I. Cauchy's Problem for Linear Wave Equations: Introductory Material

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In this chapter we touch some general aspects of Cauchy's problem for second order equations. Moreover, some tools of the theory of functions of severable real variables are presented.

§1. Real Analytic Functions. Implicit Functions. Curve Integrals

Definition I.1.1: *Let* $a_{\nu_1 \ldots \nu_N} \in \mathbb{R}$, $\nu_1 = 0, 1, 2, \ldots, \nu_N = 0, 1, 2, \ldots; N$ *is a fixed positive integer. The multiple series*

$$\sum_{\nu_1, \ldots \nu_N = 0}^{\infty} a_{\nu_1 \ldots \nu_N}$$

is said to be absolutely convergent if the finite sums satisfy

$$\sum_{\nu_1, \ldots \nu_N = 0}^{N'} |a_{\nu_1 \ldots \nu_N}| \leq M$$

for every $N' \in \mathbb{N} \cup \{0\}$ *and for some* $M \geq 0$.

It is well known then that $\sum_{\nu_1, \ldots \nu_N = 0}^{\infty} a_{\nu_1 \ldots \nu_N}$ is convergent; the value of the series does not depend on the order of summation. We now consider power series

$$\text{(I.1.1)} \qquad \sum_{\nu_1, \ldots, \nu_N = 0}^{\infty} a_{\nu_1 \ldots \nu_n} (x_1 - x_1^0)^{\nu_1} \cdot \ldots \cdot (x_n - x_n^0)^{\nu_n}$$

in $\mathbb{R}^n$ around a fixed $x_0 = (x_1^0, \ldots, x_n^0) \in \mathbb{R}^n$. We say that the series (I.1.1) is convergent in x if some arrangement into a simple series is so. Then the following proposition holds:

Proposition I.1.1: *If (I.1.1) is convergent in* x^1 *with* $|x_\nu^1 - x_\nu^0| > 0, \nu = 1, ..., n$, *then it is absolutely convergent on*

(I.1.2) $$\{x|x \in \mathbb{R}^n,\ |x_\nu - x_\nu^0| < |x_\nu^1 - x_\nu^0\}.$$

On every compact subset of (I.1.2) the series (I.1.1) converges uniformly absolutely, i. e.: If Γ is such a compact subset, if $\epsilon > 0$, then there is an $N' \in \mathbb{N}$ such that

$$\sum_{\nu_1+\ldots+\nu_n \geq N'} |a_{\nu_1\ldots\nu_n}(x_1 - x_1^0)^{\nu_1} \cdot \ldots \cdot (x_n - x_n^0)^{\nu_n}| < \epsilon,\ x \in \Gamma.$$

Definition I.1.2: *Let $\mathcal{R}$ be the open kernel of $\{x|x \in \mathbb{R}^n$, (I.1.1) absolutely convergent $\}$. $\mathcal{R}$ is called the domain of convergence of the power series (I.1.1)*

It is in fact a consequence of (I.1.1) that $\mathcal{R}$ is open and connected. $\mathcal{R}$ may be empty.

Proposition I.1.2: *Let $\mathcal{R} \underset{\neq}{\supset} \{x_0\}$. Set*

(I.1.3) $$f(x) = \sum_{\nu_1,\ldots,\nu_n=0}^{\infty} a_{\nu_1\ldots\nu_n}(x_1 - x_1^0)^{\nu_1} \cdot \ldots \cdot (x_n - x_n^0)^{\nu_n}$$

on $\mathcal{R}$. Then $f \in C^\infty(\mathcal{R})$ and

$$a_{\nu_1\ldots\nu_n} = \frac{1}{\nu_1! \cdot \ldots \cdot \nu_n!} D^\alpha f(x_0)$$

for $\alpha = (\nu_1, \ldots, \nu_n)$.

All derivatives of f on $\mathcal{R}$ can be gained by formal differentiation of the power series. The domain of convergence of the formally differentiated power series is also $\mathcal{R}$.

Definition I.1.3: *Let U be an open set of $\mathbb{R}^n$, let f be a real function on U. Then f is called real analytic on U if for every $x_0 \in U$ there is an open set $U(x_0) \subset U$ such that*

$$f(x) = \sum_{\nu_1,\ldots,\nu_n=0}^{\infty} a_{\nu_1\ldots\nu_n}(x_1 - x_1^0)^{\nu_1} \cdot \ldots \cdot (x_n - x_n^0)^{\nu_n}$$

with a power series being absolutely convergent on $U(x_0)$. The class of such functions f is devoted by $C^{\mathcal{A}}(U)$.

From Proposition I.1.2 it follows that on $\mathcal{R}$ the power series on the right side of (I.1.3) is the Taylor series of f in x_0.

We now repeat some well known facts on implicit functions which will be used frequently later on.

Proposition I.1.3: *Let $x_0 \in \mathbb{R}^n$, $y_0 \in \mathbb{R}^l$, let U, V be open subsets of $\mathbb{R}^n$, $\mathbb{R}^l$ respectively with $x_0 \in U$, $y_0 \in V$. Let $g \in (C^k(U \times V))^l$ for some $k \in \mathbb{N} \cup \{\infty, \mathcal{A}\}$. Let $g(x,y) = (g_1(x,y), \dots, g_l(x,y))$ and*

$$\text{(I.1.4)} \qquad \det \frac{\partial(g_1(x,y), \dots, g_l(x,y))}{\partial(y_1, \dots, y_l)} \neq 0, \ (x,y) \in U \times V$$

$$\text{(I.1.5)} \qquad g(x_0, y_0) = 0.$$

Then there are open neighbourhoods U_0 of x_0 and V_0 of y_0 such that $U_0 \subset U$, $V_0 \subset V$ and such that there is a unique

$$\text{(I.1.6)} \qquad f \in (C^k(U_0))^l$$

with

$$\text{(I.1.7)} \qquad f(U_0) \subset V_0,$$

$$\text{(I.1.8)} \qquad g(x, f(x)) = 0 \quad \text{on} \quad U_0,$$

$$\{(x,y) | x \in U_0, \ y \in V_0, \ g(x,y) = 0\} = \{(x, f(x)) | x \in U_0\}.$$

According to the chain rule the following formula holds

$$\text{(I.1.9)} \qquad \frac{\partial(g_1(x,f(x)), \dots, g_l(x,f(x)))}{\partial(x_1, \dots, x_n)} + \\ + \frac{\partial(g_1(x,f(x)), \dots, g_l(x,f(x))}{\partial(y_1, \dots, y_l)} \frac{\partial(f_1(x), \dots, f_l(x))}{\partial(x_1, \dots, x_n)} = 0$$

on U_0.

Closely related with Proposition I.1.3 is the problem of the existence of an inverse function. The result for arbitrary n is somewhat weaker than in the case $n = 1$.

Proposition I.1.4: *Let G be a domain of $\mathbb{R}^n$. Let*

$$f : G \to \mathbb{R}^n$$

be a continuously differentiable mapping with

$$\text{(I.1.10)} \qquad \det \frac{\partial(f_1, \dots, f_n}{\partial(x_1, \dots, x_n)} \neq 0 \text{ on } G.$$

Then $f(G)$ is also a domain. For each $x_0 \in G$ there is an open neighborhood U of x_0 and an open neighborhood V of $y_0 = f(x_0)$ such that $f|U : U \to V$ is a bijection. The inverse mapping $(f|U)^{-1}$ is continuously differentiable on V and

$$\text{(I.1.11)} \qquad \frac{\partial((f)U)_1^{-1}, \dots, (f|U)_n^{-1})}{\partial(y_1, \dots y_n)}(y) = \left(\frac{\partial(f_1, \dots, f_n)}{\partial(x_1, \dots, x_n)}\right)^{-1}(x)$$

if $y = f(x)$. If $f|U$ is from the class $C^k(U)$ for some $k \in \mathbb{N}$ then $(f|U)^{-1}$ is from $C^k(V)$ too. A mapping like $f|U$ is called a coordinate transformation.

For later reference we introduce the notion of a principal function (or potential function) for vector fields f. Moreover we give a necessary and sufficient condition under which a potential function exists.

Proposition I.1.5: *Let G be a simply connected domain of $\mathbb{R}^n$. Let $f : G \to \mathbb{R}^n$ be a continuously differentiable mapping. Necessary and sufficient for the existence of a continuously differentiable mapping $F : G \to \mathbb{R}$ with*

$$\text{(I.1.12)} \qquad f(x) = \nabla F(x) \text{ on } G$$

are the integrability conditions

$$\text{(I.1.13)} \qquad \frac{\partial f_\nu}{\partial x_\mu}(x) = \frac{\partial f_\mu}{\partial x_\nu}(x),\ x \in G,\ 1 \le \mu, \nu \le n.$$

F is called a principal function or potential function.

Let G be as in the preceding proposition. Let $x_1, x_2 \in G$, let $W_{x_1,x_2} = ([a,b], p, p([a,b])$ be a curve with

$$p([a,b]) \subset G,$$
$$p(a) = x_1,\ p_b = x_2.$$

We always assume that W_{x_1,x_2} has finite length. If $f : G \to \mathbb{R}^n$ is continuously differentiable then (I.1.13) is equivalent with

$$\text{(I.1.14)} \qquad \left\{ \begin{array}{l} \displaystyle\int_{W_{x_o,x}} f(x) \cdot dx \text{ does not depend on } W \\ \text{but only on } x \in G, \text{ provided } x_0 \text{ is some fixed point of } G. \end{array} \right.$$

In this case

(I.1.15) $$F(x) = \int_{W_{x_o,x}} f(x) \cdot dx$$

is a principal or potential function. If p is continuously differentiable then

(I.1.16) $$F(x) = \int_1^b \sum_{\nu=1}^n (f \circ p)_\nu(t) p'_\nu(t) dt,$$
$$\text{if } x_0 = p(a),\ x = p(b).$$

We mention the rule of substitution for curve integrals: Let G', G be domains of $\mathbb{R}^n$. Let $([a,b], p^*, p^*([a,b]))$ be a curve in G', let p^* be continuously differentiable. Let $g : G' \to G$ be a continuously differentiable mapping, let $p = g \circ p^*$,

$$y_1 = p^*(a),\ y_2 = p^*(b),$$
$$W_{y_1,y_2} = ([a,b], p^*, p^*([a,b]),$$

$$x_1 = p(a) = g \circ p^*(a),\ x_2 = p(b) = g \circ p^*(b),$$
$$W_{x_1,x_2} = ([a,b], p, p([a,b])).$$

Then

$$\begin{aligned}\int_{W_{x_1,x_2}} f(x) \cdot dx &= \int_a^b \sum_{\kappa=1}^n \{\sum_{\nu=1}^n (f \circ g \circ p^*)_\nu(t) \frac{\partial g_\nu}{\partial y_\kappa}(p^*(t))\} p^{*'}_\kappa(t) dt \\ &= \int_{W_{y_1,y_2}} (\frac{\partial(g_1, \ldots, g_n)}{\partial(y_1, \ldots, y_n)}(y))^* \cdot (f \circ g(y))^* \cdot dy.\end{aligned}$$

Let G be a domain and $W_{x_1,x_2}([a,b], p, p([a,b])$ be a curve in G with a continuously differentiable p. Then we set

(I.1.17) $$\int_{W_{x_1 x_2}} |f(x)||dx| = \int_a^b \{\sum_{\nu=1}^n |(f \circ p)_\nu(t)|^2\}^{\frac{1}{2}} \cdot \{\sum_{\nu=1}^n |p'_\nu(t)|^2\}^{\frac{1}{2}} dt,$$

where $p(a) = x_0$, $p(b) = x_2$.

§2. Position of the Problem

If F is a sufficiently regular function of x, u, u', u'' on an open set $\mathcal{D}$ of $\mathbb{R}^4$ with $F_{u''} \neq 0$, then the equation $F(x, u, u', u'') = 0$ locally admits a solution

$$u'' = f(x, u, u') \tag{I.2.1}$$

around a point $(x_0, u_0, u_1, u_2) \in \mathcal{D}$, $F(x_0, u_0, u_1, u_2) = 0$ according to Proposition I.1.3. In particular f is continuously differentiable if F is so. (I.2.1) is interpreted as an ordinary second order differential equation; if (x_0, u_0, u_1) are in the domain of definition of f, then (I.2.1) admits a unique solution u on an interval $[x_0, x_0 + \epsilon]$ with $u(x_0) = u_0$, $u'(x_0) = u_1$. If f is k-times continuously differentiable on its domain of definition, then u is also k-times continuously differentiable with respect to x, u_0, u_1.

The situation is different with partial differential equations as it will be clear from what follows. Let F be a continuously differentiable function of x, y, u, p, q, r, s, t on an open $\mathcal{D}$ set of $\mathbb{R}^8$. Here p, q, r, s, t stand for the derivatives

$$u_x = p, \ u_y = q$$

$$u_{xx} = r, \ u_{xy} = s, \ u_{yy} = t$$

of an unknown twice continuously differentiable function u in the variables x, y with

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0. \tag{I.2.2}$$

u is defined on a „halfopen“set $(x_0 - \epsilon, x_0 + \epsilon) \times [0, \epsilon']$ for some $\epsilon, \epsilon' > 0$, we prescribe

$$u(x, 0) = u_1(x) \tag{I.2.3}$$

$$u_y(x, 0) = u_2(x) \tag{I.2.4}$$

and look for a twice continuously differentiable function u on $(x_0 - \epsilon, x_0 + \epsilon) \times [0, \epsilon']$ with (I.2.2) on $(x_0 - \epsilon, x_0 + \epsilon) \times [0, \epsilon']$, (I.2.3), (I.2.4),

$$(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) \in \mathcal{D} \tag{I.2.5}$$

if $(x, y) \in (x_0 - \epsilon, x_0 + \epsilon) \times [0, \epsilon']$. This problem is called **Cauchy's Problem** for (I.2.2). Instead of prescribing u and u_y on $y = 0$, one can also try to fix u and ∇u on a curve $\Gamma : \varphi(x, y) = 0$ with a continuously differentiable φ

with $\nabla\varphi \neq 0$, provided this is compatible with the domain of definition of F; this means that $u(x,y) = u_1(x,y)$, $\frac{\partial u}{\partial n}(x,y) = u_2(x,y)$ on Γ with given functions u_1, u_2 (n is the normal vector on Γ in (x,y)). We will come back to this question later on.

Let $F_t \neq 0$ on $\mathcal{D}$. Then the equation $F(x,y,u,p,q,r,s,t) = 0$ locally admits a solution $t = f(x,y,u,p,q,r,s)$, and we can consider the differential equation.

(I.2.6) $$u_{yy} = f(x,y,u,u_x,u_y,u_{xx},u_{xy})$$

with $$u(x,0) = u_1(x)$$ $$u_y(x,0) = u_2(x)$$

on $(x_0 - \varepsilon, x_0 + \varepsilon)$ for some $\varepsilon > 0$. As for this problem the Cauchy-Kowalewskaja Theorem holds, which we mention without proof:

Theorem I.2.1: *Let f be real analytic on an upon set $U(x_0,0) \times V \subset \mathbb{R}^7$, where $U(x_0,0)$ is an open neighbourhood of $(x_0,0)$ in $\mathbb{R}^2$ and where V is an open set of $\mathbb{R}^5$. Let u_1, u_2 be real analytic on $U_\varepsilon(x_0) = (x_0 - \varepsilon, x_0 + \varepsilon)$. Then there is one an only function u which is real analytic on $(x_0 - \widetilde{\varepsilon}, x_0 + \widetilde{\varepsilon}) \times (-\widetilde{\varepsilon}, +\widetilde{\varepsilon}) \subset U(x_0,0) \times V$ for some $\widetilde{\epsilon} > 0$ and satisfies*

$$u_{yy} = f(x,y,u,u_x,u_y,u_{xx},u_{xy}),$$
$$u(x,0) = u_1(x),$$
$$u_y(x,0) = u_2(x)$$

on $(x_0 - \widetilde{\varepsilon}, x_0 + \widetilde{\varepsilon}) \times [0, \widetilde{\varepsilon})$.

Comparing this result with the corresponding one for ordinary differential equations, **the following questions** may be posed:

a) Is it possible to replace the analyticity of f (and of F, if $F(x,y,u,u_x,u_y,u_{xx},u_{xy},u_{yy}) = 0$ is the original problem), u_1, u_2 by the continuous differentiability of order k for some $k \geq 2$?

b) If $u = u(u_1,u_2)$ is the solution according to Theorem I.2.1 and if ρ is some positive number, is there a $\delta(\rho)$ such that

$$||u - \widehat{u}||_{C^k([x_0-\widetilde{\varepsilon}_1, x_0+\widetilde{\varepsilon}_1]\times[0,\widetilde{\varepsilon}_1])} < \rho,$$

provided $||u_1 - \widehat{u}_1||_{C^k([x_0-\widetilde{\varepsilon}_2, x_0+\widetilde{\varepsilon}_2])} + ||u_2 - \widehat{u}_2||_{C^k([x_0-\widetilde{\varepsilon}_2, x_0+\widetilde{\varepsilon}_2])} < \delta(\rho)$? Here k is some integer ≥ 2, $\widetilde{\varepsilon}_1, \widetilde{\varepsilon}_2$ are positive numbers with $0 < \widetilde{\epsilon}_1 \leq \widetilde{\epsilon}_2 < \widetilde{\epsilon}$, and $\widehat{u} = u(\widehat{u}_1, \widehat{u}_2)$ is the solution according to Theorem I.2.1 with

$\widehat{u}(x,0) = \widehat{u}_1(x)$, $\widehat{u}_y(x,0) = \widehat{u}_2(x)$. $\widetilde{\epsilon}$ in Theorem I.2.1 may depend on u_1, u_2 (besides f), thus we take for the moment the minimum of both of the values of $\widetilde{\epsilon}$ belonging to u_1, u_2 and $\widehat{u}_1, \widehat{u}_2$.

As the following **examples** show, **the answer to these questions is in general negative.** We set

$$F(x, y, u, p, q, r, s, t) = r + t.$$

Thus the equation to consider is

$$\Delta u = u_{xx} + u_{yy} = 0,$$
$$u(x,0) = u_1(x),\ u_y(x,0) = u_2(x).$$

We set $u_1(x) = 0$, $u_2(x) = e^{-1/x^2}$ on $U_\epsilon(0)$ for some $\epsilon > 0$. Without giving a proof we claim that this problem has no solution at all. The reason is that from $u_{xx} + u_{yy} = 0$ on $U(0,0)$ it follows that u is real analytic on some open neighborhood $U'(0,0) \subset U(0,0)$ of $(0,0)$ but this is a contradiction since u_2 is not real analytic in $U_2(0)$, no matter how small ϵ is. Thus **question a) has been answered in the negative sense.**

As for question b) we take the problem

$$\Delta u = u_{xx} + u_{yy} = 0,$$

$$u(x,0) = 0,\ u_y(x,0) = A_n \sin nx,$$

$n = 1, 2, \ldots$, and set $u_1^{(n)}(x) = A_n \sin nx$. A_n is a real number $\neq 0$ to be determined later on. We try to solve this problem in the form

$$u(x,y) = f(x)g(y),$$

which leads to $\Delta u = f''(x)g(y) + f(x)g''(y) = 0$,

$$0 = \frac{\Delta u(x,y)}{f(x)g(y)} = \frac{f''(x)}{f(x)} + \frac{g''(y)}{g(y)},$$

provided $f(x), g(y)$ do not vanish. To satisfy the last equation we solve the linear ordinary differential equations $f''(x) = -n^2 f(x)$, $g''(y) = n^2 g(y)$. As solutions we take

$$f(x) = \sin nx,\ g(y) = \frac{A_n}{n} \sin hny.$$

Thus we set

$$u_n(x,y) = \frac{An}{n}\sin nx \sin hny$$

An easy calculation shows that $\Delta u_n = 0$ on $\mathbb{R}^2$, $u_n(x,0) = 0$, $u_{n,y}(x,0) = A_n \sin ux$ on $\mathbb{R}$. Now we choose $A_n = e^{-\sqrt{n}}$. Then

$$\sup_{-\epsilon \le x \le \epsilon} \left| \frac{d^k}{dx^k} u_{ny}(.,0) \right| \to 0,\ n \to \infty,$$

for every $k \in \mathbb{N} \cup \{0\}$ and every $\epsilon > 0$. On the other hand

$$u_n(\frac{\pi}{2n}, y) = \frac{e^{-\sqrt{n}}}{n}\sin hny \to +\infty,\ n \to \infty,$$

for every $y > 0$. Taking $\widehat{u}_1 = 0$, $\widehat{u}_2 = 0$, $\widehat{u} = 0$ we see that for every $\widetilde{\epsilon}_1, \widetilde{\epsilon}_2$ with $0 < \widetilde{\epsilon}_1 \le \widetilde{\epsilon}_2$ we have

$$||u_n - \widehat{u}||_{C^k([-\widetilde{\epsilon}_1, \widetilde{\epsilon}_1] \times [0, \widetilde{\epsilon}_1])} \to +\infty,\ n \to \infty,$$

although

$$||u_n(.,0) - \widehat{u}_1||_{C^k([-\widetilde{\epsilon}_2, \widetilde{\epsilon}_2])} + ||u_{ny}(.,0) - \widehat{u}_2||_{C^k([-\widetilde{\epsilon}_2, \widetilde{\epsilon}_2])} \to 0,\ n \to 0;$$

Thereby question b) is also answered in the negative sense.

In order to fulfill a) and b) we have to impose an additional assumption on F namely $F_r F_t - \frac{1}{4}F_s^2 < 0$. This is called the hyperbolic case, in contrast to the elliptic one which is characterized by $F_r F_t - \frac{1}{4}F_s^2 > 0$; the case $F(x,y,u,p,q,r,s,t) = r + t$ evidently is elliptic.

§3. Auxiliary Propositions on Multiple Integrals

Let $([a,b],p,p([a,b]))$ be a curve in $\mathbb{R}^2$ with $p \in C^0([a,b],\mathbb{R}^2) \cap C^1((a,b),\mathbb{R}^2)$ and $|p'(t)| \neq 0$ on (a,b). If $|p'|$ is Riemann-integrable on (a,b), the curve has finite length

$$S = \int_a^b |p'(t)|dt.$$

We often write $p(t) = (x(t), y(t))$, $p'(t) = (x'(t), y'(t))$. In every point $t \in (a,b)$ we can define a normal by

$$\left(\frac{+y'(t)}{(x'^2(t)+y'^2(t))^{1/2}}, \frac{-x'(t)}{(x'^2(t)+y'^2(t))^{1/2}} \right).$$

Observe that this vector is orthogonal to $(x'(t), y'(t))$, the tangential vector.

The situation in $\mathbb{R}^3$ is as follows:

Let G a domain of $\mathbb{R}^2$ and let $\overline{x} : \overline{G} \to \mathbb{R}^3$ be continuous mapping with

$$\overline{G} \ni (u,v) \longmapsto \overline{x}(u,v) = \begin{pmatrix} x(u,y) \\ y(u,v) \\ z(u,v) \end{pmatrix}.$$

We moreover assume that $x_u, x_v, y_u, y_v, z_u, z_v$ exist on G, are continuous and bounded on G, and that

$$\overline{x}_u \times \overline{x}_v \neq 0$$

on G. Then the unit vector

$$n(u,v) = \frac{\epsilon(\overline{x}_u(u,v) \cdot \overline{x}_v(u,v))}{|\overline{x}_u(u,v) \times \overline{x}_v(u,v)|}, (u,v) \in G,$$

with $\epsilon = 1$ or $\epsilon = -1$ is normal to the tangential vectors $\overline{x}_u(u,v)$, $\overline{x}_v(u,v)$. If $D_i(u,v)$, $i = 1,2,3$, are the components of $\overline{x}_u(u,v) \times \overline{x}_v(u,v)$, then $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$

$$\begin{vmatrix} x_u(u,v) & y_u(u,v) & z_u(u,v) \\ x_v(u,v) & y_v(u,v) & z_v(u,v) \\ \lambda_1 & \lambda_2 & \lambda_3 \end{vmatrix} = \sum_{i=1}^{3} \lambda_i D_i(u,v).$$

Our assumption: $\overline{x}_u \times \overline{x}_v \neq 0$ simply means that

$$\sum_{i=1}^{3} D_1^2(u,v) > 0$$

on G. The surface area Ω of $\overline{x}(G)$ is defined by

$$\Omega = \int_G |\overline{x}_u \times \overline{x}_v| dudv = \int_G (\sum_{i=1}^{3} D_1^2(u,v))^{\frac{1}{2}} dudv,$$

provided the integral exists. This is the case if ∂G has Jordan measure 0.

Now want to extend our notions to arbitrary dimensions n. Let G be a domain of $\mathbb{R}^{n-1}$. We consider mappings $\overline{x} : \overline{G} \to \mathbb{R}^n$ which are continuous on $\overline{G}$, continuously differentiable on G and where

$$\text{(I.3.1)} \qquad \sum_{j=1}^{n} D_j^2(t_1, \dots, t_{n-1}) > 0,$$

$(t_1, \dots, t_{n-1}) \in G$. Here $D_1, \dots, D_{n-1}$ are defined by:

$$\text{(I.3.2)} \begin{vmatrix} \frac{\partial x_1}{\partial t_1}(t_1, \dots, t_{n-1}) & \dots & \frac{\partial x_n}{\partial t_1}(t_1, \dots, t_{n-1}) \\ & \vdots & \\ \frac{\partial x_1}{\partial t_{n-1}}(t_1, \dots, t_{n-1}) & \dots & \frac{\partial x_n}{\partial t_{n-1}}(t_1, \dots, t_{n-1}) \\ \lambda_1 & \dots & \lambda_n \end{vmatrix} = \sum_{j=1}^{n} \lambda_j D_j(t_1, \dots, t_{n-1})$$

on G for all $(\lambda_1, \dots, \lambda_n) \in \mathbb{R}$. Thus

$$\text{(I.3.3)} \quad D_j(t_1, \dots, t_{n-1}) = (-1)^{n+j} \det \frac{\partial(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n)}{\partial(t_1, \dots, t_{n-1})}(t_1, \dots, t_{n-1});$$

of course the components of $\overline{x}(t_1, \dots, t_{n-1})$ are denoted by $x_1(t_1, \dots, t_{n-1}), \dots, x_n(t_1, \dots, t_{n-1})$.

(I.3.1) simply means that the matrix $(\frac{\partial x_k}{\partial t_i}(t_1, \dots, t_{n-1}))_{\substack{1 \le i \le n-1, \\ 1 \le k \le n}}$ has maximal rank, namely $n-1$. The unit vector

$$\text{(I.3.4)} \qquad n(\overline{x}) = n(\overline{x}(t)) = n(t_1, \dots, t_{n-1}) = \\ = \frac{\varepsilon}{(\sum_{j=1}^{n} D_j^2(t_1, \dots, t_{n-1}))} \begin{pmatrix} D_1(t_1, \dots, t_{n-1} \\ \vdots \\ D_n(t_1, \dots, t_{n-1}) \end{pmatrix}$$

$t = (t_1, \dots, t_{n-1}) \in G$, with $\varepsilon = 1$ or $\varepsilon = -1$ then is normal to the tangential vectors $(\partial \overline{x}/\partial t_1)(t_1, \dots, t_{n-1}), \dots, (\partial \overline{x}/\partial t_{n-1})(t_1, \dots, t_{n-1})$. The surface area Ω of $\overline{x}(G)$ may be defined by

$$\Omega = \int_G (\int_{j=1}^{n} D_j^2(t))^{\frac{1}{2}} dt, \tag{I.3.5}$$

provided the integral exists. This is the case if ∂G has Jordan measure 0 and if $(\sum_{j=1}^{n} D_j^2(t))^{\frac{1}{2}}$ is Riemann-integrable on G (in particular this function has to be bounded on G).

In the next definition we introduce the concept of a regular hypersurface which is needed for the Theorem of Gauß:

Definition I.3.1: *Let T be a domain of $\mathbb{R}^{n-1}$ with boundary ∂T. Let ∂T have Jordan measure 0. Let there a mapping*

$$\overline{x} : \overline{T} \to \mathbb{R}^n$$

be given with the following properties: $\overline{x} \in (C^0(\overline{T}))^n \cap (C^1(T))^n$. $\overline{x}$ is injective on $\overline{T}$, $|\overline{x}(t_1) - \overline{x}(t_2)| \leq K|t_1 - t_2|, t_1, t_2 \epsilon T$ for some constant $K \geq 0$,

$$\sum_{j=1}^{n} D_j^2(t) > 0,\ t \in T,$$

where

$$D_j(t) = D_j(t_1, \ldots, t_{n-1}) = (-1)^{n+j} \cdot \det \frac{\partial(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)}{\partial(t_1, \ldots, t_{n-1}}(t)$$

and where $x_1(t), \ldots, x_n(t)$, $t \in \overline{T}$, are the components of $\overline{x}(t)$. Finally we observe that in the sense of Riemann

$$\Omega = \int_T (\sum_{j=1}^{n} D_j^2(t))^{\frac{1}{2}} dt < +\infty.$$

Ω is called the surface area of $\overline{x}(T)$, and the set of equivalent triples $(\overline{T}, \overline{x}, \overline{x}(\overline{T}))$ is called a regular hypersurface in $\mathbb{R}^n$. Two triples $(\overline{T}_1, \overline{x}_1, \overline{x}_1(\overline{T}_1))$, $(\overline{T}_2, \overline{x}_2, \overline{x}_2(\overline{T}_2))$ are called equivalent if there are open connected sets $U_1, U_2 \subset \mathbb{R}^{n-1}$ with $\overline{T}_1 \subset U_1$, $\overline{T}_2 \subset U_2$ and a continuously differentiable mapping $\overline{r} : U_1 \to U_2$ with

$$\overline{r}(\overline{T}_1) = \overline{T}_2,$$

$$\overline{r} \text{ is injective, in particular } \overline{T}_1 = r^{-1}(\overline{T}_2)$$

$$\det \frac{\partial(r_1, \ldots, r_{n-1})}{\partial(t_1, \ldots, t_{n-1}} \neq 0 \text{ on } t_1, \ldots, t_{n-1} \in U_1,$$

where $\overline{r}(t_1,\ldots,t_{n-1}) = (r_1(t_1,\ldots,t_{n-1}),\ldots,r_{n-1}(t_1,\ldots,t_{n-1}))$ *on* U_1*, and with*

$$\overline{x}_1(u) = \overline{x}_2(v)$$

for $u \in \overline{T}_1$*,* $v \in \overline{T}_2$*,* $u = \overline{r}^{-1}(v)$*. In particular the traces* $\overline{x}_1(\overline{T}_1)$*,* $\overline{x}_2(\overline{T}_2)$ *are equal.*

It is easy to show that for two equivalent triples $(\overline{T}_1, \overline{x}_1, \overline{x}_1(\overline{T}_1))$, $(\overline{T}_2, \overline{x}_2, \overline{x}_2(\overline{T}_2))$ we get

$$\text{(I.3.6)} \qquad \int_{T_1} f \circ \overline{x}_1(t) (\textstyle\sum_{j=1}^n D_j^2(t))^{\frac{1}{2}} dt = \int_{T_2} f \circ \overline{x}_2(t) (\sum_2 D_j^2(t))^{\frac{1}{2}} dt,$$

if ${}_1D_j(t) = (-1)^{n+j} \det(\partial(x_{1,1},\ldots,x_{1,j-1},x_{1,j+1},\ldots,x_{1,n})/\partial(t_1,\ldots,t_{n-1})(t)$, ${}_2D_j(t) = (-1)^{n+j} \det(\partial(x_{2,1},\ldots,x_{2,j-1},x_{2,j+1},\ldots,x_{2,n})/\partial(t_1,\ldots,t_{n-1}))(t)$, $\overline{x}_1(t) = (x_{1,1}(t),\ldots,x_{1,n}(t))$, $\overline{x}_2(t) = (x_{2,1}(t),\ldots,x_{2,n}(t))$, and if $f : \overline{x}_1(T_1) = \overline{x}_2(T_2) \to \mathbb{R}$ is a continuous bounded function. Thus we can define the integral of continuous bounded functions $f : \overline{x}(T) \to \mathbb{R}$ over regular hypersurfaces by setting

$$\text{(I.3.7)} \qquad \int_{\overline{x}(\overline{x})} f(\xi) d\Omega = \int_{\overline{x}(T)} f(\xi) d\Omega = \int_T f \circ \overline{x}(t) (\sum_{j=1}^n D_j^2(t))^{\frac{1}{2}} dt,$$

where $(\overline{T}, \overline{x}, \overline{x}(\overline{T}))$ is a representative of the regular hypersurface. In what follows we write $(\overline{T}, \overline{x}, \overline{x}(\overline{T}))$ for the regular hypersurface. Moreover, we set $F = \overline{x}(T)$, $\partial F = \overline{x}(\partial T)$, $\overline{F} = F + \partial F$ (disjoint union).

Remark I.3.1: If T is a ball, say $T = \{t | t \in \mathbb{R}^{n-1},\ |t| < 1\}$, then it is possible to relax somewhat on the assumptions on $\overline{x}$. It is no longer necessary to assume that $\overline{x}$ is uniformly Lipschitz continuous to define the integral $\int_{\overline{x}(\overline{T})} f(\xi) d\Omega$ by setting

$$\int_{\overline{x}(\overline{T})} f(\xi) d\Omega = \int_T f \circ \overline{x}(t) \cdot (\sum_{j=1}^n D_j^2(t))^{\frac{1}{2}} dt$$

for continuous bounded functions $f : \overline{x}(T) \to \mathbb{R}$. If $g : T \to \mathbb{R}$ is continuous and if $\lim_{\varepsilon \to 0} \int_{|t| < 1-\varepsilon} |g(t)| dt$ exists, we set

$$\int_{|t|<1} g(t) dt = \lim_{\varepsilon \to 0} \int_{|t| < 1-\varepsilon} g(t) dt.$$

If $F : \{t | t \in \mathbb{R}^{n-1},\ |t| \leq 1\} \to \mathbb{R}$ is continuous, then it follows that $\lim_{\varepsilon \to 0} \int_{|t|<1-\varepsilon} F(t) g(t) dt$ also exists, and we set

$$\int_{|t|<1} F(t)g(t)dt = \lim_{\varepsilon\to 0}\int_{|t|<1-\varepsilon} F(t)g(t)dt.$$

Thus it has to be supposed that

$$\int_T (\sum_{j=1}^n D_j^2(t))^{\frac{1}{2}}dt$$

exists in the sense just defined. This integral is called improper Riemann integral over $T = \{t|t \in \mathbb{R}^{n-1},\ |t| < 1\}$. The reader may verify that it is linear and monotonic. If T is the ball $\{t|t \in \mathbb{R}^{n-1},\ |t| < R\}$ for some $R > 0$, the definition of the improper Riemann integral is completely analogous.

Now we consider a bounded open subset G of $\mathbb{R}^n$ on which we impose two conditions:

Condition I.3.1: *∂G admits a representation*

$$\partial G = \bigcup_{m=1}^N \overline{F_m}$$

with $\overline{F_m} = \overline{x_m}(\overline{T}_m)$, where $(\overline{T_m}, \overline{x_m}, \overline{x_m}(\overline{T_m}))$ is a regular hypersurface, $1 \leq m \leq N$; moreover, we assume that

$$\overline{F_i} \cap \overline{F_j} = \partial F_i \cap \partial F_j,\ 1 \leq i, j \leq N,\ i \neq j.$$

From Condition I.3.1 it follows in particular that ∂G has Jordan measure 0. The second condition concerns the normal vector $n(x)$ having been introduced already.

Condition I.3.2: *For every point x,*

$$\begin{aligned} x \in \partial^0 G \ &= \bigcup_{m=1}^N F_m, \\ &= \bigcup_{m=1}^N \overline{x_m}(T_m), \end{aligned}$$

the following assertion holds: There is no sequence $\{x_k\}$ with $x_k \in \mathbb{R}^n - \overline{G}(\in G$ respectively), $k = 1, 2, \ldots,$, such that

$$x - x_k = \rho_k \cdot \widehat{n}(x)$$

with $\rho_k > 0$ for infinitely many k and

$$x - x_k = \rho_k \cdot \widehat{n}(x)$$

with $\rho_k < 0$ *for infinitely many* k, *and*

$$x_k \to x, \; x \to \infty.$$

Here $\widehat{n}(x)$, $x \in \partial^0 G$ *is the vector with components*

$$\begin{aligned} \widehat{n}_i(x) &= \widehat{n}_i(\overline{x_m}(t)) = \widehat{n}_i(t), \\ &= D_i(t)/(\sum_{j=1}^{n} D_j^2(t))^{1/2} \end{aligned}$$

and $x = \overline{x_m}(t) \in F_m$. *Naturally, the quantities* $D_j(t)$ *also refer to* $(\overline{T_m}, \overline{x_m}, \overline{x_m}(\overline{T_m}))$.

Definition I.3.2:

1. *Let* G *fulfill Conditions I.3.1 and I.3.2. Let*

$$x \in \partial^\circ G = \bigcup_{m=1}^{N} F_m,$$

   e.g. $x \in F_k$ *for some* $k \in \mathbb{N}$. *Set*

$$(\overline{T}, \overline{x}, \overline{x}(\overline{T})) = (\overline{T_k}, \overline{x_k}, \overline{x_k}(\overline{T_k})).$$

   The vector $n(x) = n(x(t)) = n(t_1, \ldots, t_{n-1})$, *defined in (I.3.4), with* $\epsilon = 1$, $x = \overline{x}(t_1, \ldots, t_{n-1})$ *is called the outer normal on* ∂G *in* x *if* $x + \tau n(x) \notin G$ *for all* $\tau \in (0, h]$ *with a sufficiently small* $h > 0$; *otherwise we take* $\epsilon = -1$ *(observe that* ϵ *is uniquely determined according to Condition I.3.2, and that either for* $\epsilon = 1$ *or* $\epsilon = -1$ *we have in fact* $x + \tau n(x) \notin G$, $\tau \in (0, h]$ *for a sufficiently small* h*)*

2. *Let* $f : \partial^\circ G \to \mathbb{R}$, *where* G *fulfills Conditions I.3.1 and I.3.2, be a function, which is continuous and bounded on each* F_m. *Then we set*

$$\int_{\partial G} f(\xi) d\Omega = \sum_{m=1}^{N} \int_{\overline{F}_m} f(\xi) d\Omega_m,$$

   where $\int_{\overline{F}_m} f(\xi) d\Omega_m$ *is defined by (I.3.7).*

3. *We say that for a* G *fulfilling Conditions I.3.1 and I.3.2, a regular hypersurface* $(\overline{T_m}, \overline{x}_m, \overline{x}_m(\overline{T_m}))$ *is positively oriented with respect to* G

if the outer normal $n(x)$, $x \in F_m = \overline{x}_m(T_m)$, is given by (I.3.4) with $\epsilon = -1$.

As for Definition I.3.2, 3., let us remark that in case we want that equivalent triples $(T_m, x_m, x_m(T_m))$, $(\widetilde{T}_m, \widetilde{x}_m, \widetilde{x}_m(\widetilde{T}_m))$ have the same orientation with respect to G, then in Definition I.3.1 we must confine us to mappings $\overline{r}$ with

$$\det \frac{\partial(r_1, \ldots, r_{n-1})}{\partial(t_1, \ldots, t_{n-1}} > 0.$$

A different characterisation of the outer normal is given by the following proposition.

Proposition I.3.1: *Let G fulfill the conditions I.3.1 and I.3.2. In particular we have*

$$\partial G = \bigcup_{m=1}^{N} \overline{F_m}$$

with regular hypersurfaces $(\overline{T_m}, \overline{x}_m, \overline{x}_m(\overline{T}_m))$, $\overline{F_m} = \overline{x}_m(\overline{T_m})$. We assume that for some index m there exists an open set U_m of $\mathbb{R}^n$ with $U_m \supset \overline{F_m}$ and a continuously differentiable function $G_m : U_m \to \mathbb{R}$ with

$$|\nabla G_m(x)| \neq 0, \qquad x \in F_m,$$

$$G_m(x) < 0, \qquad x \in G \cap U_m,$$

$$G_m(x) = 0, \qquad x \in F_m.$$

Then the components $n_i(x)$ of the outer normal in $x \in F_m$, $1 \leq i \leq n$, are given by

$$n_i(x) = \frac{1}{|\nabla G_m(x)|} \frac{\partial G_m}{\partial x_i}(x).$$

Proof: Differentiating the equation $G_m(x(t)) = 0$ with respect to $t_1, \ldots, t_{n-1}$ on T_m yields

$$\sum_{\nu=1}^{n} \frac{\partial x_\nu}{\partial t_k}(t) \frac{\partial G_m}{\partial x_\nu}(x(t)) = 0,$$

$1 \leq k \leq n-1$. Since by (I.3.2) also

$$\sum_{\nu=1}^{n} \frac{\partial x_\nu}{\partial t_k}(t) D_\nu(t) = 0,$$

$1 \leq k \leq n-1$, and since the matrix $\left(\frac{\partial x_\nu}{\partial t_k}(t)\right)_{\substack{1\leq\nu\leq n,\\ 1\leq k\leq n-1}}$, has rank $n-1$, we get

$$\xi_\nu(x(t)) = \frac{\partial G_m}{\partial x_\nu}(x(t))/|\nabla G_m(x(t))| = \delta \cdot D_\nu(x(t))/(\sum_{j=1}^{n} D_j^2(t))^{\frac{1}{2}},$$

$1 \leq \nu \leq n$, with the same $\delta = +1$ or $= -1$ for all ν and all $t \in T_m$ (observe that T_m is connected). The mean value theorem shows that for some $\vartheta = \vartheta(x, \tau, \xi) \in (0, 1)$

$$G_m(x + \tau\xi) = \sum_{\nu=1}^{n} \tau\xi_\nu \frac{\partial G_m}{\partial x_\nu}(\vartheta x + \tau(1-\vartheta)\xi) > 0$$

if $\tau \in (0, h]$ with a sufficiently small $h > 0$ and with $x = x(t) \in F_m$, $(\xi_1, \ldots, \xi_n) = (\xi_1(x(t)), \ldots, \xi_n(x(t)))$. Our proposition is proved. □

Now we formulate Gauß's Theorem:

Theorem I.3.1: *Let G be a bounded open subset of $\mathbb{R}^n$ which fulfills Conditions I.3.1 and I.3.2. Let $f \in C^0(\overline{G}) \cap C^1(G)$, let $\partial f/\partial x_i$ be Riemann-integrable over G. Then*

$$\int_G \partial f/\partial x_i dx = \int_{\partial G} f(\xi) \cdot n_i(\xi) d\Omega,$$

where n_i is the i-th component of the outer normal n on ∂G. If $f \in (C^0(\overline{G}))^n \cap (C^1(G))^n$ and if $\nabla \cdot f$ is Riemann-integrable over G, then we have

$$\int_G \nabla \cdot f dx = \int_{\partial G} (f(\xi), n(\xi)) d\Omega.$$

We give an application of Gauß's Theorem in the case $n = 2$ The regular hypersurfaces $C_m = (\overline{T}_m, \overline{x}_m, \overline{x}_m(\overline{T_m}))$ then are curves, T_m being an open interval (a_m, b_m) (observe that T_m is an open connected subset of $\mathbb{R}$), we have $\dot{p}_m = \frac{d\overline{x}m}{dt} \neq 0$ on T_m moreover $\dot{p}_m$ is Riemann-integrable on T_m. We assume that each C_m is positively oriented with respect to G, i.e. the outer normal on $F_m = \overline{x}_m(T_m)$ is given by

$$n_m = \frac{1}{(x_m'(t)^2 + y_m'(t)^2)^{1/2}}(y_m'(t), -x_m'(t))$$

if $\overline{x}_m(t) = (x_m(t), y_m(t))$.

If $f, g \in C^0(\overline{G}) \cap C^1(G)$ and if f_x, g_y are Riemann-integrable on G we get

$$\text{(I.3.8)} \quad \int_G (f_x + g_y) dx dy = \int_{m=1}^{N} \int_{a_m}^{b_m} (f \circ \overline{x}_m(t) y_m'(t) - g \circ \overline{x}_m(t) x_m'(t)) dt.$$

If we set

$$\begin{aligned} \int_{\partial G} g dx &= \sum_{m=1}^{N} \int_{a_m}^{b_m} g \circ \overline{x}_m(t) x_m'(t) dt, \\ \int_{\partial G} f dy &= \sum_{m=1}^{N} \int_{a_m}^{b_m} f \circ \overline{x}_m(t) y_m'(t) dt, \end{aligned}$$

we arrive at the formula

$$\text{(I.3.9)} \qquad \int_G (f_x + g_y) dx dy = \int_{\partial G} -g dx + \int_{\partial G} f dy.$$

If we set $f = q$, $g = -p$, we get the well known formula

$$\text{(I.3.10)} \qquad \int_G (q_x - p_y) dx dy = \int_{\partial G} p dx + \int_{\partial G} q dy.$$

Wir wollen den Begriff des regulären Hyperflächenstücks aus unserer Vorlesung „Partielle Differentialgleichungen I“, der bisher $(n-1)$-dimensionalen Mannigfaltigkeiten vorbehalten war, auf k-dimensionale Mannigfaltigkeiten verallgemeinern. Sei $k \leq n - 1$.

Definition I.3.3: *Sei $T \subset \mathbb{R}^k$ ein beschränktes Gebiet mit Rand ∂T, der das Jordan-Maß besitze. Sei*

$$\overline{x} : \overline{T} \to \mathbb{R}^n$$

eine Abbildung mit den folgenden Eigenschaften: $\overline{x} \in C^0(\overline{T}) \cap C^1(T)$, $\overline{x}$ ist injektiv auf $\overline{T}$, $|\overline{x}(t_1) - \overline{x}(t_2)| \leq K|t_1 - t_2|$, $t_1, t_2 \in T$, mit einer Konstante $K \geq 0$,

$$g(t) = \sum_{1 \leq i_1 < \ldots < i_k \leq n} \left(\det \frac{\partial(\overline{x}^{i_1}, \ldots, \overline{x}^{i_k})}{\partial(t_1, \ldots, t_k)} \right)^2 > 0, \ t = (t_1, \ldots, t_k) > 0.$$

Dann heißt

$$\Omega = \int_T \sqrt{g(t)} dt$$

der (Oberflächen)inhalt von $\overline{x}(T)$. *Das Tripel* $(\overline{T}, \overline{x}, \overline{x}(\overline{T})$ *heißt* k*-dimensionales reguläres Hyperflächenstück im* $\mathbb{R}^n$. *Die Äquivalenz zweier Tripel* $(\overline{T}_1, \overline{x}_1, \overline{x}_1(\overline{T}_1))$, $(\overline{T}_2, \overline{x}_2, \overline{x}_2(\overline{T}_2))$ *wird analog zu Definition I.3.1 erklärt (mit* k *statt* $n-1$*).*

Sei

$$g_i(t) = \sum_{1 \le j_1 < \ldots < j_k \le n} \left(\det \frac{\partial(\overline{x}_i^{j_1}, \ldots, \overline{x}_i^{j_k})}{\partial(t_1, \ldots, t_k)} \right)^2, \; i = 1, 2.$$

Dann ist (Vgl. [(I.3.6), Forster, Analysis III])

$$\int_{T_1} f \circ \overline{x}_1 \sqrt{g_1} dt = \int_{T_2} f \circ \overline{x}_2 \sqrt{g_2} dt,$$

wobei wir die unabhängige Variable in T_i, $i = 1, 2$, jedesmal mit t bezeichnen. f ist aus $C^0(\overline{x}_1(T_1) = \overline{x}_2(T_2), \mathbb{R}) \cap L^\infty(\overline{x}_1(T_1) = \overline{x}_2(T_2), \mathbb{R})$. Insbesondere können wir (Vgl. (I.3.7))

$$\int_{\overline{x}(\overline{T})} f(\xi) d\Omega = \int_{\overline{x}(T)} f(\xi) d\Omega = \int_T f \circ \overline{x} \sqrt{g} dt$$

einführen, f wie oben. Wir setzen noch $F = \overline{x}(T)$, $\partial F = \overline{x}(\partial T)$, $\overline{F} = F + \partial F$ (disjunkte Vereinigung).

Definition I.3.4: *Eine* k*-dimensionale Fläche* M *des* $\mathbb{R}^n$ *ist eine Vereinigung*

$$M = \bigcup_{m=1}^{N} \overline{F_m}$$

mit $\overline{F}_m = \overline{x}_m(\overline{T}_m)$, *wobei die* $(\overline{T}_m, \overline{x}_m, \overline{x}_m(\overline{T}_m))$ *dimensionale reguläre Hyperflächenstücke im* $\mathbb{R}^n$ *sind, die die Eigenschaft*

$$\overline{F}_i \cap \overline{F}_j = \partial F_i \cap \partial F_j, \; 1 \le i, j \le N, \; i \ne j,$$

besitzen.

Definition I.3.5: *Sei* M *eine* k*-dimensionale Fläche. Dann setzen wir*

$$\mathring{M} = \bigcup_{m=1}^{N} F_m.$$

Für Funktionen $f : \mathring{M} \to \mathbb{R}$, *die stetig und beschränkt sind, definieren wir das Integral* $\int_M f d\Omega$ *durch*

$$\int_M f d\Omega = \sum_{m=1}^{N} \int_{F_m} f d\Omega m,$$
$$= \sum_{m=1}^{N} \int_{\overline{F_m}} f d\Omega_m.$$

Um weitergehende Schlüsse zu ziehen, setzen wir voraus, daß für $1 \leq m \leq N$

$$\partial F_m = \sum_{\nu=1}^{N(m)} \overline{\partial F_m^{(\nu)}} \text{ mit}$$

$$\partial F_m^{(\nu)} \cap \partial F_m^{(\nu')} = \emptyset \text{ für } \nu \neq \nu'$$

ist mit den (abgeschlossenen) Spuren $\overline{\partial F_m^{(\nu}}$ $(k-1)$-dimensionaler regulärer Hyperflächenstücke $(\overline{R_m^{(\nu)}}, \overline{r}_m^{(\nu)}, \overline{r}_m^{(\nu)}(\overline{R}_m^{(\nu)} = \overline{\partial F_m^{(\nu)}})$.

Der Tangentialraum an F_m wird bekanntlich aufgespannt von den k Vektoren $\partial \overline{x}_m / \partial t_j$, $j = 1, \ldots k$. Ein Tagentialfeld $a = a^j(\partial \overline{x}_m / \partial t_j)$ besitzt die Divergenz $(a^j \in C^1(T_m))$

$$div\, a = \frac{1}{\sqrt{g_m}} \frac{\partial}{\partial t_j} (\sqrt{g}_m a^j)$$

Dann gilt, wenn wir i. f. nach $k \leq n-1$ voraussetzen,

Hilfssatz I.3.2: *$T_m \subset \mathbb{R}^k$ erfülle die Bedingungen I.3.1 und I.3.2. Die Abbildung $\overline{r}_m^{(\nu)}$ faktorisiere in der folgenden Weise:*

$$\overline{r}_m^{(\nu)} = \overline{r}_{m2}^{(\nu)} \circ \overline{r}_{m1}^{(\nu)},$$

$$\overline{r}_{m1}^{(\nu)} : \overline{R_m^{(\nu)}} \to \overline{\partial T_m^{(\nu)}} \text{ bijektiv,}$$

$$\overline{r}_{m2}^{(\nu)} : \overline{\partial T_m^{(\nu)}} \to \overline{\partial F_m^{(\nu)}} \text{ bijektiv,}$$

mit

$$\partial T_m = \bigcup_{\nu=1}^{N(m)} \overline{\partial T_m^{(\nu)}}, \partial T_m^{(\nu)} \cap \partial T_m^{(\nu')} = \phi \; für \; \nu \neq \nu',$$

seien die $(\overline{R_m^{(\nu)}}, \overline{r}_{m1}^{(\nu)}, \overline{r}_{m1}^{(\nu)}(\overline{R_m^{(\nu)}}) = \overline{\partial T_m^{(\nu)}})$ reguläre Hyperflächenstücke gemäß Definition I.3.2 und ∂T_m besitze die äußere Normale N_m gemäß Definition I.3.2. Es seien die $\overline{x}_m$ aus $C^2(\overline{T}_m)$. Sei a ein Tangentialfeld mit $a^j \in C^1(\overline{T}_m)$. Dann ist

$$\int_{F_m} div\, a\, d\Omega_m = \sum_{\substack{\nu=1,\dots,N(m),\\ j=1,\dots k}} \int_{\partial T_m^{(\nu)}} a^j N_{mj} \sqrt{g_m} d\Omega_{\partial T_m^{(\nu)}},$$

$$= \sum_{\substack{\nu=1,\dots,N(m)\\ j=1,\dots,k}} \int_{R_m^{(\nu)}} a^j N_{mj} \sqrt{g_m} \cdot \sqrt{g_{\partial T_m^{(\nu)}}} ds_1 \dots ds_{k-1}.$$

Beweis: Es ist

$$\int_{F_m} div\ a d\Omega_m = \int_{T_m} \frac{\partial}{\partial t_j} (\sqrt{g_m} a^j) dt.$$

Nun wenden wir den Satz 1 von Gauß auf T_m an und erhalten so die Behauptung. □

Hilfssatz I.3.3: *Auf ∂F_m erklären wir den Vektor $\widehat{N}_m$ durch*

$$\widehat{N}_m = \widehat{N}_{ml} \frac{\partial \overline{x}_m}{\partial t_l},$$

$$\begin{aligned} (\widehat{N}_{m1}, \dots, \widehat{N}_{mk})^T &= G^{-1} (N_{m1}, \dots, N_{mk})^T \\ G &= (g_{mik}). \end{aligned}$$

Dann ist

$$\int_{\partial F_m} (a, \widehat{N}_m) d\partial F_m = \sum_{\substack{\nu=1,\dots,m\\ j=1,\dots,k}} \int_{R_m(\nu)} a^j N_{mj} \sqrt{g_{\partial F_m}} ds_1 \dots, ds_{k-1}.$$

$\widehat{N}_m$ ist orthogonal zu den Tangentialvektoren $\partial \overline{x}_m / \partial s_\rho$ an ∂F_m, $\rho = 1, \dots, k-1$.

Beweis: Zur Orthogonalität: Es ist

$$N_{mj} \frac{\partial t_j}{\partial s_\rho} = 0,\ 1 \le \rho \le k-1,$$

$$\frac{\partial \overline{x}_m}{\partial s_\rho} = \frac{\partial \overline{x}_m}{\partial t_j} \frac{\partial t_j}{\partial s_\rho},$$

$$\begin{aligned} (\widehat{N}_m, \frac{\partial \overline{x}_m}{\partial s_\rho}) &= \widehat{N}_{ml} g_{lj} \frac{\partial t_j}{\partial s_\rho}, \\ &= N_{mj} \frac{\partial t_j}{\partial s_\rho} = 0. \end{aligned}$$

Weiter haben wir

$$\begin{aligned}(a, \widehat{N}_m) &= a^j g_{jl} \widehat{N}_{ml}, \\ &= a^j N_{mj}.\end{aligned}$$

$\square$

Ist $J_{\overline{x}_m}$ die zu $\overline{x}_m : \overline{T}_m \to \overline{F_m}$ gehörige Funktionalmatrix, so ist

$$\widehat{N}_m = J_{\overline{x}_m} G^{-1}(N_{m1}, \ldots, N_{mk})^T,$$

$$\frac{\partial \overline{x_m}}{\partial s_\rho} = J_{\overline{x_m}} \frac{\partial t}{\partial s_\rho},\ 1 \leq \rho \leq k-1.$$

$\{\widehat{N}_m, \frac{\partial \overline{x}_m}{\partial s_1}, \ldots, \frac{\partial \overline{x_m}}{\partial s_{k-1}}\}$ sind stets eine Basis des Tangentialraums in $\overline{x_m}(t(s)) \in \partial F_m$, wenn sie in diesem Punkt gebildet werden. Es gebe in $t(s) \in \overline{\partial T_m^{(\nu)}}$ eine Linearkombination

$$\lambda_0 G^{-1}(N_{m1}, \ldots, N_{mk})^T + \sum_{j=1}^{k-1} \lambda_j \frac{\partial t}{\partial s_j} = 0.$$

Anwendung von $J_{\overline{x_m}}$ liefert

$$\lambda_0 \widehat{N}_m + \sum_{j=1,\ldots,k-1} \lambda_j \frac{\partial \overline{x}_m}{\partial s_\rho} = 0,$$

also $\lambda_0 = \lambda_1 = \ldots = \lambda_{k-1} = 0$. Demnach wechselt $\det(G^{-1}(N_{m1}, \ldots N_{mk})^T, \frac{\partial t}{\partial s_1}, \ldots, \frac{\partial t}{\partial s_{k-1}})$ auf ∂T_m niemals das Vorzeichen, wenn ∂T_m wegweise zusammenhängend und die Gradienten von $\overline{r}_{m1}^{(\nu)}$ sich stetig auf ∂T_m fortsetzen lassen. Insbesondere ist dann N_m stetig auf ∂T_m. Für $\overline{r}_{m2}^{(\nu)}$ können wir $\overline{x}_m|\partial T_m$ wählen. Sei

$$\det(G^{-1}N_m, \frac{\partial t}{\partial s_1}, \ldots, \frac{\partial t}{\partial s_{k-1}}) > 0.$$

Dann ist die Basis $\{\widehat{N}_m, \frac{\partial \overline{x_m}}{\partial s_1}, \ldots, \frac{\partial \overline{x_m}}{\partial s_{k-1}}\}$ positiv orientiert und wir werden $\widehat{N}_m$ als äußere Normale an ∂F_m bezeichnen (im Tangentialraum an $\overline{F}_m$. Ist nun

$$\det(G^{-1}N_m, \frac{\partial t}{\partial s_1}, \ldots, \frac{\partial t}{\partial s_{k-1}}) < 0,$$

so erklären wir $\widehat{N}_m = \widehat{N}_{ml} \frac{\partial \overline{x}_m}{\partial t_l}$ durch

$$(\widehat{N}_{m1}, \ldots, \widehat{N}_{mk})^T = -G^{-1}(N_{m1}, \ldots, N_{mk})^T$$

und erhalten in Hilfssatz I.3.3 die Formel

$$\int_{\partial F_m} (a, \widehat{N}_m) d\partial F_m = - \sum_{\substack{\nu=1,\ldots,m \\ j=1,\ldots,k}} \int_{R_m^{(\nu)}} a^j N_{mj} \sqrt{g_{\partial F_m}} ds_1 \ldots ds_{k-1}$$

Nun ist auch $\widehat{N}_m$ noch nicht auf Länge normiert. Wir erhalten

$$\int_{\partial F_m} (a, \frac{\widehat{N}_m}{|\widehat{N}_m|} d\partial F_m = \pm \sum_{\substack{\nu=1,\ldots,m \\ j=1,\ldots,k}} \int_{R_m^{(\nu)}} a^j N_{mj} \frac{\sqrt{g_{\partial F_m}}}{|\widehat{N}_m|} ds_1 \ldots ds_{k-1}.$$

Unsere Ergebnisse fassen wir zusammen in
Satz I.3.2 (Vorläufige Formulierung): *Sei*

1. $1 \leq k \leq n-1$,
2. es mögen die Annahmen des Hilfssatzes I.3.2 gelten,
3. jedes ∂T_m, $1 \leq m \leq N$, sei wegweise zusammenhängend.
4. $\nabla \overline{r}_{m1}^{(\nu)}$ lasse sich stetig auf $\partial T_m^{(\nu)}$ fortsetzen, $\nu = 1, \ldots, N(m)$, $m = 1, \ldots, N$, derart, daß das Ergebnis stetig auf ∂T_m ist.
5. Sei
   $\det(G^{-1} N_m, \frac{\partial t}{\partial s_1}, \ldots, \frac{\partial t}{\partial s_{k-1}}) > 0$
   auf ∂T_m, so daß die Basis $\{\widehat{N}_m, \frac{\partial \overline{x_m}}{\partial s_1}, \ldots \frac{\partial \overline{x_m}}{\partial_{k-1}}\}$ positiv orientiert und $\widehat{N}_m$ äußere Normale an ∂F_m ist, sei weiter auf $\partial F_m^{(\nu)}$ bzw. $\partial T_m^{(\nu)}$
6. $\frac{\sqrt{g_{\partial F_m^{(\nu)}}}}{|\widehat{N}_m|} = \sqrt{g_m} \sqrt{g_{\partial T_m^{(\nu)}}}$

Unter diesen Annahmen ist

$$\int_{F_m} div\ a\ d\Omega_m = \sum_{\partial F_m} (a, \frac{\widehat{N}_m}{|\widehat{N}_m|}) d\partial F_m,$$

d.h., daß der Satz von Gauß für das k-dimensionale reguläre Hyperflächenstück $(\overline{T}_m,\ \overline{x}_m,\ \overline{x_m}(\overline{T}_m) = \overline{F_m})$ gilt.

Beweis: Folgt aus Hilfssatz I.3.3 und den darauf folgenden Erörterungen.
□

Es ist erwünscht, die Voraussetzungen 5. und 6. im letzten Satz zu rechtfertigen, doch ist dies bisher global, d.h. mit dem obigen Ansatz nur im Fall

$n = 3, k = 2$ gelungen (s. meine Vorlesung Partielle Differentialgleichungen II (Potentialtheorie), Hilfssatz 5.3).

Wir wählen daher eine neue Methode, befassen uns zunächst mit 5. und denken uns T_m bzw. $\partial T_m^{(\nu)}$ in der folgenden Weise parametrisiert:

Dann führen wir aus praktischen Gründen die folgende Umbenennung: $s_0 \to s_1, s_1 \to s_2, \ldots, s_{k-1} \to s_k$, $t = s$ durch. Die Abbildung $r_{m1}^{(\nu)}$ in Hilfssatz I.3.2 ist die Identität und

$$\begin{aligned} (N, \frac{\partial t}{\partial s_2}, \ldots, \frac{\partial t}{\partial s_k}) &= E_k = k \times k \text{ Einheitsmatrix}, \\ (G^{-1}N, \frac{\partial t}{\partial s_2}, \ldots, \frac{\partial t}{\partial s_k}) &= \begin{pmatrix} g_m^{11} & 0 \\ & \\ \vdots & E_{k-1} \end{pmatrix}, \\ \det(G^{-1}N, \frac{\partial t}{\partial s_2}, \ldots, \frac{\partial t}{\partial s_k}) &= g_m^{11}. \end{aligned}$$

Mit $G = (g_{mik})$ ist auch G^{-1} positiv definit und $g_m^{11} > 0$. Damit ist also $\widehat{N}_m$ äußere Normale und die Voraussetzung 5. gerechtfertigt. Nun rechtfertigen wir 6. Es ist $g_{\partial T_m^{(\nu)}} = 1$,

$$\begin{aligned} g_{\partial F_m} &= \det((\frac{\partial \overline{x}_m}{\partial t_i}, \frac{\partial \overline{x}_m}{\partial t_l}))_{2 \le i,l \le m}, \\ g_m &= \det((\frac{\partial \overline{x}_m}{\partial t_i}, \frac{\partial \overline{x}_m}{\partial t_l}))_{1 \le i,l \le m}, \\ |\widehat{N}_m| &= \sqrt{g_m^{11}} = \sqrt{\frac{g_{\partial F_m}}{g_m}} \end{aligned}$$

nach den üblichen Regeln für die Inversenbildung. Damit gilt 6., d.h. $\sqrt{g_{\partial F_m}}/|\widehat{N}_m| = \sqrt{g_m}\sqrt{g_{\partial T_m^{(\nu)}}}$ für die vorhin eingeführte Parametrisierung.

Definition I.3.6: *Es mögen die Voraussetzungen 1. bis 4. aus Satz I.3.2 gelten. F_m heißt orientiert, wenn T_m bezüglich der äußeren Normalen orientiert ist, d.h. z.B. die Parametrisierung oben besitzt. Man sagt dann, F_m hat die von T_m induzierte Orientierung.*

Satz I.3.3: *Es mögen die Voraussetzungen 1. bis 4. aus Satz I.3.2 gelten. F_m sei orientiert. Sei a Tangentialfeld aus $C^1(\overline{T}_m)$. Dann gilt der Satz von Gauß:*

$$\int_{F_m} div\, a\, d\Omega = \int_{\partial F_m} (a, \frac{\widehat{N}_m}{|\widehat{N}_m|}) d\, \partial F_m$$

Beweis: Folgt aus Satz I.3.2. □

Definition I.3.7: *Es sei*

$$M = \bigcup_{m=1}^{N} \overline{F}_m$$

eine K-dimensionale Fläche des $\mathbb{R}^n$. Es mögen die Voraussetzungen 1. bis 4. des Satzes I.3.2 für jedes $m, 1 \leq m \leq N$, gelten.M heißt orientiert, wenn jedes F_m orientiert ist und für jedes stetige Tangentialfeld a auf M die folgende Aussage gilt: Es gebe m, l mit $1 \leq m, l \leq N, m \neq l$, und ν, μ mit $1 \leq \nu \leq N(m), 1 \leq \mu \leq N(l)$ derart, dass $\partial F_m^{(\nu)} = \partial F_l^{(\mu)}$gilt. Dann ist

$$\int_{\partial F_m^{(\nu)}} (a, \frac{\widehat{N}_m}{|\widehat{N}_m|}) d\partial F_m^{(\nu)} = - \int_{\partial F_l^{(\mu)}} (a, \frac{\widehat{N}_l}{|\widehat{N}_l|}) d\partial F_l^{(\mu)}.$$

M heißt geschlossen, wenn M orientiert und

$$\sum_{m=1}^{N} \int_{\partial F_m} (a, \frac{\widehat{N}_m}{|\widehat{N}_m|}) d\partial F_m^{(\nu)} = 0$$

ist für jedes stetige Tangentialfeld a auf$M$.

Offenbar folgt für geschlossenes M aus dem Satz von Gauß

$$\int_M div\, a d\Omega = 0.$$

Gemäß dem wohlbekannten minimalen Voraussetzungen im Satz von Gauß im Euklidischen, auf den wir den Beweis des Satzes I.3.2 zurückgeführt haben, benötigt man im Satz I.3.3 nur

$$a \in C^1(F_m) \cap C^0(\overline{F_m}),$$

$$div\, a \in L^1(F_m).$$

Die folgende Skizze erleichtert das Verständnis:

§4 n-Dimensional Polar Coordinates. The Laplace-Beltrami Operator

We start with the following definition:

Definition I.4.1: *Let $x \in \mathbb{R}^n$, $x \neq 0$. We set $r = |x|$, $\xi = \frac{x}{|x|}$, and call $r, \xi_1, \ldots, \xi_{n-1}$ the n-dimensional polar coordinates of x. Here $\xi_1, \ldots, \xi_{n-1}, \xi_n$ with*

$$\begin{aligned} \xi_n &= (1 - \sum_{i=1}^{n-1} \xi_i^2)^{\frac{1}{2}} \ \textit{or} \\ \xi_n &= -(1 - \sum_{i=1}^{n-1} \xi_i^2)^{\frac{1}{2}} \end{aligned}$$

are the coordinates of ξ.

Proposition I.4.1: *The integral*

$$\int_{|x|<1} \frac{dx}{\sqrt{1 - |x|^2}}$$

exists as an improper integral as defined in I.3.

Proof: Setting $\rho(x_n) = \sqrt{1 - x_n^2}$ we can restrict us to the consideration of

$$\int_{-1}^{+1} \int_{|\widetilde{x}|<\rho} \frac{d\widetilde{x}}{\sqrt{\rho^2 - |\widetilde{x}|^2}} dx_n,\ \rho = \rho(x_n), |x_n| < 1,$$

with $\widetilde{x} = (x_1, \ldots, x_{n-1})$ (cf. Tonelli's Theorem). We have

$$\int_{|\widetilde{x}|<\rho} \frac{d\widetilde{x}}{\sqrt{\rho^2 - |\widetilde{x}|^2}} = \rho^{n-2} \int_{|\widetilde{x}|<1} \frac{d\widetilde{x}}{\sqrt{1 - |\widetilde{x}|^2}},$$

provided the integral on the right hand side exists; this follows from the transformation rule. Since for $n = 1$ the assertion of our proposition is valid, we can prove the assertion for arbitrary n by induction. □

We now want to find an appropriate partition of $S^{n-1} = \{\xi | \xi \in \mathbb{R}^n, |\xi| = 1\}$ into regular hypersurfaces. We set

$$T_1 = T_2 = \{t | t = (t_1, \dots, t_{n-1}) \in \mathbb{R}^{n-1},\ |t| < 1\}.$$

$$\overline{x}_1(t) = (x_{11}(t) = \xi_1 = t_1,\ x_{12}(t) = \xi_2 = t_2, \dots,$$

$$x_{1n-1}(t) = \xi_{n-1} = t_{n-1},$$

$$x_{1n}(t) = (1 - \sum_{i=1}^{n-1} \xi_i^2)^{1/2} = (1 - \sum_{i=1}^{n-1} t_i^2)^{1/2}),$$

$$\overline{x}_2(t) = (x_{21}(t) = \xi_1 = t_1,\ x_{22}(t) = \xi_2 = t_2, \dots,$$

$$x_{2n-1}(t) = \xi_{11-1} = t_{n-1},$$

$$x_{2n}(t) = -(1 - \sum_{i=1}^{n-1} \xi_i^2)^{1/2} = -(1 - \sum_{i=1}^{n-1} t_i^2)^{1/2}),$$

$(\xi_1, \dots, \xi_{n-1}) = t \in \overline{T}_1, \overline{T}_2$ respectively

Proposition I.4.2: $(\overline{T}_1, \overline{x}_1, \overline{x}_1, (\overline{T}_1)), (\overline{T}_2, \overline{x}_2, \overline{x}_2, (\overline{T}_2))$ *are almost regular hypersurfaces,*

(I.4.1) $$S^{n-1} = \overline{F}_1 \cup \overline{F}_2$$

with $\overline{F}_1 = \overline{x}_1(\overline{T}_1)$, $\overline{F}_2 = \overline{x}_2(\overline{T}_2)$, $F_1 = \overline{x}_1(T_1)$, $F_2 = \overline{x}_2(T_2)$, $\partial F_1 = \overline{x}_1(\partial T_1)$, $\partial F_2 = \overline{x}_2(\partial T_2)$; *moreover*

$$\overline{F}_1 \cap \overline{F}_2 = \partial F_1 \cap \partial F_2.$$

Proof: We calculate the determinant (I.3.2). This gives

$$
\text{(I.4.2)} \qquad \begin{vmatrix} 1 & 0 & \dots & 0 & \partial\xi_n/\partial t_1 \\ 0 & 1 & \dots & 0 & \partial\xi_n/\partial t_2 \\ & & \vdots & & \\ 0 & 0 & \dots & 1 & \partial\xi_n/\partial t_{n-1} \\ \lambda_1 & \lambda_2 & \dots & \lambda_{n-1} & \lambda_n \end{vmatrix} = \sum_{i=1}^{n} \lambda_i D_i(t).
$$

Setting $\lambda_1 = \ldots = \lambda_{n-1} = 0$, but $\lambda_n = 1$ yields

$$D_n(t) = 1.$$

Thus

$$\sum_{i=1}^{n} D_i^2(t) \geq 1,\ t \in T_j,\ j = 1, 2.$$

Employing Proposition I.3.1 with $G_1(x) = |x|^2 - 1$ for F_1, $G_2(x) = |x|^2 - 1$ for F_2 furnishes

$$n_i(\xi) = \xi_i,\ |\xi| = 1,\ \xi \in F_1 \text{ or } \zeta \in F_2.$$

Thus

$$\xi_n(t) = \epsilon_j \frac{D_n(t)}{(\sum_{i=1}^{n} D_i^2(t)^{1/2}},\ t \in T_j,$$

with $\epsilon_j = 1$ or $\epsilon_j = -1$ on T_j. Since $D_n(t) = 1$ we get

$$(\sum_{i=1}^{n} D_i^2(t))^{\frac{1}{2}} = \frac{1}{|\xi_n|},$$

$$= \frac{1}{(1 - \sum_{i=1}^{n-1} \xi_i^2)^{1/2}} = \frac{1}{(1 - \sum_{i=1}^{n-1} t_i^2)^{1/2}}.$$

According to Proposition I.4.1 the function $\sum_{i=1}^{n} D_i^2(t))^{\frac{1}{2}}$ is improperly Riemann-integrable on T_j , $j = 1, 2$. All other conditions of Definition I.3.1 and Remark I.3.1 respectively are trivially fulfilled. Thus $(\overline{T_1}, \overline{x_1}, \overline{x_1}(\overline{T_1}))$, $(\overline{T_2}, \overline{x_2}, \overline{x_2}(\overline{T_2}))$ are regular hypersurfaces. The remaining assertions of Proposition I.4.2 are trivial. □

Now we are in a position to define the surface integral for a function f with

$$f : F_1 \to \mathbb{R},$$
$$f : F_2 \to \mathbb{R},$$

which is continuous and bounded on F_i, $i = 1, 2$. We set

$$\int_{|\xi|=1} f(\xi) d\Omega = \int_{F_1} f(\xi) d\Omega_1 + \int_{F_2} f(\xi) d\Omega_2,$$

$$= \int_{T_1} \frac{f(\xi_1, ..., \xi_{n-1}, \sqrt{1 - \sum_{i=1}^{n-1} \xi_i^2})}{\sqrt{1 - \sum_{i=1}^{n-1} \xi_i^2)}} \, d\xi_1 ... d\xi_{n-1} +$$

$$+ \int_{T_2} \frac{f(\xi_1, ..., \xi_{n-1}, -\sqrt{1 - \sum_{i=1}^{n-1} \xi_i^2})}{\sqrt{1 - \sum_{i=1}^{n-1} \xi_i^2)}} \, d\xi_1 ... d\xi_{n-1} \, ,$$

$(I.4.3)$

$$= \int_{T_1} \frac{f(\xi_1, ..., \xi_{n-1}, \sqrt{1 - \sum_{i=1}^{n-1} \xi_i^2}) +}{\sqrt{1-}}$$

$$\frac{+ f(\xi_1, ..., \xi_{n-1}, -\sqrt{1 - \sum_{i=1}^{n-1} \xi_i^2})}{\sqrt{- \sum_{i=1}^{n-1} \xi_i^2)}} \, d\xi_1 ... d\xi_{n-1} \, .$$

It is also clear how the Theorem of Gauß has to be applied to G $= \{x | |x| < 1\}$. In what follows we will write $d\omega$ instead of $d\Omega$ if $S^{n-1} = \{\xi | \, |\xi \, | = 1\}$ or F_1 or F_2 are concerned. Remind that the outer normal in S^{n-1} has been calculated in the proof of Proposition I.4.2. If we consider $G_R = \{x | \, |x| < R\}$ and $\partial G_R = \{\xi | \, |\xi| = R\}$ for some $R > 0$ then a simple calculation shows that

$(I.4.4)$

$$\int_{|\xi|=R} f(\tilde{\xi}) d\Omega = R^{n-1} \int_{|\xi|=1} f(R\xi) d\omega,$$

if f:$\partial G_R \to \{\xi | \xi_n = 0\} \to \mathbb{R}$ is continuous and bounded. The outer normal has the components $\frac{1}{R}\xi_i$.

Now we derive some rules for integrals over G, G_R, ∂G, ∂G_R or related sets. We start with

Proposition I.4.3: *Let* $D \leq a \leq b$. *Let* f *be a continuous real valued function on* $\{x | a \leq |x| \leq b\} \subset \mathbb{R}^n$. *Then*

$$\int_{a<|x|<b} f(x)dx = \int_a^b r^{n-1} \int_{|\xi|=1} f(r\xi) d\omega \; dr$$

Proof: We assume that $0 < a < b$. *Setting*

$$\begin{aligned} \xi_i &= t_i, 1 \leq i \leq n-1 \\ \xi_n &= \sqrt{1 - \sum_{i=1}^{n-1} \xi_i^2} \;, \\ &= \sqrt{1 - \sum_{i=1}^{n-1} t_i^2} \;, \end{aligned}$$

$$(t_1, ...t_{n-1}) \in \overline{T_1} = \overline{T_2}\,, t_n = |x|,\; a \leq t_n \leq b,$$

$$x_i = t_n \xi_i(t_1, \ldots, t_n - 1)$$

we have defined a diffeomorphism between $\overline{T_1} \times [a, b]$ *and* $\{x | a \leq |x| \leq b\} \cap \{x_n \geq 0\}$ whose Jacobian is $(r = |x|)$

$$det \frac{\partial(x_1, \ldots, x_n)}{\partial(t_1, \ldots, t_n)} = \begin{vmatrix} r\frac{\partial \xi_1}{\partial t_1} & \cdots & r\frac{\partial \xi_n}{\partial t_1} \\ & \vdots & \\ r\frac{\partial \xi_1}{\partial t_{n-1}} & \cdots & r\frac{\partial \xi_n}{\partial tn-1} \\ xi_1 & \cdots & \xi_n \end{vmatrix}$$

$$= r^{n-1}\sum_{i=1}^{n}\xi_i(t_1,\ldots,t_{n-1})D_i(t_1,\ldots,t_{n-1})$$

according to (I.4.2). From the proof of Proposition I.4.2 we get

$$\xi_i(t_1,\ldots,t_{n-1} = \varepsilon\frac{D_i(t_1,\ldots,t_{n-1})}{(\sum_{i=1}^{n}D_i^2(t_1,\ldots,t_{n-1}))^{\frac{1}{2}}},\ 1\le i\le n,$$

for $(t_1,\ldots,t_{n-1})\in T_1$ and with $\varepsilon = +1$ or $\varepsilon = -1$.

This means that on T_1

$$|det\frac{\partial(x_1,\ldots,x_n)}{\partial(t_1,\ldots,t_n)}| = r^{n-1}(\sum_{i=1}^{n}D_i^2(t_1,\ldots,t_{n-1}))^{\frac{1}{2}} = \frac{r^{n-1}}{|\xi_n|}$$

thus the transformation rule gives

$$\int_{a\le|x|\le b,x_n\ge 0} f(x)dx = \int_a^b r^{n-1}\int_{T_1} f(r\xi_1(t_1,\ldots,t_{n-1}),$$
$$\ldots r\xi n(t_1,\ldots,t_{n-1}))\cdot$$
$$\cdot(\sum_{i=1}^{n}D_i^2(t_1,\ldots,t_{n-1}))^{\frac{1}{2}})$$
$$d(t_1\ldots t_{n-1})dr$$

Setting $\xi_n = -\sqrt{1-\sum_{i=1}^{n-1}t_i^2}, (t_1,\ldots,t_{n-1})\,\epsilon\,\overline{T_2} = \overline{T_1}$ we arrive at a similar formula for $\int_{a\le|x|\le b,x_n\le 0} f(x)dx$. Formula (I.4.3) completes the proof, since the case a = b is the trivial one and since the case a = 0 is covered by a limit process for $\int_{\epsilon\le|x|\le b} f(x)dx, \epsilon\to 0$. □

If n = 2 we easily get the well known formula

$$\int_{a\le|x|\le b} f(x)dx = \int_a^b\int_0^{2\Pi} f(r\cos\varphi, r\sin\varphi)r\,dr\,d\varphi.$$

Also the surface area ω_n of S^{n-1} is easily calculated with the aid of Proposition I.4.3. Setting $f(t) = e^{-t^2}, t\epsilon\mathbb{R}$,
we get

$$\int_{\mathbb{R}^n} f(|x|)dx = (\int_{-\infty}^{\infty} e^{-t^2} dt)^n = \omega_n \int_{-\infty}^{\infty} r^{n-1} e^{-r^2} dr$$

Employing in both of the last integrals the substitution $x \mapsto \sqrt{x}(x \geq 0)$ and the formulara $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, Rez > 0$, we arrive at $\omega_n = 2(\Gamma(\frac{1}{2}))^n / \Gamma(\frac{n}{2})$. Since $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ we obtain

$$w_n = \frac{2(\sqrt{\pi}^n)}{\Gamma(\frac{n}{2})}.$$

The next consequence of Proposition I.4.3 is

Proposition I.4.4 *Let* $f\epsilon C([-1,+1],\mathbb{R})$. *Then the equation*

$$\int_{S^{n-1}} f(\xi_n) d\omega = \omega_{n-1} \int_{-1}^{+1} f(t)(1-t^2)^{(n-3)/2} dt$$

holds.

Proof: We have

$$\int_{S^{n-1}} f(\xi_n) d\omega = \int_{\sum_{i=1}^{n-1} \xi_i^2 < 1} \frac{1}{\sqrt{1 - \sum_{i=1}^{n-1} \xi_i^2}}$$

$$(f(\sqrt{1 - \sum_{i=1}^{n-1} \xi_i^2}) + f(-\sqrt{1 - \sum_{i=1}^{n-1} \xi_i^2}) \; d(\xi_1 ... \xi_{n-1})$$

by $(I.4.3)$. Using Proposition $I.4.3$ in $n-1$ dimensions we get

$$\int_{S^{n-1}} f(\xi_n)\, d\omega = \omega_{n-1} \int_0^1 \varrho^{n-2} \frac{f(\sqrt{1-\varrho^2}) + f(-\sqrt{1-\varrho^2})}{\sqrt{1-\varrho^2}}\, dg$$

$$= \omega_{n-1} \int_0^1 (1-s^2)^{\frac{1}{2}(n-2)} \frac{f(s)+f(-s)}{s} \cdot \frac{+s}{\sqrt{1-s^2}}\, ds,$$

$$= \omega_{n-1} \int_0^1 (1-s^2)^{\frac{1}{2}n-\frac{3}{2}} (f(s)+f(-s))\, ds,$$

$$= \omega_{n-1} \int_{-1}^{+1} (1-s^2)^{\frac{n-3}{2}} f(s) ds,$$

where, to get the second equation, we have introduced new variables by setting $s = \sqrt{1-\varrho^2}$. □

Proposition I.4.5 *Let* $f : S^{n-1} \to \mathbb{R}$ *be continuous. Let* S *be real (n,n) - matrix with* $S^*S = SS^* = I$, *i.e.* S *is orthogonal. Then*

$$\int_{S^{n-1}} f(S\xi) d\omega = \int_{S^{n-1}} f(\xi) d\omega$$

Proof: We set $g(x) = |x| f(\frac{x}{|x|}), |x| > 0$. The transformation rule furnishes

$$\int_{\epsilon \le |x| \le 1} g(x) dx = \int_{\epsilon \le |x| \le 1} g(Sx) dx, 0 < \epsilon < 1;$$

from Proposition I.4.3 it follows that

$$\int_\epsilon^1 r^{n-1} \int_{S^{n-1}} r f(\xi) d\omega \ dr = \int_\epsilon^1 r^{n-1} \int_{S^{n-1}} r f(S\xi) d\omega \ dr,$$

$$\int_{S^{n-1}} f(\xi)d\omega = \int_{S^{n-1}} f(S\xi)d\omega,$$

since $\int_\epsilon^1 r^n dr > 0$.

Proposition I.4.6: *Let $f \epsilon C^o([-1,+1],\mathbb{R})$. Let x be an element of $\mathbb{R}^n$ with $|x| = 1$. Then*

$$\int_{S^{n-1}} f((x,\xi))d\omega = \omega_{n-1} \int_{-1}^{+1} f(t)(1-t^2)^{\frac{n-3}{2}} dt.$$

Proof: Let S be an orthogonal (n,n) - matrix with $S^*x = (0,...0,1)$. From the preceding proposition it follows that

$$\int_{S^{n-1}} f((x,y))d\omega = \int_{S^{n-1}} f((x,Sy))d\omega,$$

$$= \int_{S^{n-1}} f((S^*x,y))d\omega,$$

$$= \int_{S^{n-1}} f((\xi_n))d\omega;$$

thus we get with Proposition I.4.4

$$\int_{S^{n-1}} f((x,y))d\omega = \omega_{n-1} \int_{-1}^{+1} f(t)(1-t^2)^{(n-3)/2} dt$$

and our assertion is proved. □

Now we introduce the Laplace-Beltrami operator on S^{n-1}.

Definition I.4.1: *Let $f : S^{n-1} \rightarrow \mathbb{R}$ be a continuous function. Let $\varphi : \{x | x \epsilon \mathbb{R}^n, x \neq 0\} \rightarrow S^{n-1}$ be the function defined by $\varphi(x) = \frac{x}{|x|}$. We assume that $f \circ \varphi$ is twice continuously differentiable for $x \neq 0$ and set*

$$\wedge(f)(\xi) = \triangle f \circ \varphi(\xi), \xi \epsilon S^{n-1}.$$

$\wedge$ *is called the Laplace-Beltrami operator.*

Proposition I.4.7: *We have*

$$\triangle f \circ \varphi(x) = \frac{1}{r^2} \wedge (f)(\xi)$$

if $x \neq 0, x = r\xi$ *with* $r = |x|, \xi \epsilon S^{n-1}$.

Proof: Setting $g(x) = f \circ \varphi(x),\ x \neq 0\,,\ \overline{g}(r, \widetilde{\xi}) = g(r\widetilde{\xi}), r > 0,\ \widetilde{\xi} \epsilon \mathbb{R}^n$ with $\widetilde{\xi} = (\widetilde{\xi}_1, ..., \widetilde{\xi}_n)$ we get $(\partial \overline{g}/\partial \widetilde{\xi}_i)(r, \widetilde{\xi}) = r(\partial g/\partial x_i)(r\widetilde{\xi}), (\partial^2 \overline{g}/\partial \widetilde{\xi}_i^2)(r, \widetilde{\xi}) = r^2(\partial^2 g/\partial x_i^2)(r\xi)$. Consequently

$$\begin{aligned} \sum_{i=1}^{n} \frac{1}{r^2} \frac{\partial^2 \overline{g}}{\partial \widetilde{\xi}_i^2} (r, \widetilde{\xi}) &= \triangle g(r\widetilde{\xi}) \\ &= \triangle f \circ \partial(r\widetilde{\varphi}). \end{aligned}$$

If $\widetilde{\xi} \epsilon S^{n-1}, \xi = \widetilde{\xi}, x = r\xi$ we get

$$\sum_{i=1}^{n} \frac{1}{r^2} \frac{\partial^2 \overline{g}}{\partial \widetilde{\xi}_i^2} (r, \xi) = \triangle f \circ \varphi(x).$$

Since $\overline{g}(r, \widetilde{\xi}) = \overline{g}(1, \widetilde{\xi}), \widetilde{\xi} \epsilon \mathbb{R}^n$, we obtain first in the general case and then for $\widetilde{\xi} = \xi \epsilon S^{n-1}$ the relations

$$\begin{aligned} \sum_{i=1}^{n} \frac{\partial^2 \overline{g}}{\partial \widetilde{\xi}_i} (r, \widetilde{\xi}) &= \sum_{i=1}^{n} \frac{\partial^2 \overline{g}}{\partial \widetilde{\xi}_i} (1, \widetilde{\xi}) \\ &= \triangle g(\widetilde{\xi}), \\ \sum_{i=1}^{n} \frac{\partial^2 \overline{g}}{\partial \widetilde{\xi}_i} (1, \xi) &= \triangle g(\xi), \\ &= \wedge(f)(\xi). \end{aligned}$$

Our proposition is proved. $\square$

Proposition I.4.8: *Let:* $f : \mathbb{R}^n - \{0\} \to \mathbb{R}$ *be twice continuously differentiable. We set* $x = r\xi$ *if* $x \epsilon \mathbb{R}^n - \{0\}$ *with* $r = |x|, \xi \epsilon S^{n-1}$,

$$\widetilde{f(r, \widetilde{\xi})} = f(r\widetilde{\xi}), r > 0, \widetilde{\xi} \epsilon \mathbb{R}^n - \{0\}.$$

Then

(I.4.5) $\quad \triangle f(x) = \frac{\partial^2 \widetilde{f}}{\partial_{r^2}}(r, \xi) + \frac{n-1}{r} \frac{\partial \widetilde{f}}{\partial r}(r, \xi) +$

$$+\frac{1}{r^2} \wedge (\widetilde{f}(r, .))(\xi)$$

if $x = r\xi$ *with* r, ξ *as above.*

Proof: We introduce some auxiliary functions. Let

$$g(r, x) = f(r\frac{x}{|x|}), r > 0, x \neq 0.$$

If we consider r as to be the function $r : \mathbb{R}^n \to \mathbb{R}^+$,$r(x) = |x|$, then we get for $\widetilde{g}$ with $\widetilde{g}(x) = g(r(x), x)$ the relations

$$\frac{\partial \widetilde{g}}{\partial x_i}$$

if $x = r\xi$ *with* r, ξ *as above.*

Proof: We introduce some auxiliary functions. Let

$$g(r, x) = f(r\frac{x}{|x|}), r > 0, x \neq 0.$$

If we consider r as to be the function $r : \mathbb{R}^n \to \mathbb{R}^+, r(x) = |x|$, then we get for $\widetilde{g}$ with $\widetilde{g}(x) = g(r(x), x)$ the relations

$$\frac{\partial \widetilde{g}}{\partial x_i} = \frac{\partial g}{\partial r} \frac{\partial r}{\partial x_i} + \frac{\partial g}{\partial x_i}$$

$$\frac{\partial^2 \widetilde{g}}{\partial x_i^2} = \frac{\partial^2 g}{\partial r^2}(\frac{\partial r}{\partial x_i})^2 + \frac{\partial g}{\partial r} \frac{\partial^2 r}{\partial x_i^2} + \frac{\partial^2 g}{\partial x_i^2} + 2\frac{\partial^2 g}{\partial r \partial x_i} \frac{\partial r}{\partial x_i}.$$

Since we have $\frac{\partial r}{\partial x_i} = \frac{x_i}{r}, \frac{\partial^2 r}{\partial x_i^2} = \frac{1}{r} + x_i(-\frac{1}{r^2})\frac{x_i}{r} = \frac{1}{r} - \frac{x_i^2}{r^3}$ we arrive at

$$\frac{\partial^2 \widetilde{g}}{\partial x_i^2} = \frac{\partial^2 g}{\partial r^2} \frac{x_i^2}{r^2} + \frac{\partial g}{\partial r}(\frac{1}{r} - \frac{x_i^2}{r^3}) + \frac{\partial^2 g}{\partial x_i^2} + \frac{2}{r}\frac{\partial}{\partial r}(x_i \frac{\partial g}{\partial x_i}).$$

Since $\widetilde{g} = f$ we obtain

$$\triangle f(x) = \frac{\partial^2 g}{\partial r^2}(r, x) + \frac{n-1}{r} \frac{\partial g}{\partial r}(r, x) + \sum_{i=1}^{n} \frac{\partial^2 g}{\partial x_i^2}(r, x) +$$

$$+\sum_{i=1}^{n}\frac{2}{r}\,\frac{\partial}{\partial r}(x_i\frac{\partial g}{\partial x_i})(r,x)$$

Since $g(r,\lambda x)=g(r,x),\lambda>0$, the equation

(I.4.6) $\frac{\partial g}{\partial\lambda}(r,\lambda x)=\sum_{i=1}^{n}\frac{\partial g}{\partial x_i}(r,\lambda x)\lambda x_i=0$

holds for $\lambda=1$. Consequently

$$\triangle f(x)=\frac{\partial^2 g}{\partial r^2}(r,x)+\frac{n-1}{r}\,\frac{\partial g}{\partial r}(r,x)+$$

$$+\sum_{i=1}^{n}\frac{\partial^2 g}{\partial x_i^2}(r,x).$$

Since $g(r,.)=\widetilde{f}(r,\varphi(.))$ with φ as in Proposition I.4.8 the expression $\sum_{i=1}^{n}\frac{\partial^2 g}{\partial x_i^2}(r,x)$ equals to $\frac{1}{r^2}\Lambda(\widetilde{f}(r,.))(\xi)$ if $x=r\xi$ with $\xi\epsilon S^{n-1},r=|x|$. But since$\widetilde{f}(r,\xi)=g(r,\xi),r>0,\ \xi\epsilon S^{n-1}$, our proposition is proved. □

Formula (I.4.5) can also be written in the form

(I.4.7) $\triangle f(x)=\frac{\partial^2 f}{\partial r^2}(r\xi)+\frac{n-1}{r}\,\frac{\partial f}{\partial r}(r\xi)+\frac{1}{r^2}\Lambda(f(r.))(\xi)$.

Application of the chain rule yields in the case $n=2$ the formula $\Lambda(f(r.))(\xi)=\frac{\partial^2 f}{\partial\varphi^2}\,(r,\varphi)$ if $x=r\ cos\ \varphi,y=r\ sin\ \varphi,r=|x|,\xi=\frac{x}{|x|}=(cos\ \varphi,sin\ \varphi),r>0,\ 0\leq\varphi\leq\ 2\pi$ and if we set $f(r,\varphi)=f(r\ cos\ \varphi,r.sin\ \varphi)$ (which, however, is not quite correct).

Proposition I.4.9: *Let* $f:S^{n-1}\to\mathbb{R}$ *be a continuous function. Let* $\varphi:\{x|x\epsilon\mathbb{R}^n,x\neq 0\}\to\mathbb{R}^n$ *be the function defined by* $\varphi(x)=\frac{x}{|x|}$. *We assume that* $f\cdot\varphi$ *is twice continuously differentiable for* $x\neq 0$; *then*

$$\int_{S^{n-1}}\Lambda(f)(\xi)d\omega=0.$$

Proof: We set $g = f \cdot \varphi$. Then

$$\int_{\frac{1}{2}\le|x|\le 2} \Delta g dx = \int_{|x|=2} (\nabla g, \gamma) d\Omega -$$

$$- \int_{|x|=\frac{1}{2}} (\nabla g, \gamma) d\omega.$$

In the first integral on the right side of the last equation γ is the outer normal on G_2, in the second me $-\gamma$ is the outer normal on $G_{\frac{1}{2}}$. On $|x| = 2$ we have

$$(\nabla g, \gamma) = \frac{1}{2} \sum_{i=1}^{n} x_i \frac{\partial g}{\partial x_i}(x),$$

and m $|x| = \frac{1}{2}$ we obtain

$$(\nabla g, \gamma) = -2 \sum_{i=1}^{n} x_i \frac{\partial g}{\partial x_i}(x).$$

Formula (I.4.6) shows that in both cases $(\nabla g, \gamma)$ vanishes. Thus

$$0 = \int_{\frac{1}{2}\le|x|\le 2} \Delta g dx = \int_{\frac{1}{2}}^{2} r^{n-1} \int_{S_{n-1}} \frac{1}{r^2} \Lambda(f)(\xi) d\omega \; dr$$

$$= \int_{\frac{1}{2}}^{2} r^{n-3} \int_{S_n - 1} \Lambda(f)(\xi) d\omega \; dr$$

where we have applied Propositions I.4.3, I.4.7. Since $\int_{1/2}^{2} r^{n-3} dr \neq 0$ we obtain the assertion. □

Problem I.4.1: Let f be as in Proposition I.4.9. Shows that

$$\int_{S_{n-1}} \Lambda(f)(\xi) f(\xi) \; d\omega \le 0.$$

Problem I.4.2: Let f, h be as f in Proposition I.4.9. Shows that

$$\int_{S_{n-1}} (\Lambda(f)(\xi)) h(\xi) - \Lambda(h)(\xi) f(\xi) \; d\omega = 0.$$

Mit ♡-lichen Grüßen