$$

Then we have for all multiindices α, $|\alpha|\leqq(n-3)/2+2+k$,

$$(D^\alpha v)(.)\in C^0_{\mathrm{loc}}(\mathbb{R}^+,L^q(\mathbb{R}^n)),\quad q\geqq 1,$$

and we get the following theorem:

Theorem 1. *Let $1/(1-\mu-\lambda)\geqq 2n/(n-2)$ with $1>\mu\geqq 0$, $1\geqq\lambda>0$, $1\geqq\lambda+\mu$, let*

$$v\colon\ \mathbb{R}^+\to H^{(n-3)/2+2+k,\infty}(\mathbb{R}^n),\quad k\in\mathbb{N},$$

be a mapping with the properties listed above. Then we have the inequalities ($t\geqq 1$)

$$\begin{aligned}&\Big\|\int_0^t(-\Delta)^{-\frac12}\sin(-\Delta)^{\frac12}(t-\sigma)\,v(\sigma)\,d\sigma\Big\|_{H^{k,1/(1-\lambda-\mu)}(\mathbb{R}^n)}\\&\leqq c_4\sum_{0\leqq|\alpha|\leqq(n-3)/2+1+k}\int_0^t(t-\sigma+1)^{n(1-\lambda)-(n-1)/2}\\&\quad\cdot\{\|D^\alpha v(\sigma)\|+\|D^\alpha v(\sigma)\|_{L^{1/(1-\mu)}(\mathbb{R}^n)}\}\,d\sigma,\end{aligned}\tag{2}$$

$$\begin{aligned}&\Big\|\int_0^t\cos(-\Delta)^{\frac12}(t-\sigma)\,v(\sigma)\,d\sigma\Big\|_{H^{k,1/(1-\lambda-\mu)}(\mathbb{R}^n)}\\&\leqq c_5\sum_{0\leqq|\alpha|\leqq(n-3)/2+2+k}\int_0^t(t-\sigma+1)^{n(1-\lambda)-(n-1)/2}\\&\quad\cdot\{\|D^\alpha v(\sigma)\|+\|D^\alpha v(\sigma)\|_{L^{1/(1-\mu)}(\mathbb{R}^n)}\}\,d\sigma.\end{aligned}\tag{3}$$

Proof. In order to prove (2) we split up the integration into two parts: one from 0 to $t-1$ and one from $t-1$ to t. For the treatment of the first part one only has to use (1). For the second part we have to use the estimate ($g\in H^{[n/2]+1,2}(\mathbb{R}^n)$)

$$\begin{aligned}\|g\|_{L^{1/(1-\lambda-\mu)}(\mathbb{R}^n)} &\leq \|g\|_{C^0(\mathbb{R}^n)}^{(1-\lambda-\mu)\left(\frac{1}{1-\lambda-\mu}-\frac{2n}{n-2}\right)} \left\{\frac{n-1}{n-2}\prod_{i=1}^{n}\left\|\frac{\partial g}{\partial x_i}\right\|^{\frac{1}{n}}\right\}^{\frac{2n}{n-2}(1-\lambda-\mu)},\\ &\leq c_6\|\nabla g\|_{[n/2]}^{(1-\lambda-\mu)\left(\frac{1}{1-\lambda-\mu}-\frac{2n}{n-2}\right)} \left\{\frac{n-1}{n-2}\prod_{i=1}^{n}\left\|\frac{\partial g}{\partial x_i}\right\|^{\frac{1}{n}}\right\}^{\frac{2n}{n-2}(1-\lambda-\mu)},\end{aligned}$$

which follows from a well-known Sobolev-inequality and from Hilfssatz 4 in [5]. Inequality (3) follows directly from (1).

Remark 1. For $2\leq 1/(1-\lambda-\mu)<2n/(n-2)$ the right side of (2) changes into

$$c_7\sum_{0\leq|\alpha|\leq(n-3)/2+1+k}\int_0^t(t-\sigma+1)^{1+n(1-\lambda)-(n-1)/2}\{\|D^\alpha v(\sigma)\|+\|D^\alpha v(\sigma)\|_{L^{1/(1-\mu)}(\mathbb{R}^n)}\}\,d\sigma.$$

The right side of (3) changes analogously.

II. The Case of Even Space Dimensions and Vanishing Mass

Let $\varphi\in C_0^{n/2+2}(\mathbb{R}^n)$, $\psi\in C^{(n-2)/2+2}(\mathbb{R}^n)$. Then the solution of the homogeneous wave-equation

$$\frac{\partial^2}{\partial t^2}u-\Delta u=0$$

with initial-data φ and ψ is given by

$$\begin{aligned}\Phi(x,t)=&\sum_{v=0}^{(n-2)/2}(v+1)\,b_v t^v\left(\frac{\partial^v}{\partial t^v}G_1\right)(x,t)\\ &+t\sum_{v=0}^{(n-2)/2}b_v t^v\left(\left(\frac{\partial^{v+1}}{\partial t^{v+1}}G_1\right)(x,t)+\left(\frac{\partial^v}{\partial t^v}G_2\right)(x,t)\right)\end{aligned}$$

(see [2], p. 394). The b_v's are constants and we have

$$G_1(x,t)=\frac{2\Gamma\left(\frac{n+1}{2}\right)}{\pi\Gamma\left(\frac{n}{2}\right)t^{n-1}}\int_0^t\frac{r^{n-1}}{\omega_n(t^2-r^2)^{\frac{1}{2}}}\int_{\Omega_n}\varphi(x+r\xi)\,d\xi\,dr,$$

$$G_2(x,t)=\frac{2\Gamma\left(\frac{n+1}{2}\right)}{\pi\Gamma\left(\frac{n}{2}\right)t^{n-1}}\int_0^t\frac{r^{n-1}}{\omega_n(t^2-r^2)^{\frac{1}{2}}}\int_{\Omega_n}\psi(x+r\xi)\,d\xi\,dr.$$

Since

$$G_1(x,t)=\frac{1}{\omega_{n+1}}\int_{\Omega_{n+1}}\varphi(x_1+t\xi_1,\dots,x_n+t\xi_n)\,d\xi,$$

$$G_2(x,t)=\frac{1}{\omega_{n+1}}\int_{\Omega_{n+1}}\psi(x_1+t\xi_1,\dots,x_n+t\xi_n)\,d\xi,$$

we can easily carry out the differentiation with respect to t in the formula for $\Phi(x,t)$. We get as a standard term

$$t^{-|\tilde\gamma|}\int_0^t\frac{r^{n-1}}{\omega_n(t^2-r^2)^{\frac12}}\int_{\Omega_n}(D^\alpha\varphi)(x+r\xi)(r\xi)^{\tilde\gamma}\,d\xi\,dr,\qquad 0\leqq|\tilde\gamma|=|\alpha|\leqq(n-2)/2+1.$$

For $t\geqq 1$ and $0<\varepsilon<t$ we obtain as in [6], Chapter I.A.,

$$\begin{aligned}&\left|\int_{t-\varepsilon}^t\frac{r^{n-1}}{\omega_n(t^2-r^2)^{\frac12}}\int_{\Omega_n}(D^\alpha\varphi)(x+r\xi)(r\xi)^{\tilde\gamma}\,d\xi\,dr\right|\\&\quad\leqq\frac{2\sqrt{\varepsilon}}{\sqrt{t}}\sup_{t-\varepsilon\leqq r\leqq t}\left|r^{n-1}\int_{\Omega_n}(D^\alpha\varphi)(x+r\xi)(r\xi)^{\tilde\gamma}\,d\xi\right|\\&\quad\leqq\frac{2\sqrt{\varepsilon}}{\sqrt{t}}\cdot c_7\Big\{\sum_{\substack{2\leqq|\bar\alpha|\leqq|\alpha|+1\\0\leqq|\bar\gamma|=|\alpha|-1=|\tilde\gamma|-1}}t^{-(|\tilde\gamma|-1)}\int_{K_t(0)}|(D^{\bar\alpha}\varphi)(x+y)||y^{\bar\gamma}|\,dy\\&\qquad+\sum_{\substack{0\leqq|\bar\alpha|\leqq 1\\0\leqq|\bar\gamma|\leqq 1}}t^{-|\bar\gamma|}\int_{K_t(0)}|(D^{\bar\alpha}\varphi)(x+y)||y^{\bar\gamma}|\,dy\Big\}.\end{aligned}\tag{4}$$

Furthermore, the trivial relation

$$\int_0^{t-\varepsilon}\frac{r^{n-1}}{\omega_n(t^2-r^2)^{\frac12}}\int_{\Omega_n}(D^\alpha\varphi)(x+r\xi)(r\xi)^{\tilde\gamma}\,d\xi=\frac{1}{\omega_n}\int_{K_{t-\varepsilon}(0)}(D^\alpha\varphi)(x+y)\,y^{\tilde\gamma}\frac{dy}{(t^2-|y|^2)^{\frac12}}$$

and the inequality

$$\begin{aligned}&\left|\int_0^{t-\varepsilon}\frac{r^{n-1}}{\omega_n(t^2-r^2)^{\frac12}}\int_{\Omega_n}(D^\alpha\varphi)(x+r\xi)(r\xi)^{\tilde\gamma}\,d\xi\right|\\&\quad\leqq\frac{1}{\omega_n\sqrt{\varepsilon t}}\int_{K_{t-\varepsilon}(0)}|(D^\alpha\varphi)(x+y)|\,|y^{\tilde\gamma}|\,dy\end{aligned}\tag{5}$$

hold. Summing up all these terms and proceeding as in Chapter I we get

Theorem 2. *Let a mapping*

$$v:\ \mathbb{R}^+\to C_0^{n/2}(\mathbb{R}^n)$$

be given with $\operatorname{supp} v(t)\subset K_{\rho(t)}(0)$, $\rho(t)$ *a positive continuous function. Let*

$$v(\cdot)\in C^0_{\mathrm{loc}}(\mathbb{R}^+,H^{n/2+1+k,\infty}(\mathbb{R}^n)).$$

Let $1/(1-\lambda-\mu) \geqq 2n/(n-2)$ *with* $1>\mu\geqq 0$, $1\geqq\lambda>0$, $1\geqq\lambda+\mu$. *Then the following estimates hold:*

$$\left\| \int_0^t (-\Delta)^{-\frac{1}{2}} \sin(-\Delta)^{\frac{1}{2}}(t-\sigma)\, v(\sigma)\, d\sigma \right\|_{H^{k,1/(1-\lambda-\mu)}(\mathbb{R}^n)} \leqq c_8 \sum_{0\leqq|\alpha|\leqq n/2+k} \int_0^t (t-\sigma+1)^{n(1-\lambda)-(n-1)/2} \cdot \{\|D^\alpha v(\sigma)\| + \|D^\alpha v(\sigma)\|_{L^{1/(1-\mu)}(\mathbb{R}^n)}\}\, d\sigma, \tag{6}$$

$$\left\| \int_0^t \cos(-\Delta)^{\frac{1}{2}}(t-\sigma)\, v(\sigma)\, d\sigma \right\|_{H^{k,1/(1-\lambda-\mu)}(\mathbb{R}^n)} \leqq c_9 \sum_{0\leqq|\alpha|\leqq n/2+1+k} \int_0^t (t-\sigma+1)^{n(1-\lambda)-(n-1)/2} \cdot \{\|D^\alpha v(\sigma)\| + \|D^\alpha v(\sigma)\|_{L^{1/(1-\mu)}(\mathbb{R}^n)}\}\, d\sigma. \tag{7}$$

Remark 2. For the function $\Phi(x,t)$ we have proved the inequality

$$\|\Phi(t)\|_{L^{1/(1-\lambda-\mu)}(\mathbb{R}^n)} \leqq c_{10}\, t^{n(1-\lambda)-(n-1)/2} \{ t^{-n/2} \|\varphi\|_{L^{1/(1-\mu)}(\mathbb{R}^n)} + t^{-(n-2)} \|\psi\|_{L^{1/(1-\mu)}(\mathbb{R}^n)} + \sum_{1\leqq|\alpha|\leqq n/2+1} \|D^\alpha \varphi\|_{L^{1/(1-\mu)}(\mathbb{R}^n)} + \sum_{1\leqq|\alpha|\leqq n/2} \|D^\alpha \psi\|_{L^{1/(1-\mu)}(\mathbb{R}^n)} \}, \quad t\geqq 1, \tag{8}$$

where $1>\mu\geqq 0$, $1\geqq\mu+\lambda$, $\lambda>0$. If $2\leqq 1/(1-\lambda-\mu)<2n/(n-2)$ in Theorem 2 we have to replace the integral on the right side of (6) by

$$\int_0^t (t-\sigma+1)^{1+n(1-\lambda)-(n-1)/2} \{\|D^\alpha v(\sigma)\| + \|D^\alpha v(\sigma)\|_{L^{1/(1-\mu)}(\mathbb{R}^n)}\}\, d\sigma.$$

The right side of (7) changes analogously.

III. The Case of Nonvanishing Mass

This case is easily reduced to the case of vanishing mass by Hadamard's descent-method: The solution v of the wave-equation

$$\frac{\partial^2 v}{\partial t^2} - \Delta v + m v = 0, \quad m>0,$$

with sufficiently smooth initial-data $v(0,x)=0$, $(\partial v/\partial t)(0,x)=\psi(x)$ fulfills the relation

$$\frac{\partial^2 \tilde{v}}{\partial t^2} - \Delta\tilde{v} = 0 \quad \text{over } \mathbb{R}^+ \times \mathbb{R}^{n+1},$$

where $\tilde{v} = e^{-i\sqrt{m}x_{n+1}} v$, $\tilde{v}(0,x)=0$, $\left(\frac{\partial\tilde{v}}{\partial t}\right)(0,x) = e^{-i\sqrt{m}x_{n+1}}\psi$. This guarantees an at $\frac{1}{2}$ higher decay-rate if we could show that the factor $e^{-i\sqrt{m}x_{n+1}}$ does not affect the estimates of Chapters I and II.

Firstly, let $n+1$ be even. Let $\psi \in C_0^{(n+1)/2+1}(\mathbb{R}^n)$,

$$\tilde{\psi} := e^{-i\sqrt{m}x_{n+1}}\psi .$$

We have to estimate a term

$$\int_0^t \frac{r^n}{\omega_{n+1}(t^2-r^2)^{\frac{1}{2}}} \int_{\Omega_{n+1}} (D^\alpha \tilde{\psi})(x+r\xi)\,\xi^\gamma\,d\xi. \tag{9}$$

Proceeding as in Chapter II one easily gets the inequality

$$\begin{aligned} t^{-(|\bar{\gamma}|-1)} & \Big| \int_{K_r(0)} (D^{\tilde{\alpha}} \tilde{\psi})(x+y)\, y^{\bar{\bar{\gamma}}}\, dy \Big| \\ & \leqq t^{-(|\bar{\bar{\gamma}}|-\bar{\bar{\gamma}}_{n+1}-1)} \int_{\sum_{\nu=1}^n y_\nu^2 \leqq r^2} |D^{(\tilde{\alpha}_1,\dots,\tilde{\alpha}_n)} \psi(x_1+y_1,\dots,x_n+y_n)\, y^{(\bar{\bar{\gamma}}_1,\dots,\bar{\bar{\gamma}}_n)}| \\ & \quad \cdot \Big| \int_{y_{n+1}^2 \leqq r^2 - \sum_{\nu=1}^n y_\nu^2} e^{-i\sqrt{m}(x_{n+1}+y_{n+1})} (y_{n+1}/r)^{\bar{\bar{\gamma}}_{n+1}}\, dy_{n+1} \Big|\, dy_1 \dots dy_n \end{aligned} \tag{10}$$

where

$$\tilde{\alpha} = (\tilde{\alpha}_1, \dots, \tilde{\alpha}_{n+1}), \quad \bar{\bar{\gamma}} = (\bar{\bar{\gamma}}_1, \dots, \bar{\bar{\gamma}}_{n+1}), \quad 2 \leqq |\tilde{\alpha}| \leqq |\alpha|+1,$$
$$0 \leqq |\bar{\bar{\gamma}}| = |\alpha|-1, \quad 1 \leqq t, \quad t-\varepsilon \leqq r \leqq t, \quad \varepsilon > 0.$$

Since the second integral on the right side of (10) is bounded by a constant which depends only on m we get the same estimates as in Chapter II. The terms with $0 \leqq |\tilde{\alpha}| \leqq 1$, $0 \leqq |\bar{\bar{\gamma}}| \leqq 1$ can be treated analogously by supplementing the factor $\xi_1^2 + \dots + \xi_{n+1}^2 = 1$ in (9). Furthermore, we have

$$\begin{aligned} & \left| t^{-|\bar{\gamma}|} \int_0^{t-\varepsilon} \frac{r^n}{\omega_{n+1}(t^2-r^2)^{\frac{1}{2}}} \int_{\Omega_{n+1}} (D^{\tilde{\alpha}} \tilde{\psi})(x+r\xi)\,\xi^{\bar{\gamma}}\,d\xi \right| \\ & \leqq \frac{1}{\omega_{n+1}\sqrt{\varepsilon t}} \int_{\sum_{\nu=1}^n y_\nu^2 \leqq (t-\varepsilon)^2} \left| D^{(\alpha_1,\dots,\alpha_n)} \psi(x_1+y_1,\dots,x_n+y_n) \frac{y^{(\bar{\gamma}_1,\dots,\bar{\gamma}_n)}}{t^{|\bar{\gamma}|-\bar{\gamma}_{n+1}}} \right| \\ & \quad \cdot \left| \int_{y_{n+1}^2 \leqq r^2 - \sum_{\nu=1}^n y_\nu^2} e^{-i\sqrt{m}(x_{n+1}+y_{n+1})} \frac{\sqrt{\varepsilon t}\, y_{n+1}^{\bar{\gamma}_{n+1}}}{t^{\bar{\gamma}_{n+1}} \left\{ t^2 - \sum_{\nu=1}^{n+1} y_\nu^2 \right\}^{\frac{1}{2}}} dy_{n+1} \right| dy_1 \dots dy_n . \end{aligned} \tag{11}$$

As before the second integral on the right side of (11) is bounded by a constant which depends only on m and ε and we obtain the same estimates as in II.

Secondly, let $n+1$ be odd. This case is much simplier than the foregoing one. As in (10) we only have to integrale separately a product which contains $e^{-i\sqrt{m}(x_{n+1}+y_{n+1})}$ and then can proceed as in Chapter I.

We conclude with two theorems:

Theorem 3. *Let*

$$N = \begin{cases} n+2, & n \text{ odd}, \\ n+1, & n \text{ even}. \end{cases}$$

Let $1 \geqq \lambda > 0$, $1 > \mu \geqq 0$, $1 \geqq \lambda + \mu$, $1/(1-\lambda-\mu) \geqq 2$. *Let* $\varphi \in C_0^{(N-1)/2+2}(\mathbb{R}^n)$, $\psi \in C_0^{(N-1)/2+1}(\mathbb{R}^n)$. *The solution of the homogeneous wave-equation*

$$\frac{\partial^2 \Phi}{\partial t^2} - \Delta \Phi + m\Phi = 0, \quad m > 0,$$

with initial-data φ *and* ψ *fulfills the estimate*

$$\begin{aligned} \|\Phi(t)\|_{L^{1/(1-\lambda-\mu)}(\mathbb{R}^n)} \leqq c_{11}\, t^{n(1-\lambda)-n/2} \{ & t^{-N/2} \|\varphi\|_{L^{1/(1-\mu)}(\mathbb{R}^n)} \\ & + t^{-(N-3)/2} \|\psi\|_{L^{1/(1-\mu)}(\mathbb{R}^n)} + \sum_{1 \leqq |\alpha| \leqq (N+1)/2} \|D^\alpha \varphi\|_{L^{1/(1-\mu)}(\mathbb{R}^n)} \\ & + \sum_{1 \leqq |\alpha| \leqq (N-1)/2} \|D^\alpha \psi\|_{L^{1/(1-\mu)}(\mathbb{R}^n)} \}, \quad t \geqq 1. \end{aligned}$$

Theorem 4. *Let* N, λ, μ *be as in Theorem* 3. *Let a mapping*

$$v:\ \mathbb{R}^+ \to C_0^{(N-1)/2}(\mathbb{R}^n)$$

be given with $\operatorname{supp} v(t) \subset K_{\rho(t)}(0)$, $\rho(t)$ *a positive continuous function. Let*

$$v(.) \in C^0_{\text{loc}}(\mathbb{R}^+, H^{(N-1)/2+1+k,\infty}(\mathbb{R}^n)), \quad k \in \mathbb{N}.$$

Then the following estimates hold (n odd e.g.)

$$\begin{aligned} & \left\| \int_0^t (-\Delta+m)^{-\frac{1}{2}} \sin(-\Delta+m)^{\frac{1}{2}}(t-\sigma)\, v(\sigma)\, d\sigma \right\|_{H^{k,1/(1-\lambda-\mu)}(\mathbb{R}^n)} \\ & \leqq c_{12} \sum_{0 \leqq |\alpha| \leqq (N-1)/2+k} \int_0^t (t-\sigma+1)^{n(1-\lambda)-n/2} \\ & \quad \cdot \{ \|D^\alpha(-\Delta+m)^{-\frac{1}{2}} v(\sigma)\| + \|D^\alpha v(\sigma)\|_{L^{1/(1-\mu)}(\mathbb{R}^n)} \}\, d\sigma . \end{aligned} \tag{12}$$

$$\begin{aligned} & \left\| \int_0^t \cos(-\Delta+m)^{\frac{1}{2}}(t-\sigma)\, v(\sigma)\, d\sigma \right\|_{H^{k,1/(1-\lambda-\mu)}(\mathbb{R}^n)} \\ & \leqq c_{13} \sum_{0 \leqq |\alpha| \leqq (N-1)/2+1+k} \int_0^t (t-\sigma+1)^{n(1-\lambda)-n/2} \\ & \quad \cdot \{ D^\alpha(-\Delta+m)^{-\frac{1}{2}} v(\sigma)\| + \|D^\alpha v(\sigma)\|_{L^{1/(1-\mu)}(\mathbb{R}^n)} \}\, d\sigma . \end{aligned} \tag{13}$$

The proofs follow directly from the preceding calculations. With regard to (12) we remark that now we have not to avoid a difficulty at $t=\sigma$ since $(-\Delta+m)^{\frac{1}{2}}$ has a bounded inverse.

IV. Further Estimates in the Case of Nonvanishing Mass

There is a gap between our estimates in Chapters I, II and III since our estimates in III involve derivatives of one order higher than the corresponding ones in Chapters I and II but give a better decay-rate. It is the aim of this chapter to show that the estimates of I and II remain still valid in the case of nonvanishing mass or are even better.

Let n be odd. Then the solution of

$$\frac{\partial^2}{\partial t^2} u - \Delta u + m u = 0, \quad m > 0,$$

with initial-data

$$u(0, x) = 0, \quad \left(\frac{\partial u}{\partial t}\right)(0, x) = \psi(x) \in C^{(n-1)/2+1}(\mathbb{R}^n)$$

has the representation

$$u(t, x) = t \prod_{(n-1)/2} (t H)$$

with the differential operator

$$\prod_{(n-1)/2} (t v) = \sum_{v=0}^{(n-1)/2} a_v t^v \frac{\partial^v}{\partial t^v} v(t, x).$$

The a_v's are constants and we have

$$H(x, t) = \frac{n}{t^n} \int_0^t \rho^{n-1} J_0\left(-\sqrt{m}\sqrt{t^2-\rho^2}\right) Q(x, \rho)\, d\rho,$$

$$Q(x, \rho) = \frac{1}{\omega_n} \int_{\Omega_n} \psi(x + r\xi)\, d\xi$$

(compare [2], p. 408). J_0 is the Bessel-function of order 0. We list some well known facts on Bessel-functions

$$J_\lambda\left(-\sqrt{m}\sqrt{t^2-\rho^2}\right) = \left(\frac{-\sqrt{m}\sqrt{t^2-\rho^2}}{2}\right)^\lambda \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{(m(t^2-\rho^2))^k}{2^k \Gamma(k+\lambda+1)}, \quad \lambda \in \mathbb{N} \cup \{0\}, \tag{14}$$

which we will need later on:

$$\frac{d}{dx}\left(J_\lambda(x)/x^\lambda\right) = -J_{\lambda+1}(x)/x^\lambda, \quad x \in \mathbb{R}, \tag{15}$$

$$|J_0(x)| \leqq 1, \quad x \in \mathbb{R}, \tag{16}$$

$$|J_\lambda(x)| \leqq \frac{1}{\sqrt{2}}, \quad x \in \mathbb{R},\ \lambda \geqq 1 \tag{17}$$

We have for $\mathbb{N} \ni \bar{v} \geqq 1$:

$$\frac{\partial^{\bar{v}}}{\partial t^{\bar{v}}} \int_0^t \rho^{n-1} \frac{\partial}{\partial t} J_0\left(-\sqrt{m}\sqrt{t^2-\rho^2}\right) Q(x, \rho)\, d\rho$$

$$= \frac{\partial^{\bar{v}-1}}{\partial t^{\bar{v}-1}} \left\{ t^{n-1} J_0(0)\, Q(x, t) + \int_0^t \rho^{n-1} \frac{\partial}{\partial t} J_0\left(-\sqrt{m}\sqrt{t^2-\rho^2}\right) Q(x, \rho)\, d\rho \right\}.$$

On applying (15) we get

$$\begin{aligned}
&\int_0^t \rho^{n-1} \frac{\partial}{\partial t} J_0\left(-\sqrt{m}\sqrt{t^2-\rho^2}\right) Q(x,\rho)\, d\rho \\
&\quad = -t \int_0^t \rho^{n-1} \frac{J_1\left(-\sqrt{m}\sqrt{t^2-\rho^2}\right)}{\sqrt{t^2-\rho^2}} \sqrt{m}\, Q(x,\rho)\, d\rho \\
&\quad = -\frac{t^n}{2} \int_{\Omega_{n+1}} J_1\left(-t\sqrt{m}\sqrt{\xi_{n+1}^2}\right) \psi(x_1+t\,\xi_1, \dots, x_n+t\,\xi_n)\, d\xi .
\end{aligned}$$

Let us regard a term

$$\begin{aligned}
&\left| t^{n+1-(\bar{\nu}-1-\mu)} \frac{\partial^\mu}{\partial t^\mu} \int_{\Omega_{n+1}} J_1\left(-t\sqrt{m}\sqrt{\xi_{n+1}^2}\right) \psi(x_1+t\,\xi_1, \dots, x_n+t\,\xi_n)\, d\xi \right| \\
&\quad \leq c_{14}\, t^{n+1-(\bar{\nu}-1-\mu)} \sum_{\substack{0\leq\rho\leq\mu \\ \rho+|\alpha|=\mu,\ |\gamma|=|\alpha| \\ \nu=1,\dots,n+1}} \left| \int_{\Omega_{n+1}} J_1^{(\rho)}\left(-t\sqrt{m}\sqrt{\xi_{n+1}^2}\right) \right. \\
&\qquad \left. \cdot (D^\alpha \psi)(x_1+t\,\xi_1, \dots, x_n+t\,\xi_n)\, \xi_\nu^2\, \xi_1^{\gamma_1} \cdot \dots \cdot \xi_n^{\gamma_n} \left(\sqrt{m}\sqrt{\xi_{n+1}^2}\right)^{\gamma_{n+1}} d\xi \right| .
\end{aligned}$$

Using the theorem of Gauss we obtain the relation

$$\begin{aligned}
&\int_{\Omega_{n+1}} J_1^{(\rho)}\left(-t\sqrt{m}\sqrt{\xi_{n+1}^2}\right)(D^\alpha \psi)(x_1+t\,\xi_1, \dots, x_n+t\,\xi_n) \\
&\qquad \cdot \xi_1^{\gamma_1} \cdot \dots \cdot \xi_\nu^{\gamma_\nu+1} \cdot \dots \cdot \xi_n^{\gamma_n} \left(\sqrt{m}\sqrt{\xi_{n+1}^2}\right)^{\gamma_{n+1}} \xi_\nu\, d\xi \\
&\quad = \frac{1}{t^{|\alpha|+1+n}} \int_{K_t(0)} \frac{\partial}{\partial y_\nu} \left\{ J_1^{(\rho)}\left(-\sqrt{m}\sqrt{y_{n+1}^2}\right)(D^\alpha \psi)(x_1+y_1, \dots, x_n+y_n) \right. \\
&\qquad \left. \cdot y_1^{\gamma_1} \cdot \dots \cdot y_\nu^{\gamma_\nu+1} \cdot \dots \cdot y_n^{\gamma_n} \left(\sqrt{m}\sqrt{y_{n+1}^2}\right)^{\gamma_{n+1}} \right\} dy_1 \dots dy_{n+1} .
\end{aligned} \tag{18}$$

Let us regard the terms

$$\frac{1}{t^{\gamma_{n+1}+1}} \int_{-\{t^2-\sum_{\nu=1}^n y_\nu^2\}^{\frac{1}{2}}}^{\{t^2-\sum_{\nu=1}^n y_\nu^2\}^{\frac{1}{2}}} \frac{\partial}{\partial y_{n+1}} \left\{ \left(J_1^{(\rho)}\left(-\sqrt{m}\sqrt{y_{n+1}^2}\right)\right) y_{n+1} \left(\sqrt{m}\sqrt{y_{n+1}^2}\right)^{\gamma_{n+1}} \right\} dy_{n+1}, \tag{19}$$

$$\frac{1}{t^{\gamma_{n+1}}} \int_{-\{t^2-\sum_{\nu=1}^n y_\nu^2\}^{\frac{1}{2}}}^{\{t^2-\sum_{\nu=1}^n y_\nu^2\}^{\frac{1}{2}}} J_1^{(\rho)}\left(-\sqrt{m}\sqrt{y_{n+1}^2}\right) \frac{\partial}{\partial y_{n+1}} \left(y_{n+1}\left(\sqrt{m}\sqrt{y_{n+1}^2}\right)^{\gamma_{n+1}}\right) dy_{n+1}, \tag{20}$$

$$\frac{1}{t^{\gamma_{n+1}}} \int_{-\{t^2-\sum_{\nu=1}^n y_\nu^2\}^{\frac{1}{2}}}^{\{t^2-\sum_{\nu=1}^n y_\nu^2\}^{\frac{1}{2}}} J_1^{(\rho)}\left(-\sqrt{m}\sqrt{y_{n+1}^2}\right)\left(\sqrt{m}\sqrt{y_{n+1}^2}\right)^{\gamma_{n+1}} dy_{n+1}, \tag{21}$$

$$\frac{1}{t^{\gamma_{n+1}}} \int_{-\{t^2-\sum_{\nu=1}^n y_\nu^2\}^{\frac{1}{2}}}^{\{t^2-\sum_{\nu=1}^n y_\nu^2\}^{\frac{1}{2}}} J_1^{(\rho)}\left(-\sqrt{m}\sqrt{y_{n+1}^2}\right) y_{n+1}\left(\sqrt{m}\sqrt{y_{n+1}^2}\right)^{\gamma_{n+1}} dy_{n+1}. \tag{22}$$

It is not difficult to see that

$$\left|\frac{1}{a^k}\int_0^a J_1^{(\rho)}(x)\,x^k\,dx\right| \leqq C(\rho,k), \qquad a>0,\ k\in\mathbb{N}\cup\{0\}$$

with a positive constant $C(\rho,k)$ independent of a.

Using the Hausdorff-Young inequality we finally obtain ($\nu\geqq\bar\nu\geqq 1$, $1\geqq\lambda>0$, $1>\mu\geqq 0$, $1\geqq\lambda+\mu$)

$$\left\| t^\nu \frac{1}{t^{n+\nu-\bar\nu}}\frac{\partial^{\bar\nu-1}}{\partial t^{\bar\nu-1}}\int_0^t \rho^{n-1}\frac{\partial}{\partial t}J_0\left(-\sqrt{m}\sqrt{t^2-\rho^2}\right)Q(x,\rho)\,d\rho\right\|_{L^{1/(1-\lambda-\mu)}(\mathbb{R}^n)}$$
$$\leqq c_{15}\, t^{(1-\lambda)n-(n-1)/2}\,\|\psi\|_{H^{(n-1)/2,\,1/(1-\mu)}(\mathbb{R}^n)}, \qquad t\geqq 1.$$

On applying the methods developed in Chapter I one sees that

$$\frac{t^{\nu+1}}{t^{n+\nu-\bar\nu}}\left|\frac{\partial^{\bar\nu-1}}{\partial t^{\bar\nu-1}}t^{n-1}J_0(0)\,Q(x,t)\right|$$

can be estimated in the same way. It remains the term

$$\frac{n}{t^{n-1}}\left|\int_0^t \rho^{n-1}J_0\left(-\sqrt{m}\sqrt{t^2-\rho^2}\right)Q(x,\rho)\,d\rho\right| \leqq \frac{n}{t^{n-1}}\int_{K_t(0)}|\psi(x+y)|\,dy \tag{23}$$

which now turns out to be the trivial one. Since

$$\frac{\partial}{\partial t}\left(\sin(-\Delta+m)^{\frac12}t\,(-\Delta+m)^{\frac12}\psi\right)=\cos(-\Delta+m)^{\frac12}t\,\psi$$

we have proved the following theorem:

Theorem 5. *Let n be odd. Let*

$$\varphi\in C_0^{k+(n-1)/2+2}(\mathbb{R}^n),$$
$$\psi\in C_0^{k+(n-1)/2+1}(\mathbb{R}^n), \qquad k\in\mathbb{N}\cup\{0\}.$$

Let $1\geqq\lambda>0$, $1>\mu\geqq 0$, $1\geqq\lambda+\mu$. Then the following inequalities hold:

$$\|\sin(-\Delta+m)^{\frac12}t\,(-\Delta+m)^{-\frac12}\psi\|_{H^{k,1/(1-\lambda-\mu)}(\mathbb{R}^n)}$$
$$\leqq c_{16}\,t^{n(1-\lambda)-(n-1)/2}\,\|\psi\|_{H^{k+(n-1)/2,\,1/(1-\mu)}(\mathbb{R}^n)},$$
$$\|\cos(-\Delta+m)^{\frac12}t\,\varphi\|_{H^{k,1/(1-\lambda-\mu)}(\mathbb{R}^n)}$$
$$\leqq c_{17}\,t^{n(1-\lambda)-(n-1)/2}\|\varphi\|_{H^{k+1+(n-1)/2,\,1/(1-\mu)}(\mathbb{R}^n)}, \qquad t\geqq 1.$$

There are of course corresponding estimates for the terms

$$\int_0^t \sin(-\Delta+m)^{\frac12}(t-\sigma)(-\Delta+m)^{-\frac12}f(\sigma)\,d\sigma,$$
$$\int_0^t \cos(-\Delta+m)^{\frac12}(t-\sigma)\,f(\sigma)\,d\sigma,$$

if $f(t,x)$ is a sufficiently regular function with compact support in x (compare Chapter III).

Now we have to treat the case of even space-dimensions. The solution of

$$\frac{\partial^2 u}{\partial t^2} - \Delta u + m u = 0$$

with initial-data $u(0,x)=0$, $(\partial u/\partial t)(0,x)=\psi(x)$ has the representation

$$u(t,x) = t\, P_{(n-2)/2}(t\, G)$$

with the differential operator

$$P_{(n-2)/2}(t\, v) = \sum_{\nu=0}^{(n-2)/2} b_\nu\, t^\nu \frac{\partial^\nu}{\partial t^\nu} v(t,x).$$

Here the b_ν's are constants and moreover, we have

$$G(t,x) = \frac{2\Gamma\left(\frac{n+1}{2}\right)}{t^{n-1}\sqrt{\pi}\,\Gamma\left(\frac{n}{2}\right)} \int_0^t \frac{\rho^{n-1}}{\sqrt{t^2-\rho^2}} \cosh\left(i\sqrt{m}\sqrt{t^2-\rho^2}\right) Q(x,\rho)\, d\rho,$$

$$Q(x,\rho) = \frac{1}{\omega_n} \int_{\Omega_n} \psi(x_1+\rho\,\xi_1, \ldots, x_n+\rho\,\xi_n)\, d\xi.$$

Let us look into the question whether this case could be treated as the foregoing one. We have

$$\frac{2}{t^{n-1}} \int_0^t \frac{\rho^{n-1}}{\sqrt{t^2-\rho^2}} \cosh\left(i\sqrt{m}\sqrt{t^2-\rho^2}\right) Q(x,\rho)\, d\rho$$
$$= \int_{\Omega_{n+1}} \cosh\left(i\sqrt{m}\, t\sqrt{\xi_{n+1}^2}\right) \psi(x_1+t\,\xi_1, \ldots, x_n+t\,\xi_n)\, d\xi.$$

Now we can carry out the differentiation with respect to t. Remembering that

$$\cosh i\sqrt{m}\, t\sqrt{\xi_{n+1}^2} = \cos\sqrt{m}\, t\sqrt{\xi_{n+1}^2}$$

we can apply the theorem of Gauss and then proceed as in the foregoing case. This yields

Theorem 6. *Let n be even. Let $t \geqq 1$,*

$$\varphi \in C_0^{(n-2)/2+k+3}(\mathbb{R}^n),$$
$$\psi \in C_0^{(n-2)/2+k+2}(\mathbb{R}^n).$$

Let $1 > \lambda \geqq 0$, $1 > \mu \geqq 0$, $1 \geqq \lambda + \mu$. Then the following inequalities hold:

$$\|\sin(-\Delta+m)^{\frac{1}{2}}\, t(-\Delta+m)^{-\frac{1}{2}}\,\psi\|_{H^{k,1/(1-\lambda-\mu)}(\mathbb{R}^n)}$$
$$\leqq c_{18}\, t^{n(1-\lambda)-n/2}\, \|\psi\|_{H^{k+1+(n-2)/2,\,1/(1-\mu)}(\mathbb{R}^n)},$$

$$\|\cos(-\Delta+m)^{\frac{1}{2}}\, t\,\varphi\|_{H^{k,1/(1-\lambda-\mu)}(\mathbb{R}^n)}$$
$$\leqq c_{19}\, t^{n(1-\lambda)-n/2}\, \|\varphi\|_{H^{k+2+(n-2)/2,\,1/(1-\mu)}(\mathbb{R}^n)}.$$

As before, we have corresponding estimates for the terms

$$\int_0^t \sin(-\Delta+m)^{\frac{1}{2}}(t-\sigma)(-\Delta+m)^{-\frac{1}{2}} f(\sigma)\,d\sigma,$$

$$\int_0^t \cos(-\Delta+m)^{\frac{1}{2}}(t-\sigma)\, f(\sigma)\,d\sigma.$$

In a letter dated from Nov. 5th 1970 Prof. W.A. Strauss communicated to the author that he has proved estimates similar to ours. His proofs too are based on methods developed in [6]. Among these estimates is the following interesting one, as he states:

$$\|\sin(-\Delta+m)^{\frac{1}{2}}\, t(-\Delta+m)^{-\frac{1}{2}}\,\varphi\|_{C^0(\mathbb{R}^3)} \leqq \frac{c_{20}}{t}\,\|\nabla\varphi\|_{L^2(\mathbb{R}^3)}, \quad \varphi\in C_0^\infty(\mathbb{R}^3),\ t\geqq 1,$$

with only derivatives of φ appearing on the right side.

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