Vorlesung „Funktionalanalysis II “ (Functional Analysis II)

I. Unbounded Operators in Hilbert Spaces. General Theory

II. Spectral Theory of Selfadjoint Operators

I. Unbounded Operators in Hilbert Spaces. General Theory

§ 1. Closed Operators

Definition I.1.1: Let T be a linear operator in a Hilbert space $\mathcal{H}$ with domain of definition $\mathcal{D}(T)$. T is called closed if and only if the following implication is valid: Let $\{f_n\}$ be a sequence in $\mathcal{H}$ with

$$f_n \to f, \quad n = 1,2,\ldots,$$
$$f_n \in \mathcal{D}(T),$$
$$Tf_n \to g, \quad n = 1,2,\ldots .$$

Then $f \in \mathcal{D}(T)$ and $g = Tf$.

Of course every bounded operator $\tilde{T}$ with $\mathcal{D}(\tilde{T}) = \mathcal{H}$ is closed. If T is a linear operator in $\mathcal{H}$ with domain of definition $\mathcal{D}(T)$ and if $\tilde{T} \in L(\mathcal{H},\mathcal{H})$, we set

$$\mathcal{D}(T+\tilde{T}) = \mathcal{D}(T),$$
$$(T+\tilde{T})u = Tu + \tilde{T}u, \quad u \in \mathcal{D}(T).$$

If $\tilde{T} = cI$, $c \in \mathbb{C}$, we often write cu instead of $\tilde{T}u = cIu$ and $(T+c)u$ instead of $(T+cI)u$.

Definition I.1.2: Let T_1 be a linear operator in a Hilbert space $\mathcal{H}$ with domain of definition $\mathcal{D}(T_1)$. Let T_2 be a second linear operator in H with domain of definition $\mathcal{D}(T_2)$. Let

$$\mathcal{D}(T_1) \subseteq \mathcal{D}(T_2),$$

$T_1 x = T_2 x,\ x \in \mathcal{D}(T_1)$.

Then T_2 is called a continuation of T_1.

We now pose the question, under which conditions on T_1 it is possible to construct a closed continuation of T_1. If T_2 is such a closed continuation, then the following implication holds: Let $f_n \to 0$, $n = 1,2,\ldots$, $T_1 f_n \to g$, then $g = T_1 0 = 0$. In other words: $\|f_n\| \to 0$, $f_n \in \mathcal{D}(T_1)$, $\|T_1(f_n - f_m)\| \to 0$, $n,m \to \infty$ implies $\|T_1 f_n\| \to 0$.

Definition I.1.3: A linear operator T with domain of definition $\mathcal{D}(T)$ is called closeable if and only if the following implication holds: If for a sequence $\{f_n\}$ with $f_n \in \mathcal{D}(T)$ the $\{Tf_n\}$ are a Cauchy sequence and if $f_n \to 0$, $n = 1,2,\ldots$, then $Tf_n \to 0$.

Theorem I.1.1: Let T_1 be a linear operator with domain of definition $\mathcal{D}(T_1)$. Then T_1 has a closed continuation T_2 if and only if T_1 is closeable. If T_1 is closeable then there is a smallest closed continuation $\overline{T}_1$. This means that $\overline{T}_1$ has the following properties:

1. $\overline{T}_1$ is a continuation of T_1.
2. Any closed continuation T_2 of T_1 is a continuation of $\overline{T}_1$.

$\overline{T}_1$ is called the closure of T_1.

Proof: From what was said before Definition I.1.3 it is evident that our condition is necessary. Now we assume that T_1 is closeable. We set

$$\mathcal{D}(\overline{T}_1) = \{f \mid f \in \mathcal{H}, \text{ there exists a sequence } \{f_n\} \text{ with } f_n \in \mathcal{D}(T_1),\ f_n \to f,\ n=1,2,\dots, \text{ and } \|T_1(f_n - f_m)\| \to 0,\ n,m \to \infty\}.$$

It is clear that $\mathcal{D}(\overline{T}_1)$ is a linear subspace of $\mathcal{H}$. We set

$$\overline{T}_1 f = \lim_{n\to\infty} T_1 f_n,\quad f \in \mathcal{D}(\overline{T}_1).$$

If $\{f_n'\}$ is another sequence contained in $\mathcal{D}(T_1)$ with $f_n' \to f$, $n=1,2,\dots$, and $\|T_1(f_n'-f_m')\| \to 0$, $n,m \to \infty$, set $h_n = f_n - f_n'$; then $h_n \to 0$, $n \to \infty$, and $\|T_1(h_n - h_m)\| \to 0$, $n,m \to \infty$. Since T_1 is closeable we obtain $T_1 h_n \to 0$. The proof that $\overline{T}_1$ is linear may be omitted. It must be shown now that $\overline{T}_1$ is closed. Let $\{f_n\}$ be a sequence contained in $\mathcal{D}(\overline{T}_1)$ with $f_n \to f$, $\overline{T}_1 f_n \to g$, then we can choose for each f_n a $f_n' \in \mathcal{D}(T_1)$ with $\|f_n - f_n'\| \leq \frac{1}{n}$ and $\|\overline{T}_1 f_n - T_1 f_n'\| \leq \frac{1}{n}$. Therefore $f_n' \to f$, $T_1 f_n' \to g$, $n \to \infty$, and consequently $f \in \mathcal{D}(\overline{T}_1)$, $g = \overline{T}_1 f = \lim_{n\to\infty} T_1 f_n'$. If T_2 is any closed continuation of T_1 then necessarily $\mathcal{D}(T_2) \supset \mathcal{D}(\overline{T}_1)$ and $\overline{T}_1 f = T_2 f$, $f \in \mathcal{D}(\overline{T}_1)$. □

As the following example shows there are Hilbert spaces $\mathcal{H}$ and operators T in $\mathcal{H}$ which are not closeable: Set $\mathcal{H} = L^2((-1,+1))$, $\mathcal{D}(T) = C^0([-1,+1])$,

$$(Tf)(x) = f(0),\quad f \in \mathcal{D}(T),\quad x \in [-1,+1].$$

One easily constructs a sequence $\{f_n\}$ contained in $\mathcal{D}(T)$ with

$$\|f_n\|_{L^2((-1,+1))} \to 0,\quad n \to \infty,$$

$$1 = f_n(0) = (Tf_n)(x),\quad x \in [-1,+1].$$

Take e.g. $f_n(x) = 0$, $-1 \leq x \leq -\frac{1}{n}$, $f_n(x) = nx+1$, $-\frac{1}{n} \leq x \leq 0$, $f_n(x) = -nx+1$, $0 \leq x \leq \frac{1}{n}$, $f_n(x) = 0$, $\frac{1}{n} \leq x \leq 1$. Since

$$\|Tf_n\|_{L^2((-1,+1))} = \sqrt{2},$$

the operator T is not closeable.

Definition I.1.4: A linear subspace $\mathcal{D}$ of a Hilbert space $\mathcal{H}$ is called dense in $\mathcal{H}$ if for each $f \in \mathcal{H}$ there is a sequence $\{g_n\}$ contained in $\mathcal{D}$ such that

$$g_n \to f, \ n \to \infty$$

We can now define the notion of the adjoint of an operator T.

Definition I.1.5: Let T be a linear operator in a Hilbert space $\mathcal{H}$ with domain of definition $\mathcal{D}(T)$. Let $\mathcal{D}(T)$ be dense in $\mathcal{H}$. $\mathcal{D}(T^*)$ is the set of all $g \in \mathcal{H}$ such that there exists a $g^* \in \mathcal{H}$ with

$$\text{(I.1.1)} \quad (Tf,g) = (f,g^*), \ f \in \mathcal{D}(T).$$

To complete our definition we need

Theorem I.1.2: g^* in (I.1.1) is determined uniquely. If we set

$$g^* =: T^*g, \ g \in \mathcal{D}(T^*)$$

then T^* is a linear closed operator in $\mathcal{H}$ with domain of definition $\mathcal{D}(T^*)$.

Proof: From the density of $\mathcal{D}(T)$ it follows that g^* is determined uniquely. The linearity of T^* does not need a proof. Now let $g_n \to g$, $T^*g_n \to h$, $n \to \infty$, with $g_n \in \mathcal{D}(T^*)$, $n = 1,2,\ldots$. Then

$$(Tf,g_n) = (f,T^*g_n),$$
$$(Tf,g) = (f,h),\ f \in \mathcal{D}(T).$$

The proof is completed. □

We give some examples. The first one concerns ordinary differential operators. Let $H = L^2((a,b))$, let

$$\mathcal{D}(T) = C_0^N((a,b))$$

for some $N \in \mathbb{N}$, and let $p_k \in C^k((a,b))$, $1 \leq k \leq N$. Then we set

$$Tf(x) = \sum_{k=0}^{N} p_k(x) f^{(k)}(x),\ f \in \mathcal{D}(T).$$

$\mathcal{D}(T)$ is dense in H since already $C_0^\infty((a,b))$ is dense in $L^2((a,b))$, and we have for $f,g \in \mathcal{D}(T)$

$$(Tf,g) = \int_a^b f(x) \sum_{k=0}^{N} \overline{(-1)^k (\bar{p}_k \cdot g)^{(k)}(x)}\, dx,$$

$$= (f,T^*g).$$

Thus $\mathcal{D}(T^*) \supset \mathcal{D}(T)$ and

$$T^*g = \sum_{k=0}^{N} (-1)^k (\bar{p}_k \cdot g)^{(k)},\ g \in \mathcal{D}(T).$$

The second example stems from the field of partial differential operators. Let Ω be a bounded open set of $\mathbb{R}^n$, let $H = L^2(\Omega)$. Let $m \in \mathbb{N}$. For each multiindex α of $\mathbb{R}^n$ let there be given functions $A_\alpha \in C^{|\alpha|}(\Omega)$. We set

$$Tf(x) = \sum_{|\alpha| \leq 2m} A_\alpha(x) D^\alpha f(x),\ f \in \mathcal{D}(T) = C_0^{2m}(\Omega).$$

Then $C_0^{2m}(\Omega)$ is dense in H, since already $C_0^\infty(\Omega)$ is dense in H. The Theorem of Gauß furnishes

$$(Tf,g) = \int_\Omega f(x) \sum_{|\alpha| \leq 2m} (-1)^{|\alpha|} \overline{D^\alpha (\overline{A}_\alpha g)(x)}\, dx, \quad g \in \mathcal{D}(T).$$

Thus $\mathcal{D}(T^*) \supset \mathcal{D}(T)$,

$$T^* g = \sum_{|\alpha| \leq 2m} (-1)^{|\alpha|} \cdot D^\alpha (\overline{A}_\alpha g),$$

$g \in \mathcal{D}(T)$.

The next theorem gives a criterion for the existence of a closed continuation of a given linear operator.

Theorem I.1.3: Let T be a linear operator in H with domain of definition $\mathcal{D}(T)$. Let $\mathcal{D}(T)$ and $\mathcal{D}(T^*)$ be dense. Then T is closeable and

$$T^* = \overline{T}^*.$$

Proof: First we prove that T is closeable. Let $\{f_n\}$ be a sequence in $\mathcal{D}(T)$ with $f_n \to 0$, $Tf_n \to g$, $n = 1, 2, \ldots$. Let $h \in \mathcal{D}(T^*)$. Then

$$(g,h) = \lim_{n\to\infty} (Tf_n, h)$$

$$= \lim_{n\to\infty} (f_n, T^* h) = 0.$$

Since $\mathcal{D}(T^*)$ is dense in H, we get $g = 0$. Thus T is closeable. Now we prove the second assertion: Let $g \in \mathcal{D}(\overline{T}^*)$. Then

$$(\overline{T} f, g) = (f, \overline{T}^* g), \quad f \in \mathcal{D}(\overline{T}).$$

In particular, we get for $f \in \mathcal{D}(T)$

$$(Tf,g) = (f, \overline{T}^* g),$$

$$g \in \mathcal{D}(T^*),$$
$$T^*g = \overline{T}^*g,$$
$$\mathcal{D}(\overline{T}^*) \subseteq \mathcal{D}(T^*).$$

As for the opposite direction let $g \in \mathcal{D}(T^*)$. Then

$$(Tf,g) = (f,T^*g), \quad f \in \mathcal{D}(T).$$

If $f \in \mathcal{D}(\overline{T})$, there is a sequence $\{f_n\}$ with $f_n \in \mathcal{D}(T)$, $f_n \to f$, $Tf_n \to \overline{T}f$, $n = 1,2,\ldots$. Consequently, if f_n is inserted instead of f, the preceding equality furnishes

$$(\overline{T}f,g) = (f,T^*g).$$

Thus $g \in \mathcal{D}(\overline{T}^*)$,

$$\mathcal{D}(\overline{T}^*) \supseteq \mathcal{D}(T^*),$$
$$\overline{T}^*g = T^*g.$$

The theorem is proved. □

§ 2. The Graph of a Linear Operator

The set $H \times H = \{\{f,g\} | \ f \in H, \ g \in H\}$ can be made a Hilbert space by the following definitions:

(I.2.1) $\alpha\{f,g\} + \beta\{h,k\} := \{\alpha f+\beta h, \ \alpha g+\beta k\}, \ \alpha,\beta \in \mathbb{C}$,

(I.2.2) $(\{f,g\},\{h,k\}) := (f,h) + (g,k)$,

$\|\{f,g\}\| := (\|f\|^2+\|g\|^2)^{1/2}$.

By (I.2.1) $H \times H$ becomes a vector space over $\mathbb{C}$ with $\{0,0\}$ as the element zero. By (I.2.2) the structure of a Hilbert space is imposed on $H \times H$: It is easily shown that $H \times H$ is complete with the norm just defined.

Definition I.2.1: Let T be a linear operator in H with domain of definition $\mathcal{D}(T)$. The set

$G(T) = \{\{f,Tf\} | \ f \in \mathcal{D}(T)\}$

is a linear subspace of $H \times H$ and called the graph of T.

If T_1, T_2 are linear operators in H with domains of definition $\mathcal{D}(T_1)$, $\mathcal{D}(T_2)$, and if T_2 is a continuation T_1 (cf. Definition I.1.2), then this is evidently equivalent with

$G(T_1) \subseteq G(T_2)$.

We also write in this case

(I.2.3) $T_1 \subseteq T_2$.

<u>Proposition I.2.1:</u> T <u>is closed if and only if</u> $G(T)$ <u>is closed</u>.

<u>Proof:</u> Let $G(T)$ be closed. Let $\{f_n, Tf_n\} \to \{f,g\}$, $n = 1,2,\ldots$, in $H \times H$. Then $\{f,g\} \in G(T)$, $f_n \to f$, $Tf_n \to g$, $n = 1,2,\ldots$. Thus $f \in \mathcal{D}(T)$, $g = Tf$, and T is closed. If T is closed and if $\{f_n, Tf_n\} \to \{f,g\}$, $n = 1,2,\ldots$, then $f \in \mathcal{D}(T)$, $g = Tf$. Our proposition is proved. □

Now we introduce an operator $U \in L(H \times H, H \times H)$. U is defined by

$$U: \{f,g\} \mapsto \{-g,f\}.$$

Then evidently

$$U^2 = -I.$$

<u>Proposition I.2.2:</u> <u>Let</u> T <u>be a closed operator in</u> H <u>with dense domain of definition</u> $\mathcal{D}(T)$. <u>Then</u>

$$G(T)^{\perp} = U(G(T^*)), \tag{I.2.4}$$

$$(UG(T))^{\perp} = G(T^*). \tag{I.2.5}$$

<u>Proof:</u> 1. Let $\{\varphi,\psi\} \in G(T)^{\perp}$. Then $(\{f,Tf\},\{\varphi,\psi\}) = 0$ for all $f \in \mathcal{D}(T)$. Thus

$$(f,\varphi) + (Tf,\psi) = 0,$$

$$(Tf,\psi) = (f,-\varphi), \quad f \in \mathcal{D}(T).$$

Consequently $\psi \in \mathcal{D}(T^*)$, $T^*\psi = -\varphi$ and $\{\varphi,\psi\} \in U(G(T^*))$. If $\{-T^*\psi,\psi\} \in U(G(T^*))$, then

$$(f,-T^*\psi) + (Tf,\psi) = 0,$$

$f \in \mathcal{D}(T)$. Thus

$$\{-T^*\psi,\psi\} \in G(T)^{\perp}.$$

Thus (I.2.4) is proved.

2. Let $\{-Tf,f\} \in UG(T)$. Then we take an element $\{\varphi,\psi\}$ with

$$(\{-Tf,f\},\{\varphi,\psi\}) = 0, \text{ thus}$$

$$-(Tf,\varphi) + (f,\psi) = 0.$$

If f runs through all of $\mathcal{D}(T)$, then $\{-Tf,f\}$ runs through all of $UG(T)$, and if $\{\varphi,\psi\}$ is in $UG(T)^{\perp}$ we get: $\varphi \in \mathcal{D}(T^*)$, $\psi = T^*\varphi$, $\{\varphi,\psi\} \in G(T^*)$. For the second direction we have to go through our relations from backward. Our proposition is proved. □

Proposition I.2.3: Let T be a linear operator in H with domain of definition $\mathcal{D}(T)$. Let T be closeable. Then

$$\overline{G(T)} = G(\overline{T}).$$

Proof: The proof may be left to the reader. □

Now we want to characterize closed operators in terms of their adjoints.

Theorem I.2.1: Let T be a closed linear operator in H with dense domain of definition $\mathcal{D}(T)$. Then $\mathcal{D}(T^*)$ is also dense in H, and moreover

$$T^{**} = T. \quad \text{(I.2.6)}$$

Proof: We first show that $\mathcal{D}(T^*)$ is dense in H. If $\mathcal{D}(T^*)$ is not dense in H, then there is an $h \in H$ such that

$$h \neq 0,$$
$$(g,h) = 0, \; g \in \mathcal{D}(T^*).$$

Thus $(\{-T^*g,g\},\{0,h\}) = 0$ for all $g \in \mathcal{D}(T^*)$ and $\{0,h\} \in (UG(T^*))^{\perp}$. According to Proposition I.2.2 we have

$$\{0,h\} \in ((G(T))^{\perp})^{\perp} = G(T),$$

$$\{0,h\} = \{f,Tf\}$$

for some $f \in \mathcal{D}(T)$, and consequently $f = 0$, $Tf = h = 0$. Therefore $\mathcal{D}(T^*)$ is dense, and we can construct T^{**}. Since $G(T^{**}) = (UG(T^*))^{\perp}$ $= G(T)$ by Proposition I.2.2 we arrive at $T^{**} = T$, and our Theorem is proved.

Remark: If $T_1 \subseteq T_2$, $\mathcal{D}(T_1)$ dense in $\mathcal{H}$, then $T_2^* \subseteq T_1^*$.

Theorem I.2.2: Let T be a closeable operator in H with dense domain of definition $\mathcal{D}(T)$. Then $\mathcal{D}(T^*)$ is dense in H and

$$T^{**} = \overline{T}.$$

Proof: From the preceding theorem it follows that $\overline{T}^*$ has dense domain of definition $\mathcal{D}(\overline{T}^*)$ and that $\overline{T}^{**} = \overline{T}$. From Theorem I.1.3 we get with $\overline{T}^* \subseteq T^*$ that $\mathcal{D}(T^*)$ is dense and that $(\overline{T}^*)^* = T^{**}$. □

§ 3. Hermitian Operators

We start with

Definition I.3.1: A linear operator H in a Hilbert space $\mathcal{H}$ with domain of definition $\mathcal{D}(H)$ is called hermitian if and only if

1. $\mathcal{D}(H)$ is dense in $\mathcal{H}$,
2. $(Hf,g) = (f,Hg)$, $f,g \in \mathcal{D}(H)$.

Shortly spoken, an operator H is hermitian if $H^* \supseteq H$. We give an example: Let $\mathcal{H} = L^2(\Omega)$, where Ω is a bounded open set of $\mathbb{R}^n$, let $\mathcal{D}(H) = C_0^\infty(\Omega)$. Let $m \in \mathbb{N}$. Let there be given functions $A_{\alpha\beta} \in C^m(\Omega)$ for all multiindices α,β of $\mathbb{R}^n$ with $|\alpha|,|\beta| \leq m$. Moreover we assume that

$$(I.3.1)\quad A_{\alpha\beta} = (-1)^{|\alpha|+|\beta|}\overline{A_{\beta\alpha}}.$$

We set

$$Hu = \sum_{\substack{|\alpha|\leq m,\\ |\beta|\leq m}} D^\alpha(A_{\alpha\beta}D^\beta u),$$

$u \in \mathcal{D}(H)$. By Gauß's Theorem we get

$$(Hf,g) = (f, \sum_{\substack{|\alpha|\leq m,\\ |\beta|\leq m}} (-1)^{|\alpha|+|\beta|} D^\beta(\overline{A_{\alpha\beta}}D^\alpha g)),$$

provided $f,g \in C_0^\infty(\Omega)$. Thus $(Hf,g) = (f,Hg)$, $f,g \in \mathcal{D}(H)$ by (I.3.1).

Definition I.3.2: Let A be a linear operator in $\mathcal{H}$ with dense domain of definition $\mathcal{D}(A)$. A is called selfadjoint if and only if $A^* = A$.

Proposition I.3.1: Let A be a linear operator in H with dense domain of definition $\mathcal{D}(A)$. Then A is selfadjoint if and only if

1. A is hermitian.

2. If $(Au,v) = (u,v^*)$ for all $u \in \mathcal{D}(A)$ and some $v,v^* \in H$, then $v \in \mathcal{D}(A)$ and $v^* = Av$. - Every selfadjoint operator is hermitian and closed.

Since the proof of this proposition is trivial, we omit it. Now we give an example for a selfadjoint operator. Let $H = l_2$, i.e. the space of all sequences $x = (x_1, x_2, \ldots)$ of complex numbers with $\sum_{k=1}^{\infty} |x_k|^2 < +\infty$ and scalar product $(x,y) = \sum_{k=1}^{\infty} x_k \bar{y}_k$. We set $\mathcal{D}(A) = \{x | x \in l_2, \sum_{k=1}^{\infty} k^2 |x_k|^2 < +\infty\}$ and

$$Ax = (x_1, 2x_1, 3x_3, \ldots),\ x \in \mathcal{D}(A).$$

Evidently, A is hermitian. Now let $(Ax,y) = (x,y^*)$ for all $x \in \mathcal{D}(A)$ and some y, $y^* \in l_2$. Then we have

$$\sum_{k=1}^{\infty} k x_k \bar{y}_k = \sum_{k=1}^{\infty} x_k \bar{y}_k^*$$

if $y = (y_1, y_2, \ldots)$, $y^* = (y_1^*, y_2^*, \ldots)$. We set: $x^{(\mu)}$ is the sequence whose components are 0 with the exception of the μ-th one, which is 1. Then we get: $\mu y_\mu = y_\mu^*$, $\mu = 1,2,\ldots$, and consequently $y \in \mathcal{D}(A)$, $y^* = Ay$.

Theorem I.3.1: Let H be a hermitian operator in H with domain of definition H. Then H is selfadjoint.

Proof: It follows that $H^* = H$.

Definition I.3.3: Let H be a hermitian operator in $\mathcal{H}$ with domain of definition $\mathcal{D}(H)$. H is called essentially selfadjoint if and only if its closure $\overline{H}$ is selfadjoint.

In order to explain this definition we first remark that with H also its closure $\overline{H}$ is hermitian. Namely, let $f,g \in \mathcal{D}(\overline{H})$, $f_n \to f$, $g_n \to g$, $n \to \infty$, with $f_n, g_n \in \mathcal{D}(H)$, and $Hf_n \to \overline{H}f$, $Hg_n \to \overline{H}g$, then $(Hf_n, g_n) = (f_n, Hg_n)$ and consequently $(\overline{H}f,g) = (f,\overline{H}g)$. Our definition is equivalent with each of the following statements:

(I.3.2) $\overline{H}^* = \overline{H}$,

(I.3.3) $H^* = \overline{H}$,

(I.3.4) $H^* = H^{**}$.

(I.3.2) is clear, Theorem I.1.3 furnishes the relation $\overline{H}^* = H^*$, and from (I.3.2) we get $H^* = \overline{H}$. From (I.3.3) we get then with Theorem I.2.2 the relation (I.3.4). From (I.3.4) it follows with $H^* = \overline{H}^*$ and $H^{**} = \overline{H}$ that (I.3.2) holds.

Next we want to characterize the selfadjointness of a hermitian operator H in terms of the subspaces $(H+i)(\mathcal{H})$, $(H-i)(\mathcal{H})$. For a linear operator T in $\mathcal{H}$ with domain of definition $\mathcal{D}(T)$ we set

(I.3.5) $\mathcal{R}(T) = T(\mathcal{H}) = \{Tf \mid f \in \mathcal{D}(T)\}$.

Evidently $\mathcal{R}(T)$ is a linear subspace of $\mathcal{H}$, it's called the range of T.

Theorem I.3.2: Let H be a hermitian operator in $\mathcal{H}$ with domain of definition $\mathcal{D}(H)$. Then H is selfadjoint if and only if

$$\mathcal{R}(H+i) = \mathcal{H} \text{ and } \mathcal{R}(H-i) = \mathcal{H}.$$

Let us remark that in general we write T+λ for the operator T+λI being defined on $\mathcal{D}(T)$; λ is any complex number.

Proof: Let us first assume that H is selfadjoint. For $f \in \mathcal{D}(H)$ we have

$$\begin{aligned}\|(H\pm i)f\|^2 &= (Hf\pm if, Hf\pm if)\\ &= \|Hf\|^2 + \|f\|^2 \pm (if,Hf) \pm (Hf,if),\\ &= \|Hf\|^2 + \|f\|^2 \pm i(f,Hf) \mp i(Hf,f),\\ &= \|Hf\|^2 + \|f\|^2.\end{aligned}$$

Thus $\|(H\pm i)f\| \geq \|f\|$. Thus we can define the inverse $(H\pm i)^{-1}$ of $(H\pm i)$ on $\mathcal{R}(H\pm i)$. We have $\|(H\pm i)^{-1}\| \leq 1$. Now we show that $\mathcal{R}(H\pm i)$ are closed subspaces of $\mathcal{H}$. Let $g \in \overline{\mathcal{R}(H\pm i)}$, $g_n \to g$, $n \to \infty$, with $g_n \in \mathcal{R}(H\pm i)$. Then $g_n = (H\pm i)f_n$ with uniquely determined $f_n \in \mathcal{D}(H)$. Since

$$\|g_n - g_m\| = \|(H\pm i)(f_n - f_m)\| \geq \|f_n - f_m\|$$

we obtain that $f_n \to f$, $n \to \infty$. Thus $Hf_n \to g \pm if$, $n \to \infty$; since H is closed we arrive at $f \in \mathcal{D}(H)$, $Hf = g \pm if$, $(H\pm i)g = g$, $g \in \mathcal{R}(H\pm i)$. Consequently, $\mathcal{R}(H\pm i)$ is closed. If $\mathcal{R}(H+i) \neq \mathcal{H}$, then there exists a $g \in \mathcal{H}$ with

$$\begin{aligned}((H+i)f,g) &= 0, \quad f \in \mathcal{D}(H),\\ \|g\| &= 1.\end{aligned}$$

Therefore $(Hf,g) = (f,ig)$, $g \in \mathcal{D}(H)$, $Hg = ig$, since we have assumed H to be selfadjoint. Since $0 = \|(H-i)g\| \geq \|g\|$ we arrive at $g = 0$ which is a contradiction. The case $\mathcal{R}(H-i) \neq \mathcal{H}$ is treated analogously. Thus we have proved that $\mathcal{R}(H\pm i) = \mathcal{H}$. As for the second direction we assume that $\mathcal{R}(H\pm i) = \mathcal{H}$. Now let

$$(Hf,g) = (f,g^*), \quad f \in \mathcal{D}(H).$$

Thus

$$((H+i)f,g) = (f,g^*-ig), \quad f \in \mathcal{D}(H),$$

and there is an $h \in \mathcal{D}(H)$ with $g^*-ig = (H-i)h$; it follows that

$$\begin{aligned} ((H+i)f,g) &= (f,(H-i)h), \\ &= ((H+i)f,h), \quad f \in \mathcal{D}(H). \end{aligned}$$

Since $\mathcal{R}(H+i) = \mathcal{H}$ we get $g = h \in \mathcal{D}(H)$. Thus H is selfadjoint. Our theorem is proved. □

<u>Proposition I.3.2:</u> <u>Let</u> H <u>be a hermitian operator in</u> $\mathcal{H}$ <u>with domain of definition</u> $\mathcal{D}(H)$. <u>Then</u> $\overline{\mathcal{R}(H\pm i)} = \mathcal{R}(\overline{H}\pm i)$.

<u>Proof:</u> Let $g \in \overline{\mathcal{R}(H+i)}$, $g = \lim_{n\to\infty} (H+i)f_n$. Since $\|f_n-f_m\| \leq \|(H+i)(f_n-f_m)\|$, the sequence $\{f_n\}$ is convergent; we set $f = \lim_{n\to\infty} f_n$. Then $\{Hf_n\}$ is also convergent; we set

$$\tilde{g} = \lim_{n\to\infty} Hf_n$$

and obtain $Hf_n \to g-if=\tilde{g}, n \to \infty$, $f \in \mathcal{D}(\overline{H})$, $\overline{H}f = g-if$, $(\overline{H}+i)f = g$. Thus $g \in \mathcal{R}(\overline{H}+i)$. Analogously we can prove that $\overline{\mathcal{R}(H-i)} \subset \mathcal{R}(\overline{H}-i)$. As for the second part of the proof let $g \in \mathcal{R}(\overline{H}+i)$. Then $g = \overline{H}f+if$ with a unique $f \in \mathcal{D}(\overline{H})$ (observe that $\overline{H}$ is also hermitian). Thus there is a sequence $\{f_n\}$ with $f_n \in \mathcal{D}(H)$, $n = 1,2,\dots$, $f_n \to f$, $Hf_n \to \overline{H}f$, $n \to \infty$. Thus $(H+i)f_n \to (\overline{H}+i)f = g$, $n \to \infty$, and $\mathcal{R}(\overline{H}+i) \subset \overline{\mathcal{R}(H+i)}$. In the same way it is shown that $\mathcal{R}(\overline{H}-i) \subset \overline{\mathcal{R}(H-i)}$. Our proposition is proved. □

Proposition I.3.3: Let H be hermitian with domain of definition $\mathcal{D}(H)$. Then

$$\|(H-z)f\| \geq |\mathrm{Im}\ z|\,\|f\|, \quad f \in \mathcal{D}(H),\ z \in \mathbb{C}.$$

Proof: Let $z = a+ib$ with $a,b \in \mathbb{R}$, $b \neq 0$. Then $H-a$ is hermitian and

$$\|(H-z)f\| = \|((H-a)-ib)f\|,$$
$$= |b|\,\|(\tfrac{1}{b}(H-a)-i)f\| \geq |b|\,\|f\|.$$ □

As a consequence from Proposition I.3.3 we get that for every hermitian operator H with domain of definition $\mathcal{D}(H)$ the operator $(H-z)^{-1}$ exists with domain of definition $\mathcal{R}(H-z)$ and is bounded, i.e.

$$\|(H-z)^{-1}f\| \leq \frac{1}{|\mathrm{Im}\ z|}\|f\|, \quad f \in \mathcal{R}(H-z),$$

provided $\mathrm{Im}\ z \neq 0$.

Theorem I.3.3: Let H be hermitian with domain of definition $\mathcal{D}(H)$. H is essentially selfadjoint if and only if

$$\overline{\mathcal{R}(H\pm i)} = \mathcal{H}.$$

Proof: $\overline{H}$ is selfadjoint if and only if $\mathcal{R}(\overline{H}\pm i) = \mathcal{H}$, but according to Proposition I.3.3 this is equivalent to $\overline{\mathcal{R}(H\pm i)} = \mathcal{H}$. □

Next we treat some examples:

1st Example: We set $\mathcal{H} = L^2((-\pi,+\pi))$,

$$\mathcal{D}(H_o) = C_o^2((-\pi,+\pi)),$$
$$H_o u = -u'', \quad u \in \mathcal{D}(H_o).$$

Then $\mathcal{D}(H_o)$ is dense in $\mathcal{H}$ and H_o is hermitian, since by partial integration

$$\int_{-\pi}^{+\pi} (-u'')\overline{v}\, dx = \int_{-\pi}^{+\pi} u\overline{(-v'')}\, dx,$$

$u,v \in C_o^2((-\pi,+\pi))$. H_o, however, is not essentially selfadjoint. This is seen as follows: For $u \in \mathcal{D}(H_o)$ we even have

$$\begin{aligned} ((H_o-i)u,v) &= \int_{-\pi}^{+\pi} (-u''-iu)\overline{v}\, dx, \\ &= \int_{-\pi}^{+\pi} (-u\overline{v''}-iu\overline{v})\, dx, \\ &= \int_{-\pi}^{+\pi} u\overline{(-v''+iv)}\, dx, \quad v \in C^2((-\pi,+\pi)). \end{aligned}$$

$v = e^{\sqrt{+i}\cdot}$ is a twice continuously differentiable function on $[-\pi,+\pi]$ with $-v''+iv = 0$ on $[-\pi,+\pi]$. Thus $v \in L^2((-\pi,+\pi))$ is orthogonal to $\mathcal{R}(H_o-i)$, but $v \neq 0$. Therefore $\mathcal{R}(H_o-i)$ is not dense in $\mathcal{H}$.

2nd Example: Again $\mathcal{H} = L^2((-\pi,+\pi))$, but

$$\mathcal{D}(H_1) = \{u \mid u \in C^2([-\pi,+\pi]),\ u(-\pi) = u(+\pi),\ u'(-\pi) = u'(+\pi)\},$$

$$H_1 u = -u'',\ u \in \mathcal{D}(H_1).$$

By partial integration it can be easily shown that H_1 is hermitian. We claim that H_1 is essentially selfadjoint. To prove this we first observe that the functions

$$\varphi_k(x) = \frac{1}{\sqrt{2\pi}} e^{ikx},\ k = 0, \pm 1, \pm 2, \ldots,$$

have the following properties: $\varphi_k \in \mathcal{D}(H_1)$,

$$H_1\varphi_k = k^2\varphi_k,$$

$\{\varphi_k \mid k = 0, \pm 1, \pm 2, \ldots\}$ is a complete orthonormal system in $\mathcal{H}$.

Our assertion is then furnished by the

Proposition I.3.4: Let H be a hermitian operator in a Hilbert space $\mathcal{H}$ with domain of definition $\mathcal{D}(H)$. If there is a complete orthonormal system $\{\tilde{\varphi}_1, \tilde{\varphi}_2, \ldots\}$ in $\mathcal{H}$ with

$$\text{(I.3.6)} \quad H\tilde{\varphi}_k = \lambda_k\tilde{\varphi}_k, \quad k = 1,2,\ldots$$

for some $\lambda_k \in \mathbb{C}$, $k = 1,2,\ldots$, then H is essentially selfadjoint.

Proof: From (I.3.6) it follows that $\lambda_k \in \mathbb{R}$. The set

$$\mathcal{D} = \{f \mid f = \sum_{k=1}^{N} c_k\tilde{\varphi}_k \text{ for some } N \in \mathbb{N} \text{ and some } c_1,\ldots,c_N \in \mathbb{C}\}$$

is contained in $\mathcal{D}(H)$ and dense in $\mathcal{H}$. Let $g \in \mathcal{H}$. Thus for each $\varepsilon > 0$ there are a $N(\varepsilon) \in \mathbb{N}$ and $d_1,\ldots,d_{N(\varepsilon)} \in \mathbb{C}$ such that

$$\| g - \sum_{k=1}^{N(\varepsilon)} d_k\tilde{\varphi}_k \| < \varepsilon.$$

If we set $c_k = \frac{d_k}{\lambda_k + i}$ and $f_{N(\varepsilon)} = \sum_{k=1}^{N(\varepsilon)} c_k\tilde{\varphi}_k$, then

$$\| g - (H+i)f_{N(\varepsilon)} \| < \varepsilon.$$

Thus $\mathcal{R}(H+i)$ is dense in $\mathcal{H}$. Replacing $\lambda_k + i$ by $\lambda_k - i$ we get that also $\mathcal{R}(H-i)$ is dense in $\mathcal{H}$. Our proposition is proved. □

Of course H_1 is not selfadjoint since $\mathcal{R}(H+i) \subset C^o([-\pi,+\pi]) \subset L^2((-\pi,+\pi))$.

<u>3^{rd} Example:</u> We set

$$\mathcal{D}(H_2) = \{u \mid u \in L^2((-\pi,+\pi)). \text{ There are a } N \in \mathbb{N} \text{ and complex numbers } u_1,\dots,u_N \text{ such that}$$

$$u(x) = \frac{1}{\sqrt{2\pi}} \sum_{k=-N}^{+N} u_k e^{ikx} \text{ a.e. on } (-\pi,+\pi)\},$$

$$H_2 u = -u'', \ u \in \mathcal{D}(H_2).$$

Then $H_2 \subseteq H_1$. Of course the numbers u_k in the definition of $\mathcal{D}(H_2)$ are determined uniquely, namely we get

$$u_k = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{+\pi} u(x) e^{-ikx}\, dx.$$

$\mathcal{D}(H_2)$ is dense in $\mathcal{H} = L^2((-\pi,+\pi))$, H_2 is hermitian. Proposition I.3.4 shows that H_2 is essentially selfadjoint. We have $\overline{H_2} \subseteq \overline{H_1}$, but in view of the proposition to follow we even get

$$\overline{H_2} = \overline{H_1}.$$

<u>Proposition I.3.5:</u> 1. <u>Let</u> T_1, T_2 <u>be densely defined linear operators in</u> $\mathcal{H}$ <u>with domain of definition</u> $\mathcal{D}(T_1), \mathcal{D}(T_2)$. <u>Let</u> $T_2 \subseteq T_1$. <u>Then</u>

$$T_1^* \subseteq T_2^*.$$

2. <u>Let</u> A <u>be a selfadjoint operator in</u> $\mathcal{H}$ <u>with domain of definition</u> $\mathcal{D}(A)$. <u>Let</u> T <u>be a hermitian operator in</u> $\mathcal{H}$ <u>with domain of defini-</u>

tion $\mathcal{D}(T)$ and with

$$A \subseteq T.$$

Then

$$A = T.$$

Proof: Let $y \in \mathcal{D}(T_1^*)$. Then we have

$$(T_2x,y) = (T_1x,y) = (x,T_1^*y), \quad x \in \mathcal{D}(T_2).$$

Thus $y \in \mathcal{D}(T_2^*)$ and $T_2^*y = T_1^*y$. As for the second assertion, it now follows with the first one:

$$A \subseteq T \subseteq T^* \subseteq A^* = A$$

which implies $A = T$. □

4th Example: We set

$$\mathcal{D}(H_3) = \{u | \; u \in L^2((-\pi,+\pi)), \sum_{k=-\infty}^{+\infty} k^4 |\int_{-\pi}^{+\pi} u(x) e^{-ikx} dx|^2 < +\infty\},$$

$$H_3 u = \sum_{k=-\infty}^{+\infty} k^2 (\frac{1}{\sqrt{2\pi}} \int_{-\pi}^{+\pi} u(x) e^{-ikx} dx) \frac{1}{\sqrt{2\pi}} e^{ikx}, \quad u \in \mathcal{D}(H_3).$$

Evidently, $H_2 \subseteq H_3$. We now show that H_3 is selfadjoint. First H_3 is hermitian since

$$(H_3u,f) = \sum_{k=-\infty}^{+\infty} k^2 ((\frac{1}{\sqrt{2\pi}} \int_{-\pi}^{+\pi} u(x) e^{-ikx} dx) \frac{e^{ik.}}{\sqrt{2\pi}}, f)$$

$$= \int_{-\pi}^{+\pi} u(x) \sum_{k=-\infty}^{+\infty} k^2 \int_{-\pi}^{+\pi} \overline{f(y)} \frac{e^{iky}}{\sqrt{2\pi}} dy \frac{e^{-ikx}}{\sqrt{2\pi}} dx,$$

$$= \int_{-\pi}^{+\pi} u(x) \sum_{k=-\infty}^{+\infty} k^2 \overline{\int_{-\pi}^{+\pi} f(y) \frac{e^{-iky}}{\sqrt{2\pi}} dy \frac{e^{ikx}}{\sqrt{2\pi}}} dx,$$

$$= (u,H_3f),$$

$u \in \mathcal{D}(H_3)$, $f \in \mathcal{D}(H_3)$. Now we want to show that H_3 is selfadjoint. For any $f \in L^2((-\pi,+\pi))$ we have to solve the equations

$$(H_3+i)u = f,$$
$$(H_3-i)u = f.$$

If we use the notation $\varphi_k = \frac{1}{\sqrt{2\pi}} e^{ik\cdot}$, $k \in \mathbb{Z}$, we have

$$f = \sum_{k=-\infty}^{+\infty} f_k\varphi_k$$

and $u_1 = \sum_{k=-\infty}^{\infty} u_{1k}\varphi_k$ with $u_{1k} = \frac{f_k}{k^2+i}$ is the solution of $(H_3+i)u = f$ (observe that $k^4|u_{1k}|^2 \leq |f_k|^2$ and consequently $u_1 \in \mathcal{D}(H_3)$); analogously we get that $u_2 = \sum_{k=-\infty}^{\infty} u_{2k}\varphi_k$ with $u_{2k} = \frac{f_k}{k^2-i}$ is in $\mathcal{D}(H_3)$ and solves $(H_3-i)u = f$. We get $H_3 = \overline{H_2} = \overline{H_1}$ with Proposition I.3. .

The next theorems are important for applications.

<u>Theorem I.3.4: Let</u> H <u>be hermitian in</u> $\mathcal{H}$ <u>with domain of definition</u> $\mathcal{D}(H)$. <u>If there is a real number</u> c <u>such that</u>

$$\mathcal{R}(H+c) = \mathcal{H},$$

<u>then</u> H <u>is selfadjoint</u>.

Proof: Let $g \in \mathcal{D}(H^*)$. Then we have for all $f \in \mathcal{D}(H) = \mathcal{D}(H+c)$ the equations

$$(Hf,g) = (f,H^*g),$$
$$((H+c)f,g) = (f,H^*g) + (f,cg),$$
$$= (f,(H^*+c)g).$$

Since $\mathcal{R}(H+c) = \mathcal{H}$ there is a $\varphi \in \mathcal{D}(H)$ such that

$$(H+c)\varphi = (H^*+c)g,$$
$$((H+c)f,g) = (f,(H+c)\varphi) = ((H+c)f,\varphi).$$

Consequently $\mathcal{D}(H^*) \subseteq \mathcal{D}(H)$, and our theorem is proved. □

Theorem I.3.5: Let H be hermitian in $\mathcal{H}$ with domain of definition $\mathcal{D}(H)$. If there is a real number c such that $(H+c)^{-1}$ is densely defined in $\mathcal{H}$ and bounded, i.e. $\|(H+c)f\| \geq a\|f\|$, $f \in \mathcal{D}(H)$, for some positive constant a, then H is essentially selfadjoint.

Proof: If we can show that $\mathcal{R}(\overline{H}+c) \supseteq \overline{\mathcal{R}(H+c)} = \mathcal{H}$, then it follows from Theorem I.3.4 that $\overline{H}$ is selfadjoint. Let now $g \in \overline{\mathcal{R}(H+c)}$. Then there is a sequence $\{f_n\}$ with $f_n \in \mathcal{D}(H+c) = \mathcal{D}(H)$, $n = 1,2,\ldots,$ such that

$$(H+c)f_n \to g, \quad n \to \infty.$$

Since $\|f_n - f_m\| \leq a\|(H+c)(f_n-f_m)\|$ we obtain that $f_n \to f$, $n \to \infty$, for some $f \in \mathcal{H}$. Thus

$$\|H(f_n-f_m)\| = \|(H+c)(f_n-f_m) - c(f_n-f_m)\|$$
$$\leq \|(H+c)(f_n-f_m)\| + |c|\,\|f_n-f_m\|.$$

Consequently the sequence $\{Hf_n\}$ is also convergent and f is in $\mathcal{D}(\overline{H})$ with

$$(\overline{H}+c)f = g.$$

Our theorem is proved. □

II. Spectral Theory of Selfadjoint Operators

§ 1. The Resolvent of Selfadjoint Operators

Definition II.1.1: Let T be a linear operator in a Hilbert space $\mathcal{H}$ with domain of definition $\mathcal{D}(T)$. The resolvent set of T is the set of all $z \in \mathbb{C}$ such that

$$\mathcal{R}(T-z) = \mathcal{H},$$
$(T-z)x = 0$ implies $x = 0$,
$(T-z)^{-1}$ is bounded.

We denote the resolvent set of T by $\Sigma(T)$. Its complement

$$S(T) = \mathbb{C}-\Sigma(T)$$

is called the spectrum of T. If $z \in \Sigma(T)$, then $(T-z)^{-1}$ is called the resolvent of T in z.

In view of Proposition I.3.3 the theorem to follow is close by

Theorem II.1.1: Let A be selfadjoint in $\mathcal{H}$ with domain of definition $\mathcal{D}(A)$. Let $z \in \mathbb{C}$, $\text{Im } z \neq 0$. Then $z \in \Sigma(A)$, and we have

$$\|(A-z)^{-1}\| \leq \frac{1}{|\text{Im } z|}.$$

Proof: According to Proposition I.3.3 we have

(II.1.1) $\|(A-z)f\| \geq |\text{Im } z|\,\|f\|, \quad f \in \mathcal{D}(A).$

As in the first part of the proof of Theorem I.3.2 we can show that $\mathcal{R}(A-z)$ is a closed subspace of $\mathcal{H}$ if $\text{Im } z \neq 0$. If $\mathcal{R}(A-z) \neq \mathcal{H}$, then there is a $g \in \mathcal{H}$ such that $g \neq 0$,

$$((A-z)f,g) = 0, \quad f \in \mathcal{D}(A).$$

Thus

$$(Af,g) = (zf,g) = (f,\bar{z}g),$$
$$g \in \mathcal{D}(A),$$
$$Ag = \bar{z}g.$$

From (II.1.1) it follows that $g = 0$. Our theorem is proved. □

Theorem II.1.2: A real number λ_o is in $\Sigma(A)$, for a selfadjoint operator A in $\mathcal{H}$ with domain of definition $\mathcal{D}(A)$, if and only if

$$\|(A-\lambda_o)f\| \geq c\|f\|, \quad f \in \mathcal{D}(A),$$

with some positive constant c.

Proof: Let $\lambda_o \in \Sigma(A)$. Then $\|(A-\lambda_o)f\| \geq c\|f\|$, $f \in \mathcal{D}(A)$, for some $c > 0$. Now, let

$$\|(A-\lambda_o)f\| \geq c\|f\|, \quad f \in \mathcal{D}(A),$$

for some $c > 0$. As in the first part of the proof of Theorem I.3.2 one shows that $\mathcal{R}(A-\lambda_o)$ is a closed subspace of $\mathcal{H}$. From this it follows as in the proof of the preceding theorem that $\mathcal{R}(A-\lambda_o) = \mathcal{H}$. Theorem I.3.4 completes the proof. □

For the resolvent of a selfadjoint operator A in $\mathcal{H}$ we often write

(II.1.2) $R_z = R_z(A) = (A-z)^{-1}, \ z \in \Sigma(A)$.

Next we prove the resolvent equation.

<u>Theorem II.1.3:</u> <u>Let</u> A <u>be selfadjoint with domain of definition</u> $\mathcal{D}(A)$. <u>Let</u> $z_1, z_2 \in \Sigma(A)$. <u>Then</u>

(II.1.3) $R_{z_1} - R_{z_2} = (z_1-z_2)R_{z_1}R_{z_2}$.

<u>Proof:</u> For $z \in \Sigma(A)$ we have

$$(A-z)R_z f = f, \ f \in \mathcal{H},$$

$$R_z(A-z)f = f, \ f \in \mathcal{D}(A).$$

Thus

$$\begin{aligned}(R_{z_1}-R_{z_2})g &= (A-z_1)^{-1}g - (A-z_2)^{-1}g,\\ &= (A-z_1)^{-1}(A-z_2)(A-z_2)^{-1}g - (A-z_1)^{-1}(A-z_1)(A-z_2)^{-1}g,\\ &= (A-z_1)^{-1}((A-z_2)(A-z_2)^{-1} - (A-z_1)(A-z_2)^{-1})g,\\ &= (A-z_1)^{-1}(A-z_2-(A-z_1))(A-z_2)^{-1}g,\\ &= (z_1-z_2)R_{z_1}R_{z_2}g, \ g \in \mathcal{H}.\end{aligned}$$

Our theorem is proved. □

The preceding theorem has far reaching consequences, namely the analyticity of R_z. We prefer to give another proof of this property of the resolvent.

Theorem II.1.4: Let $z_o \in \Sigma(A)$ and $|z-z_o| < \| R_{z_o} \|^{-1}$. Then $z \in \Sigma(A)$ and

$$\text{(II.1.4)} \quad R_z = \sum_{k=0}^{\infty} (z-z_o)^k R_{z_o}^{k+1}.$$

The series in (II.1.4) converges with respect to the norm of $L(H,H)$.

Before we give the proof we remark that $\| R_z \| > 0$ if $z \in \Sigma(A)$. Thus it in particular follows from Theorem II.1.4 that $\Sigma(A)$ is open. We also remind the reader of a general theorem in Banach spaces $\mathcal{B}$. If $B \in L(\mathcal{B},\mathcal{B})$ and if $\|B\| < 1$, then

$$\text{(II.1.5)} \quad (I-B)^{-1} = \sum_{k=0}^{\infty} B^k,$$

where the series in (II.1.5) converges in the topology of the space of bounded operators $L(\mathcal{B},\mathcal{B})$ from $\mathcal{B}$ into itself.

Proof of Theorem II.1.4: Since $|z-z_o| \| R_{z_o} \| < 1$ we can apply (II.1.5) to the operator $(z-z_o)R_{z_o}$. This yields the expansion

$$(I-(z-z_o)R_{z_o})^{-1} = \sum_{k=0}^{\infty} (z-z_o)^k R_{z_o}^k .$$

We set

$$C = \sum_{k=0}^{\infty} (z-z_o)^k R_{z_o}^{k+1},$$

$$= R_{z_o} (I-(z-z_o)R_{z_o})^{-1},$$

$$= (I-(z-z_o)R_{z_o})^{-1} R_{z_o} .$$

We want to show that $C = R_z$. Firstly we have $Cf \in \mathcal{D}(A)$, $f \in \mathcal{H}$. Then

$$(A-z)Cf = (A-z_o)(I-(z-z_o)R_{z_o}) \cdot (I-(z-z_o)R_{z_o})^{-1} R_{z_o} f,$$

$$= (A-z_o)R_{z_o} f = f.$$

Thus $\mathcal{R}(A-z) = \mathcal{H}$. On the other hand

$$C(A-z)f = (I-(z-z_o)R_{z_o})^{-1} R_{z_o} (A-z)f,$$

$$= (I-(z-z_o)R_{z_o})^{-1} R_{z_o} (A-z_o)(I-(z-z_o)R_{z_o})f,$$

$$= f, \quad f \in \mathcal{D}(A),$$

and we obtain the uniqueness of A-z. Since C is the bounded inverse of A-z our theorem is proved. □

As a consequence of Theorem II.1.4 we obtain that for $g,f \in \mathcal{H}$ the function $\phi(.) = (R_z f,g)$ is holomorphic in $\Sigma(A)$ and admits the expansion

$$\phi(z) = \sum_{k=0}^{\infty} (R_{z_o}^{k+1} f,g)(z-z_o)^k$$

around z_o where $|z-z_o| < \| R_{z_o} \|^{-1}$.

Next we characterize the adjoint of R_z.

<u>Theorem II.1.5:</u> <u>Let</u> A <u>be a selfadjoint operator in</u> $\mathcal{H}$ <u>with domain of definition</u> $\mathcal{D}(A)$. <u>Let</u> $z \in \Sigma(A)$. <u>Then</u>

$$R_z^* = R_{\bar{z}}.$$

<u>Proof:</u> Let $f,g \in \mathcal{D}(A)$. Then

$$((A-z)f,g) = (f,(A-\bar{z})g).$$

If we set $f = R_z u$, $g = R_{\bar{z}} v$, then

$$(u,R_{\bar{z}}v) = (R_z u,v) = (u,R_z^* v)$$

Since u,v run through all of $\mathcal{H}$ if f,g do so in $\mathcal{D}(A)$ our theorem is proved. □

§ 2. Spectral Families

First we concentrate on the case of a finite dimensional Euclidean space H. Let dim $H = n$, let A be a hermitian operator in H, i.e. A is represented by a hermitian (n,n)-matrix which is also denoted by A. If $\{\varphi_1,\ldots,\varphi_n\}$ is a complete orthonormal system in H of eigenfunctions of A belonging to the (real) eigenvalues $\lambda_1 \leqq \lambda_2 \leqq \ldots \leqq \lambda_n$, then

$$f = \sum_{i=1}^{n} (f,\varphi_i)\varphi_i,$$

$$Af = \sum_{i=1}^{n} \lambda_i (f,\varphi_i)\varphi_i,$$

$$R_z f = (A-z)^{-1} f = \sum_{i=1}^{n} \frac{1}{\lambda_i - z}(f,\varphi_i)\varphi_i, \qquad \operatorname{Im} z \neq 0;$$

here the eigenvalues are counted according to their multiplicities. We set

$$E(\lambda)f = \begin{cases} \sum\limits_{i,\lambda_i \leqq \lambda} (f,\varphi_i)\varphi_i & \text{if } \lambda \geqq \lambda_1 \\ 0, & \text{if } \lambda < \lambda_1, \ f \in H. \end{cases}$$

Then $E(\lambda)$ is bounded, everywhere defined and constant on $[-\infty,\lambda_1)$, $[\lambda_n,+\infty)$, $[\lambda_i,\lambda_{i+1})$ if $\lambda_i \neq \lambda_{i+1}$. In particular we have

$$E(\lambda) = 0, \ \lambda < \lambda_1,$$

$$E(\lambda) = I, \ \lambda \geqq \lambda_n,$$

$$(E(\lambda_i+\varepsilon)-E(\lambda_i-\varepsilon))f = \sum_{j,\lambda_j=\lambda_i} (f,\varphi_j)\varphi_j$$

if $\varepsilon > 0$ is sufficiently small. With what was said before on the $E(\lambda)$ we obtain

$$E(\lambda+0)f = \lim_{\substack{\varepsilon>0,\\ \varepsilon\to 0}} E(\lambda+\varepsilon)f = E(\lambda)f, \ f \in \mathcal{H}.$$

Thus $E(\lambda)f$ is strongly continuous from the right. We also easily get

$$E(\lambda)E(\mu) = E(\min(\lambda,\mu)),$$

$$E(\lambda)^* = E(\lambda).$$

Thus each $E(\lambda)$ is a projection; the set $\{E(\lambda) \mid \lambda \in \mathbb{R}\} \subset L(\mathcal{H},\mathcal{H})$ is called a spectral family. In what follows this notion is carried over to infinite dimensional Hilbert spaces. We start with

Proposition II.2.1: Let $\mathcal{H}$ be any Hilbert space. Let $\mathcal{M}_1,\mathcal{M}_2$ be two closed subspaces of $\mathcal{H}$. Let P_1,P_2 be the projections from $\mathcal{H}$ onto $\mathcal{M}_1,\mathcal{M}_2$ resp. Then

$\mathcal{M}_1 \subseteq \mathcal{M}_2$ if and only if $P_2P_1 = P_1$.

Proof: Let first $\mathcal{M}_1 \subseteq \mathcal{M}_2$. Then $P_1f \in \mathcal{M}_1$, $P_2P_1f = P_1f$, $f \in \mathcal{H}$. If conversely $P_2P_1 = P_1$ we get for $f \in \mathcal{M}_1$: $P_1f = f$, $P_2P_1f = P_1f = f$ and consequently $P_1f = f \in \mathcal{M}_2$. □

From now on $\mathcal{H}$ is again an arbitrary (possibly infinite dimensional) Hilbert space. We want to define the notion of a spectral family in $\mathcal{H}$:

Definition II.2.1: Let there be given a projection $E(\lambda)$ in $\mathcal{H}$ for each $\lambda \in \mathbb{R}$. Let the $E(\lambda)$, $\lambda \in \mathbb{R}$, have the following properties:

(II.2.1) $E(\lambda)E(\mu) = E(\min(\lambda,\mu))$, (II.2.2) $E(\lambda+0)f = \lim_{\substack{\varepsilon>0, \\ \varepsilon\to 0}} E(\lambda+\varepsilon)f$

exists for every $f \in H$ and every $\lambda \in \mathbb{R}$ and is equal to $E(\lambda)f$,

(II.2.3) $E(\lambda)f \to 0,\ f \in H,\ \lambda \to -\infty$,

(II.2.4) $E(\lambda)f \to f,\ f \in H,\ \lambda \to +\infty$.

Then the set $\{E(\lambda) | \lambda \in \mathbb{R}\}$ is called a spectral family.

Proposition II.2.2: Let $\{E(\lambda) | \lambda \in \mathbb{R}\}$ be a spectral family. Then

$$E(\lambda)E(\mu) = E(\mu)E(\lambda),$$

$\mu, \lambda \in \mathbb{R}$.

Proof: Follows from (II.2.1). □

Definition II.2.2: Let $\{E(\lambda) | \lambda \in \mathbb{R}\}$ be a spectral family. Let $-\infty < a \leq b < +\infty$, $\Delta = [a,b]$. Then we set

$$E(\Delta) = E(b) - E(a).$$

Proposition II.2.3: For a spectral family $\{E(\lambda) | \lambda \in \mathbb{R}\}$ the operator $E(\Delta)$ is always a projection. If Δ', Δ'' are two closed finite intervals with

$$\mathring{\Delta}' \cap \mathring{\Delta}'' = \phi$$

then

$$E(\Delta')E(\Delta'') = 0 = E(\Delta'')E(\Delta').$$

<u>This is equivalent to</u>

$$E(\Delta')H = M(\Delta') \perp M(\Delta'') = E(\Delta'')\mathcal{H}.$$

<u>Proof:</u> For $f,g \in H$ we get

$$(E(\Delta)f,g) = (f,E(\Delta)g),$$

$$\begin{aligned} E(\Delta)E(\Delta) &= E(b)^2 + E(a)^2 - 2E(a)E(b), \\ &= E(b) + E(a) - 2E(a), \\ &= E(\Delta), \end{aligned}$$

where we have applied Proposition II.2.2 and (II.2.1). If $\Delta' = [a,b]$, $\Delta'' = [c,d]$ we can assume that $b \leq c$. Then with (II.2.1)

$$\begin{aligned} E(\Delta')E(\Delta'') &= (E(b)-E(a))(E(d)-E(c)) \\ &= E(b) - E(a) - E(b) + E(a) \\ &= 0. \end{aligned}$$

With Proposition II.2.2 we get $E(\Delta'')E(\Delta') = 0$. □

Next we define the integral $\int_a^b f(\lambda)dE(\lambda)$ for continuous functions f.

<u>Proposition II.2.4:</u> <u>Let</u> $f:[a,b] \to \mathbb{C}$ be continuous. We set

$$\delta(\varepsilon) = \sup \{\delta \mid |f(\lambda_1)-f(\lambda_2)| \leq \varepsilon \text{ für } \lambda_1,\lambda_2 \in [a,b] \text{ mit } |\lambda_1-\lambda_2| \leq \delta\}$$

for $\varepsilon > 0$. <u>Let</u> $\mathfrak{z}' = (\lambda_1', \dots, \lambda_{m+1}')$, $\mathfrak{z}'' = (\lambda_1'', \dots, \lambda_{n+1}'')$ <u>be two partitions of the interval</u> $[a,b]$ <u>with</u> $a = \lambda_1' < \lambda_2' < \dots < \lambda_{m+1}' = b$, $a = \lambda_1'' < \lambda_2'' < \dots < \lambda_{n+1}'' = b$,

$$\max_{1\leq i\leq m} |\lambda'_{i+1}-\lambda_i| \leq \delta(\varepsilon),$$

$$\max_{1\leq k\leq n} |\lambda''_{k+1}-\lambda''_k| \leq \delta(\varepsilon).$$

If we set

$$T' = \sum_{i=1}^{m} f(\lambda_i^*)(E(\lambda'_{i+1})-E(\lambda'_i)),$$

$$T'' = \sum_{k=1}^{n} f(\lambda_k^{**})(E(\lambda''_{k+1})-E(\lambda''_k))$$

with points $\lambda_i^* \in [\lambda'_i,\lambda'_{i+1}]$, $\lambda_k^{**}\in[\lambda''_k,\lambda''_{k+1}]$, *then*

$$\|T'-T''\| \leq 2\varepsilon.$$

Proof: Let $\mathfrak{Z}'''$ be the partition of [a,b] with the points $\lambda'_1,\ldots,\lambda'_{m+1},\lambda''_1,\ldots,\lambda''_{n+1}$. Let us assume that in $[\lambda'_k,\lambda'_{k+1}]$ the points $\lambda'_k,\lambda_{k_1},\ldots,\lambda_{k_{p_k}},\lambda_{k_{p_k+1}} = \lambda'_{k+1}$ with

$$\lambda'_k = \lambda_{k_1} < \lambda_{k_2} < \ldots < \lambda_{k_{p_k}} < \lambda_{k_{p_k+1}} = \lambda'_{k+1}$$

belong to $\mathfrak{Z}'''$ and that $\mu_{kl} \in [\lambda_{k_l},\lambda_{k_{l+1}}]$, $1 \leq k \leq m$, $1 \leq l \leq p_k$; then we set

$$T''' = \sum_{k=1}^{m} \sum_{l=1}^{p_k} f(\mu_{kl})(E(\lambda_{k_{l+1}})-E(\lambda_{k_l})).$$

Evidently

$$T' = \sum_{k=1}^{m} \sum_{l=1}^{p_k} f(\lambda_k^*)(E(\lambda_{k_{l+1}})-E(\lambda_{k_l}))$$

and

$$T'''-T' = \sum_{k=1}^{m} \sum_{l=1}^{p_k} (f(\mu_{kl})-f(\lambda_k^*))\cdot(E(\lambda_{k_{l+1}})-E(\lambda_{k_l})).$$

Now we want to make use of a more general formula, namely: Let $\Delta_1,\ldots,\Delta_q$ be closed intervals of the real axis with $\overset{\circ}{\Delta}_i \cap \overset{\circ}{\Delta}_j = \phi$ if $i \neq j$. Let $\varepsilon_1,\ldots,\varepsilon_q \in \mathbb{C}$. Then

$$\text{(II.2.5)} \quad \|\sum_{j=1}^{q} \varepsilon_j E(\Delta_j)f\|^2 = \sum_{i,j=1}^{q} \varepsilon_i \overline{\varepsilon}_j \cdot (E(\Delta_i)f, E(\Delta_j)f),$$

$$\text{(II.2.6)} \quad = \sum_{j=1}^{q} |\varepsilon_j|^2 \|E(\Delta_j)f\|^2,$$

$$\text{(II.2.7)} \quad = \sum_{j=1}^{q} |\varepsilon_j|^2 (E(\Delta_j)f,f)$$

by proposition II.2.3; (II.2.7) immediately furnishes $\|(T'-T''')f\|^2 \leq \varepsilon^2\|f\|^2$. The inequality $\|(T''-T''')f\|^2 \leq \varepsilon^2\|f\|^2$ is proved analogously. Thus

$$\|T'-T''\| \leq \|T'-T'''\| + \|T'''-T''\| \leq 2\varepsilon. \qquad \square$$

Proposition II.2.4 enables us to give the following definition:

<u>Definition II.2.3:</u> <u>Let</u> $\{E(\lambda)|\lambda \in \mathbb{R}\}$ <u>be a spectral family.</u> <u>Let</u> $\Delta = [a,b]$, $\varphi:\Delta \to \mathbb{C}$ <u>be continuous.</u> <u>For</u> $n = 1,2,..$ <u>let there be given closed intervals</u> $\Delta_1^{(n)},\ldots,\Delta_{k_n}^{(n)}$ <u>with</u>

$$\Delta = \bigcup_{j=1}^{k_n} \Delta_j^{(n)},$$

$$\overset{\circ}{\Delta}{}_i^{(n)} \cap \overset{\circ}{\Delta}{}_j^{(n)} = \phi,\ i \neq j,\ 1 \leq i,j \leq k_n,$$

(II.2.8) $\max_{1 \le i \le k_n} |\Delta_i^{(n)}| \to 0$ if $n \to \infty$;

let $\lambda_i^{(n)} \in \Delta_i^{(n)}$, $1 \le i \le k_n$. Then the operators

$$T_n = \sum_{i=1}^{k_n} \varphi(\lambda_i^{(n)}) E(\Delta_i^{(n)})$$

converge in the norm of $L(H,H)$ if $n \to \infty$. The limit does not depend on the choice of the sequence of partitions $\Delta_1^{(n)}, \dots, \Delta_{k_n}^{(n)}$, provided (II.2.8) is fulfilled, and it does also not depend on the $\lambda_i^{(n)}$. It is denoted by

$$\int_a^b \varphi(\lambda) dE(\lambda) = \int_\Delta \varphi(\lambda) dE(\lambda) = \varphi(E,\Delta).$$

Let us consider the function $\alpha : \lambda \to (E(\lambda)f,f)$, $\lambda \in \mathbb{R}$, for fixed but arbitrary $f \in H$. α only assumes real values and for $\lambda < \mu$ we get

$$\begin{aligned} \alpha(\mu)-\alpha(\lambda) &= ((E(\mu)-E(\lambda))f,f) \\ &= (E([\lambda,\mu])f,f) \\ &= \|E([\lambda,\mu])f\|^2 \geqq 0. \end{aligned}$$

Thus α is monotonically non decreasing and, in particular, it is of bounded variation. We also have

$$\alpha(\lambda) \to 0, \quad \lambda \to -\infty,$$
$$\alpha(\lambda) \to \|f\|^2, \quad \lambda \to +\infty.$$

We want to review now some facts on functions of bounded variation. Our reference is [RN, pp. 7]. Let I be a finite closed, open or halfopen interval. Let $f: I \to \mathbb{R}$ be a function. Then f is said to have bounded variation if there exists a finite number c such that

$$\text{(II.2.9)} \quad \sum_{k=1}^{n} |f(x_k)-f(x_{k-1})| \leq c$$

for every n-tuple $(x_o,x_1,\ldots,x_n)$ with $x_i \in I$, $0 \leq i \leq n$, $x_o < x_1 < \ldots < x_n$, $n=1,2,\ldots$. The infimum of all c for which (II.2.9) holds is called the total variation of f on I, shortly $T(I) = T_f(I)$. If $f:I \to \mathbb{C}$ is complex valued then f is said to have bounded variation if and only if the real and the imaginary part of f have bounded variation. Every real function $f:I \to \mathbb{R}$ having bounded variation can be decomposed into

$$\text{(II.2.10)} \quad f = f_1 - f_2$$

where $f_i:I \to \mathbb{R}$ have bounded variation, $i=1,2$, and, moreover, are monotonically non decreasing on I. We can take $f_1(x) = T_f(I \cap [a,x])$, $x \in I$, $a = \inf\{\xi \mid \xi \in I\}$, $f_2(x) = T_f(I \cap [a,x]) - f(x)$, $x \in I$. Of course, any bounded monotonically non decreasing $f:I \to \mathbb{R}$ has bounded variation. If $I = [a,b]$, $\alpha:I \to \mathbb{R}$ is monotonically non decreasing and if $f:[a,b] \to \mathbb{R}$ is continuous, then the sums

$$\sum_{l=1}^{k_n} f(\xi_l^{(n)})(\alpha(x_l^{(n)}) - \alpha(x_{l-1}^{(n)})),$$

$$a = x_o^{(n)} < x_1^{(n)} < \ldots < x_{k_n}^{(n)} = b, \quad \xi_l^{(n)} \in [x_{l-1}^{(n)}, x_l^{(n)}], \quad 1 \leq l \leq k_n,$$

tend to a limit if $n \to \infty$, provided $\delta(n) = \max\limits_{1 \leq l \leq k_n} (x_l^{(n)} - x_{l-1}^{(n)})$ tends to 0 if $n \to \infty$. This limit turns out to be independent of the choice of the sequence of partitions $(x_o^{(n)},\ldots,x_{k_n}^{(n)})$ of $[a,b]$ and of the choice of the $\xi_l^{(n)} \in [x_{l-1}^{(n)}, x_l^{(n)}]$; it's designated by

$$\int_a^b f(x)d\alpha(x) = \int_a^b f\,d\alpha = \int_{[a,b]} f\,d\alpha.$$

If $\alpha:[a,b] \to \mathbb{R}$ has bounded variation and if $\alpha = \alpha_1 - \alpha_2$ is decomposed as in (II.2.10) then we set

$$\int_a^b f(x)d\alpha(x) = \int_a^b f\, d\alpha = \int_{[a,b]} f\, d\alpha,$$

$$= \int_a^b f\, d\alpha_1 - \int_a^b f\, d\alpha_2.$$

Let us remark that for α_1, α_2 we can also take $\frac{1}{2}(\alpha(x)+T_\alpha([a,x]))$, $\frac{1}{2}(T_\alpha([a,x])-\alpha(x))$; these quantities are called the positive and the negative indefinite variation of α. A complex valued function $\alpha: I \to \mathbb{C}$ is said to have bounded variation if $\alpha_1 = \text{Re}\ \alpha$, $\alpha_2 = \text{Im}\ \alpha$ have bounded variation. If $I = [a,b]$ and if $f:[a,b] \to \mathbb{R}$ is continuous we set

$$\int_a^b f(x)d\alpha(x) = \int_a^b f\, d\alpha = \int_{[a,b]} f\, d\alpha,$$

$$= \int_a^b f\, d\alpha_1 + i \int_a^b f\, d\alpha_2.$$

is said to have bounded variation on $[a,+\infty)$, $(-\infty,b]$, $(-\infty,+\infty)$ respectively if

(II.2.11) $T_\alpha([a,b]) < c$ for all b, $a < b < +\infty$,

(II.2.12) $T_\alpha([a,b]) < c$ for all a, $-\infty < a < b$,

(II.2.13) $T_\alpha([a,b]) < c$ for all a,b, $-\infty < a < b < +\infty$,

respectively. If $\alpha:[a,b] \to \mathbb{C}$ has bounded variation we also write

$$T_\alpha([a,b]) = \int_a^b |d\alpha(x)|,$$

and consequently we set in the cases (II.2.11), (II.2.12), (II.2.13)

$$\int_a^{+\infty} |d\alpha(x)| = \inf\{c \mid c > T_\alpha([a,b])\ \forall b,\ a < b < +\infty\},$$

$$\int_{-\infty}^{b} |d\alpha(x)| = \inf\{c \mid c > T_\alpha([a,b])\ \forall a,\ -\infty < a < b\},$$

$$\int_{-\infty}^{+\infty} |d\alpha(x)| = \inf\{c \,|\, c > T_\alpha([a,b]) \;\forall a,b,\; -\infty < a < b < +\infty\}.$$

If $f:(-\infty,+\infty) \to \mathbb{R}$ is continuous and bounded, if $\alpha:(-\infty,+\infty) \to \mathbb{C}$ has bounded variation and if $\lim\limits_{b\to+\infty,\, a\to-\infty} \int_a^b f\, d\alpha$ exists, then we set

$$\int_{-\infty}^{+\infty} f\, d\alpha = \lim_{\substack{b\to+\infty,\\ a\to-\infty}} \int_a^b f\, d\alpha.$$

The integrals $\int_a^{+\infty} f\, d\alpha$, $\int_{-\infty}^{b} f\, d\alpha$ are defined analogously. All these definitions can be carried over to complex valued functions f by considering Re f and Im f.

Now we study the function $(E(\lambda)f,g)$, $f,g \in H$ by means of the theory of functions of bounded variation.

Theorem II.2.1: *Let* $\{E(\lambda) \,|\, \lambda \in \mathbb{R}\}$ *be a spectral family. Then the function* $(E(.)f,g)$ *has bounded variation on* $(-\infty,+\infty)$ *for every* $f,g \in H$. *Moreover*

$$\int_{-\infty}^{+\infty} |d(E(\lambda)f,g)| \leq \|f\|\cdot\|g\|.$$

If $\varphi:\Delta \to \mathbb{C}$ *is continuous on the closed interval* $\Delta = [a,b]$, *then*

$$(\varphi(E,\Delta)f,g) = \int_\Delta \varphi(\lambda)d(E(\lambda)f,g).$$

Proof: We decompose Δ into

$$\Delta = \bigcup_{i=1}^{n} \Delta_i$$

with closed intervals Δ_i with $\overset{\circ}{\Delta}_j \cap \overset{\circ}{\Delta}_k = \phi$, $j \neq k$. Then

$$\begin{aligned}\sum_{i=1}^{n} |(E(\Delta_i)f,g)| &= \sum_{i=1}^{n} |(E(\Delta_i)f,E(\Delta_i)g)|,\\ &\leq \sum_{i=1}^{n} \|E(\Delta_i)f\| \cdot \|E(\Delta_i)g\|\\ &\leq \left\{\sum_{i=1}^{n} \|E(\Delta_i)f\|^2\right\}^{\frac{1}{2}} \cdot \left\{\sum_{i=1}^{n} \|E(\Delta_i)g\|^2\right\}^{\frac{1}{2}},\\ &= \left\{\sum_{i=1}^{n} (E(\Delta_i)f,f)\right\}^{\frac{1}{2}} \cdot \left\{\sum_{i=1}^{n} (E(\Delta_i)g,g)\right\}^{\frac{1}{2}},\\ &= \|E(\Delta)f\| \cdot \|E(\Delta)g\| \leq \|f\| \cdot \|g\|.\end{aligned}$$

Since Δ was arbitrary, the first part of our theorem is proved. As for the second part Δ is decomposed into closed intervals $\Delta_i^{(n)}$,

$$\Delta = \bigcup_{i=1}^{k_n} \Delta_i^{(n)},$$

for each $n \in \mathbb{N}$ with $\overset{\circ}{\Delta}{}_j^{(n)} \cap \overset{\circ}{\Delta}{}_k^{(n)} = \phi$, $j \neq k$, $1 \leq i,j \leq k_n$, and with $\lim\limits_{n\to\infty} \max\limits_{1\leq i\leq k_n} |\Delta_i^{(n)}| = 0$. If $T_n = \sum\limits_{i=1}^{k_n} \varphi(\lambda_i^{(n)})E(\Delta_i^{(n)})$ with $\lambda_i^{(n)} \in \Delta_i^{(n)}$,

we get:

$$(T_n f,g) = \sum_{i=1}^{k_n} \varphi(\lambda_i^{(n)})(E(\Delta_i^{(n)}f,g)$$

$$\to (\varphi(E,\Delta)f,g) = \int_\Delta \varphi(\lambda)d(E(\lambda)f,g),$$

$n \to \infty$. □

<u>Theorem II.2.2: Let</u> $\{E(\lambda)\,|\,\lambda\in\mathbb{R}\}$ <u>be a spectral family</u>. <u>Let</u> Δ <u>be a closed interval</u>. <u>Then</u>

$$\|\varphi(E,\Delta)\| \leq \max_{\lambda\in\Delta} |\varphi(\lambda)|,$$

$$\|\varphi(E,\Delta)f\| \leq \max_{\lambda\in\Delta} |\varphi(\lambda)|\,\|E(\Delta)f\|$$

<u>for any continuous function</u> $\varphi:\Delta\to\mathbb{C}$.

<u>Proof:</u> The notations are chosen like in the proof of Theorem II.2.1. Then for $f\in H$

$$\|T_n f\|^2 = \sum_{i=1}^{k_n} |\varphi(\lambda_i^{(n)})|^2 \|E(\Delta_i^{(n)})f\|^2,$$

$$\leq \max_{\lambda\in\Delta} |\varphi(\lambda)|^2 \|E(\Delta)f\|^2.$$ □

<u>Theorem II.2.3: Let</u> $\{E(\lambda)\,|\,\lambda\in\mathbb{R}\}$ <u>be a spectral family</u>. <u>Let</u> Δ <u>be a closed interval</u>. <u>Let</u> $\varphi:\Delta\to\mathbb{C}$ <u>be a continuous function</u>. <u>Then</u>

$$\varphi(E,\Delta)^* = \overline{\varphi}(E,\Delta),$$

<u>where</u> $\overline{\varphi}$ <u>is defined by</u> $\overline{\varphi}(\lambda) = \overline{\varphi(\lambda)}$, $\lambda\in\Delta$.

<u>Proof:</u> We have for $f,g\in H$

$$(\varphi(E,\Delta)f,g) = \int_\Delta \varphi(\lambda)\,d(E(\lambda)f,g),$$

$$= \int_\Delta \varphi(\lambda)\,d(f,E(\lambda)g),$$

$$= \int_\Delta \varphi(\lambda)\,d\overline{(E(\lambda)g,f)},$$

$$= \overline{\int_{\Delta} \overline{\varphi}(\lambda) d(E(\lambda)g,f)},$$

$$= \overline{(\overline{\varphi}(E,\Delta)g,f)},$$

$$= (f,\overline{\varphi}(E,\Delta)g). \qquad \square$$

Theorem II.2.4: Let $\{E(\lambda) | \lambda \in \mathbb{R}\}$ be a spectral family, let $\varphi: \mathbb{R} \to \mathbb{C}$ be continuous and bounded. Then

$$\lim_{\substack{a\to-\infty,\\ b\to+\infty}} \int_a^b \varphi(\lambda) dE(\lambda) f$$

exists for every $f \in H$ and is denoted by

$$\varphi(E)f = \int_{-\infty}^{+\infty} \varphi(\lambda) dE(\lambda) f.$$

Moreover, $\varphi(E)$ is in $L(H,H)$ and

$$\|\varphi(E)\| \leqq \sup_{\lambda\in\mathbb{R}} |\varphi(\lambda)|.$$

If additionally $\varphi(\lambda) \to 0$ for $\lambda \to +\infty$ and for $\lambda \to -\infty$, then

$$\|\int_a^b \varphi(\lambda) dE(\lambda) - \varphi(E)\| \to 0$$

for $a \to -\infty$, $b \to +\infty$.

Proof: Let $a' < a < b < b'$. Then

$$\|\int_{a'}^{b'} \varphi(\lambda) dE(\lambda) f - \int_a^b \varphi(\lambda) dE(\lambda) f\|$$

$$= \|\int_{a'}^{a} \varphi(\lambda) dE(\lambda) f + \int_b^{b'} \varphi(\lambda) dE(\lambda) f\|,$$

$$\leq \|\int_{a'}^{a} \varphi(\lambda)dE(\lambda)f\| + \|\int_{b}^{b'} \varphi(\lambda)dE(\lambda)f\|,$$

$$\leq \sup_{\lambda\in\mathbb{R}} |\varphi(\lambda)| (\|(E(a)-E(a'))f\| + \|(E(b')-E(b))f\|)$$

$$\leq \sup_{\lambda\in\mathbb{R}} |\varphi(\lambda)| ((E(a)-E(a'))f,f))^{\frac{1}{2}} +$$

$$+ \sup_{\lambda\in\mathbb{R}} |\varphi(\lambda)| (((E(b')-E(b))f,f))^{\frac{1}{2}},$$

$$\leq \sup_{\lambda\in\mathbb{R}} |\varphi(\lambda)| (E(a)f,f)^{\frac{1}{2}} + \sup_{\lambda\in\mathbb{R}} |\varphi(\lambda)| ((I-E(b))f,f)^{\frac{1}{2}},$$

where we have used the second inequality in Theorem II.2.2 and the monotonicity of $(E(\lambda)f,f)$. Letting a tend to $-\infty$ and b tend to $+\infty$ we see that

$$\lim_{\substack{a\to-\infty,\\ b\to+\infty}} \int_{a}^{b} \varphi(\lambda)dE(\lambda)f$$

exists. The second inequality in Theorem II.2.2 also shows that $\varphi(E) \in L(H,H)$ and $\|\varphi(E)\| \leq \sup_{\lambda\in\mathbb{R}} |\varphi(\lambda)|$. As for the last assertion the preceding calculations show that for any $\varepsilon > 0$

$$\|\int_{a'}^{b'} \varphi(\lambda)dE(\lambda)f - \int_{a}^{b} \varphi(\lambda)dE(\lambda)f\| \leq \varepsilon\|f\|$$

if $a' < a < b < b'$ and if $-a,b$ are sufficiently large. □

The formula for $\|T_n f\|^2$ in the proof of Theorem II.2.2 shows that

$$\text{(II.2.14)} \quad \|\int_{a}^{b} \varphi(\lambda)dE(\lambda)f\|^2 = \int_{a}^{b} |\varphi(\lambda)|^2 d(E(\lambda)f,f)$$

for any $f \in H$, any a,b, $a < b$, and any continuous $\varphi:[a,b] \to \mathbb{C}$. If $\varphi: \mathbb{R} \to \mathbb{C}$ is continuous and bounded we get therefore

$$\text{(II.2.15)} \quad \left\| \int_{-\infty}^{+\infty} \varphi(\lambda) dE(\lambda) f \right\|^2 = \int_{-\infty}^{+\infty} |\varphi(\lambda)|^2 d(E(\lambda)f,f).$$

<u>Theorem II.2.5:</u> <u>Let</u> $\Delta_1 = [a,b]$, $\Delta_2 = [c,d]$ <u>be two closed intervals with</u> $\overset{\circ}{\Delta}_1 \cap \overset{\circ}{\Delta}_2 = \phi$. <u>Let</u>

$$\varphi: \Delta_1 \to \mathbb{C},$$
$$\psi: \Delta_2 \to \mathbb{C}$$

<u>be continuous.</u> <u>Let</u> $\{E(\lambda) | \lambda \in \mathbb{R}\}$ <u>be a spectral family.</u> <u>Then</u>

$$\varphi(E,\Delta_1)\psi(E,\Delta_2) = 0.$$

<u>Proof:</u> Taking two Riemannian sums approximating $\varphi(E,\Delta_1)$ and $\psi(E,\Delta_2)$ and using Proposition II.2.3 our theorem follows. □

<u>Theorem II.2.6:</u> <u>Let</u> $\psi,\varphi: \Delta \to \mathbb{C}$ <u>be continuous on the closed interval</u> $\Delta = [a,b]$. <u>Then</u>

$$\varphi(E,\Delta)\psi(E,\Delta) = \varphi\psi(E,\Delta).$$

<u>Proof:</u> Taking Riemannian sums as in the proof of Theorem II.2.1 we get

$$\sum_{i=1}^{k_n} \varphi(\lambda_i^{(n)}) E(\Delta_i^{(n)}) \sum_{j=1}^{k_n} \psi(\lambda_j^{(n)}) E(\Delta_j^{(n)}) =$$
$$= \sum_{i=1}^{k_n} \varphi(\lambda_i^{(n)}) \psi(\lambda_i^{(n)}) E(\Delta_i^{(n)}),$$

which proves our assertion. □

§ 3. Stieltjes's Inversion Formula. Further Properties of Functions of Bounded Variation

Let $\rho:\mathbb{R}\to\mathbb{C}$ be a function having bounded variation on $(-\infty,+\infty)$. Then it follows from (II.2.9) and (II.2.13) that ρ is bounded. As it is proved in [RN,], for each $\lambda\in\mathbb{R}$ ρ has a limit from the right

$$\rho(\lambda+0) = \lim_{\substack{\varepsilon\to 0,\\ \varepsilon>0}} \rho(\lambda+\varepsilon)$$

and a limit from the left

$$\rho(\lambda-0) = \lim_{\substack{\varepsilon\to 0,\\ \varepsilon>0}} \rho(\lambda-\varepsilon).$$

If for example ρ has jumps in $\lambda_1,\dots,\lambda_n$ with $\lambda_1<\dots<\lambda_n$ but is constant otherwise we get with

$$\rho_k = \rho(\lambda_k+0)-\rho(\lambda_k-0),$$

$$z\in\mathbb{C},\ \operatorname{Im} z\neq 0,$$

$$F(z) = \sum_{k=1}^{n} \frac{\rho_k}{\lambda_k-z}$$

the formula

$$\text{(II.3.1)}\quad -\frac{1}{2\pi i}\int_{\Gamma_\varepsilon(\lambda^{(1)},\lambda^{(2)})} F(z)\,dz = \sum_{\lambda^{(1)}<\lambda_k<\lambda^{(2)}} \rho_k$$

$$= \rho(\lambda^{(2)})-\rho(\lambda^{(1)});$$

here $\Gamma_\varepsilon(\lambda^{(1)},\lambda^{(2)})$, $0<\varepsilon$, $\lambda^{(1)}<\lambda^{(2)}$, $\lambda^{(i)}\neq\lambda_j$, $i=1,2$, $j=1,\ldots$ $\ldots,n$, is a curve as described in the figure to follow:

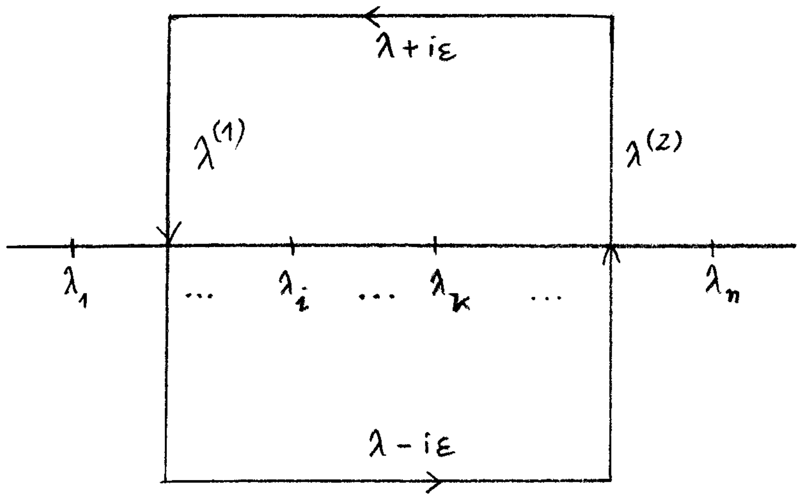

Fig. 1

$\Gamma_\varepsilon(\lambda^{(1)},\lambda^{(2)})$ is run through in the positive sense. (II.3.1) is then a simple consequence of the residuum formula. If we let ε tend to 0 the contributions of the integration over the perpendicular parts of $\Gamma_\varepsilon(\lambda^{(1)},\lambda^{(2)})$ tend to 0 and we end with

$$\lim_{\substack{\varepsilon\to 0,\\ \varepsilon>0}} \frac{1}{2\pi i}\left(\int_{\lambda^{(1)}}^{\lambda^{(2)}} F(\lambda+i\varepsilon)d\lambda - \int_{\lambda^{(1)}}^{\lambda^{(2)}} F(\lambda-i\varepsilon)d\lambda\right)$$

$$= \rho(\lambda^{(2)}) - \rho(\lambda^{(1)}).$$

The formula to follow is generalization of this simple situation.

Theorem II.3.1: ρ is as described in the beginning of this paragraph. Then $\int_{-\infty}^{+\infty} \frac{1}{\lambda-z}\, d\rho(\lambda)$ exists for $\operatorname{Im} z \neq 0$ and the function F defined by

$$F(z) = \int_{-\infty}^{+\infty} \frac{1}{\lambda - z}\, d\rho(\lambda), \quad \text{Im } z \neq 0,$$

is holomorphic. Moreover,

$$|F(z)| \leq \frac{1}{|\text{Im } z|} \int_{-\infty}^{+\infty} |d\rho(\lambda)|.$$

If $-\infty < \lambda_1 < \lambda_2 < +\infty$, then Stieltjes's inversion formula holds:

$$\frac{1}{2}(\rho(\lambda_2+0)+\rho(\lambda_2-0)) - \frac{1}{2}(\rho(\lambda_1+0)+\rho(\lambda_1-0))$$

$$= \lim_{\substack{\varepsilon \to 0, \\ \varepsilon > 0}} \frac{1}{2\pi i} \int_{\lambda_1}^{\lambda_2} (F(\lambda+i\varepsilon)-F(\lambda-i\varepsilon))\, d\lambda.$$

Proof: Let

$$a = \lambda_1 < \lambda_2 < \ldots < \lambda_n < \lambda_{n+1} = b.$$

Then for $\text{Im } z \neq 0$, $\varepsilon > 0$

$$\left| \sum_{i=1}^{n} \frac{1}{\lambda_i - z}(\rho(\lambda_{i+1})-\rho(\lambda_i)) - \int_a^b \frac{1}{\lambda - z}\, d\rho(\lambda) \right|$$

$$\leq \left| \sum_{i=1}^{n} \int_{\lambda_i}^{\lambda_{i+1}} \left(\frac{1}{\lambda_i - z} - \frac{1}{\lambda - z}\right) d\rho(\lambda) \right|$$

$$\text{(II.3.2)} \quad \leq \varepsilon \int_a^b |d\rho(\lambda)|,$$

provided $\delta = \max\limits_{1 \leq i \leq n} (\lambda_{i+1} - \lambda_i)$ is small enough. Strictly spoken, this estimate only holds if ρ is real valued and monotonically non decreasing, but (II.2.10) shows the validity of this estimate in the general case too. Since we can choose a fixed δ for any compact subset of $\{z | \text{Im } z \neq 0\}$ such that (II.3.2) holds for all z in this compact subset we have shown that

$$\int_a^b \frac{1}{\lambda - z} d\rho(\lambda)$$

is holomorphic in $\{z | \operatorname{Im} z \neq 0\}$. Since $\lim_{\lambda \to +\infty} \frac{1}{\lambda - z} = \lim_{\lambda \to -\infty} \frac{1}{\lambda - z} = 0$, $\operatorname{Im} z \neq 0$, it is easily shown that $\int_{-\infty}^{+\infty} \frac{1}{\lambda - z} d\rho(\lambda)$ exists.

Moreover, the convergence is uniform on every compact subset of $\{z | \operatorname{Im} z \neq 0\}$. Thus $F(z) = \int_{-\infty}^{+\infty} \frac{1}{\lambda - z} d\rho(\lambda)$ is holomorphic on $\{z | \operatorname{Im} z \neq 0\}$. We have for $\varepsilon > 0$, $\lambda_1 < \lambda_2$:

$$F(\lambda + i\varepsilon) - F(\lambda - i\varepsilon) = \int_{-\infty}^{+\infty} \left(\frac{1}{\mu - (\lambda + i\varepsilon)} - \frac{1}{\mu - (\lambda - i\varepsilon)} \right) d\rho(\mu),$$

$$= \int_{-\infty}^{\infty} \frac{2i\varepsilon}{(\mu - \lambda)^2 + \varepsilon^2} d\rho(\mu),$$

$$D = \frac{1}{2\pi i} \int_{\lambda_1}^{\lambda_2} (F(\lambda + i\varepsilon) - F(\lambda - i\varepsilon))\, d\lambda =$$

$$= \int_{-\infty}^{+\infty} \frac{1}{\pi} \int_{\lambda_1}^{\lambda_2} \frac{\varepsilon}{(\lambda - \mu)^2 + \varepsilon^2} d\lambda\, d\rho(\mu),$$

since the reader may easily verify by taking Riemannian sums and observing that $1/((\lambda - \mu)^2 + \varepsilon^2) \to 0$, if $\mu \to +\infty$, and, if $\mu \to -\infty$, that the order of integration can be altered. The inner integral gives

$$k(\mu; \varepsilon) = k(\mu; \lambda_1, \lambda_2, \varepsilon) = \frac{1}{\pi} \left[\arc\tan \frac{\lambda - \mu}{\varepsilon} \right]_{\lambda_1}^{\lambda_2}.$$

We now study the properties of $k(\mu; \lambda_1, \lambda_2, \varepsilon)$. We have

(II.3.3) $0 < k(\mu; \lambda_1, \lambda_2, \varepsilon) < 1$,

(II.3.4) $k(\mu;\lambda_1,\lambda_2,\varepsilon) \to 0,\ \varepsilon \to 0,$
uniformly on $\mu \leqq \lambda_1-\eta$, and
uniformly on $\mu \geqq \lambda_2+\eta$, if
$\eta > 0$ is any fixed number,

(II.3.5) $k(\mu;\lambda_1,\lambda_2,\varepsilon) \to 1,\ \varepsilon \to 0,$
uniformly on $\mu \in [\lambda_1+\eta,\lambda_2-\eta]$
if $\eta > 0$ is any fixed number
$< \lambda_2-\lambda_1$.

If $f:[a,b] \to \mathbb{C}$ is continuous and if $a < c \leqq b$, then

$$\text{(II.3.6)} \quad \int_a^{c-0} f(\lambda)d\rho(\lambda) := \lim_{\delta \downarrow 0} \int_a^{c-\delta} f(\lambda)d\rho(\lambda)$$

exists. This is seen as follows: We have

$$\int_a^{c-\delta''} f(\lambda)d\rho(\lambda) - \int_a^{c-\delta'} f(\lambda)d\rho(\lambda)$$

$$= \int_{c-\delta'}^{c-\delta''} f(\lambda)d\rho(\lambda)$$

$$= \int_{c-\delta'}^{c-\delta''} (f(\lambda)-f(c))d\rho(\lambda) + f(c)(\rho(c-\delta'')-\rho(c-\delta')),$$

$0 < \delta'' < \delta' < c-a$. Since ρ has a limit from the left in c and since f is continuous we arrive at our assertion. Similarly it is shown that

$$\text{(II.3.7)} \quad \int_{c+0}^{b} f(\lambda)d\rho(\lambda) := \lim_{\delta \downarrow 0} \int_{c+\delta}^{b} f(\lambda)d\rho(\lambda)$$

exists if $a \leqq c < b$. Since we have the relation

$$\text{(II.3.8)}\quad \int_a^b f(\lambda)d\rho(\lambda) = \int_a^{c-\delta} f(\lambda)d\rho(\lambda) + \int_{c-\delta}^{c+\delta} f(\lambda)d\rho(\lambda) + $$

$$+ \int_{c+\delta}^{b} f(\lambda)d\rho(\lambda)$$

for $c \in (a,b)$, $0 < \delta < \min\{c-a,b-c\}$ as is easily seen by going over to Riemannian sums, we arrive at

$$\text{(II.3.9)}\quad \int_a^b f(\lambda)d\rho(\lambda) = \int_a^{c-0} f(\lambda)d\rho(\lambda) + \int_{c+0}^{b} f(\lambda)d\rho(\lambda) + $$

$$+ f(c)(\rho(c+0)-\rho(c-0));$$

here we have to take into consideration that

$$\text{(II.3.10)}\quad \lim_{\delta\downarrow 0} \int_{c-\delta}^{c+\delta} f(\lambda)d\rho(\lambda) = \lim_{\delta\downarrow 0}\{f(c)(\rho(c+\delta)-\rho(c-\delta)) + $$

$$+ \int_{c-\delta}^{c+\delta} (f(\lambda)-f(c))d\rho(\lambda)\},$$

$$= f(c)(\rho(c+0)-\rho(c-0)).$$

Thus we get

$$\text{(II.3.11)}\quad D = \int_{-\infty}^{\lambda_1-\eta} k(\mu;\varepsilon)d\rho(\mu) + \int_{\lambda_1-\eta}^{\lambda_1-0} k(\mu;\varepsilon)d\rho(\mu)$$

$$+ \int_{\lambda_1+0}^{\lambda_1+\eta} k(\mu;\varepsilon)d\rho(\mu) + $$

$$+ k(\lambda_1;\varepsilon)(\rho(\lambda_1+0)-\rho(\lambda_1-0)) + $$

$$+ \int_{\lambda_1+\eta}^{\lambda_2-\eta} k(\mu;\varepsilon)d\rho(\mu) + \int_{\lambda_2-\eta}^{\lambda_2-0} k(\mu;\varepsilon)d\rho(\mu)$$

$$+ \int_{\lambda_2+0}^{\lambda_2+\eta} k(\mu;\varepsilon)d\rho(\mu) +$$

$$+ k(\lambda_2;\varepsilon)(\rho(\lambda_2+0)-\rho(\lambda_2-0)) +$$

$$+ \int_{\lambda_2+\eta}^{+\infty} k(\mu;\varepsilon)d\rho(\mu).$$

We set $D' = D - \{\frac{1}{2}(\rho(\lambda_2+0)+\rho(\lambda_2-0)) - \frac{1}{2}(\rho(\lambda_1+0)+\rho(\lambda_1-0))\}$. We have

$$\text{(II.3.12)} \quad k(\lambda_1;\varepsilon) = \frac{1}{\pi} \text{ arc tan } \frac{\lambda_2-\lambda_1}{\varepsilon} \to \frac{1}{2}, \quad \varepsilon \to 0,$$

$$\text{(II.3.13)} \quad k(\lambda_2;\varepsilon) = -\frac{1}{\pi} \text{ arc tan } \frac{\lambda_1-\lambda_2}{\varepsilon} \to \frac{1}{2}, \quad \varepsilon \to 0.$$

This yields

$$D' = \int_{-\infty}^{\lambda_1-\eta} k(\mu;\varepsilon)d\rho(\mu) + \int_{\lambda_1-\eta}^{\lambda_1-0} k(\mu;\varepsilon)d\rho(\mu) +$$

$$+ \int_{\lambda_1+0}^{\lambda_1+\eta} k(\mu;\varepsilon)d\rho(\mu) +$$

$$+ (\frac{1}{\pi} \text{ arc tan } \frac{\lambda_2-\lambda_1}{\varepsilon} - \frac{1}{2})(\rho(\lambda_1+0)-\rho(\lambda_1-0)) +$$

$$+ \frac{1}{2}(\rho(\lambda_1+0)-\rho(\lambda_1-0)) +$$

$$+ \int_{\lambda_1+\eta}^{\lambda_2-\eta} (k(\mu;\varepsilon)-1)d\rho(\mu) + \rho(\lambda_2-\eta) - \rho(\lambda_1+\eta) +$$

$$+ \int_{\lambda_2-\eta}^{\lambda_2-0} k(\mu;\varepsilon)d\rho(\mu) + \int_{\lambda_2+0}^{\lambda_2+\eta} k(\mu;\varepsilon)d\rho(\mu) +$$

$$+ \left(\frac{1}{\pi} \text{ arc tan } \frac{\lambda_2-\lambda_1}{\varepsilon} - \frac{1}{2}\right)(\rho(\lambda_2+0)-\rho(\lambda_2-0)) +$$

$$+ \frac{1}{2}(\rho(\lambda_2+0)-\rho(\lambda_2-0)) + \int_{\lambda_2+\eta}^{+\infty} k(\mu;\varepsilon)d\rho(\mu) -$$

$$- \frac{1}{2}\rho(\lambda_2+0) - \frac{1}{2}\rho(\lambda_2-0) + \frac{1}{2}\rho(\lambda_1+0) + \frac{1}{2}\rho(\lambda_1-0).$$

We set

$$T(\eta) = \frac{1}{2}(\rho(\lambda_1+0)-\rho(\lambda_1-0)) + \rho(\lambda_2-\eta) - \rho(\lambda_1+\eta) +$$

$$+ \frac{1}{2}(\rho(\lambda_2+0)-\rho(\lambda_2-0)) - \frac{1}{2}\rho(\lambda_2+0) - \frac{1}{2}\rho(\lambda_2-0) +$$

$$+ \frac{1}{2}\rho(\lambda_1+0) + \frac{1}{2}\rho(\lambda_1-0).$$

Clearly $T(\eta) \to 0$ if $\eta \to 0$. Let us set

$$D' = S(\eta;\varepsilon) + T(\eta).$$

The first integral and the last one tend to 0 if $\eta > 0$ is fixed and ε tends to 0; this follows from (II.3.4). (II.3.12) yields that

$$\left(\frac{1}{\pi} \text{ arc tan } \frac{\lambda_2-\lambda_1}{\varepsilon} - \frac{1}{2}\right)[(\rho(\lambda_1+0)-\rho(\lambda_1-0)) +$$

$$+ (\rho(\lambda_2+0)-\rho(\lambda_2-0))]$$

tends to 0 if $\varepsilon \to 0$. The fourth integral tends to 0 if $\eta > 0$ is fixed and ε tends to 0; this follows from (II.3.5). The sum of all these terms is denoted by $S_1(\eta;\varepsilon)$. The function $k(\mu;\varepsilon)$ is continuous on $[\lambda_1-\eta_0,\lambda_1]$ for any $\eta_0 > 0$; moreover $k(\mu;\varepsilon)$ is uniformly bounded by 0 from below and by 1 from above (cf. (II.3.3)). Considering the integral

$$\int_{\lambda_1-\eta}^{\lambda_1-0} k(\mu;\varepsilon)d\rho(\mu), \quad 0<\eta\leq\eta_o,$$

we can restrict ourselves to the case that ρ is real valued and monotonically non decreasing (cf. (II.2.10)). Let $\delta<\eta\leq\eta_o$ and let $\lambda_1-\eta = x_o<x_1<\ldots<x_n = \lambda_1-\delta$; we obtain

$$\sum_{l=1}^{n} k(x_l;\varepsilon)(\rho(x_l)-\rho(x_{l-1}))$$

$$\leq \rho(\lambda_1-\delta)-\rho(\lambda_1-\eta).$$

Choosing an equidistant partition of $[\lambda_1-\eta,\lambda_1-\delta]$ and letting n tend to ∞ we arrive at

$$\int_{\lambda_1-\eta}^{\lambda_1-\delta} k(\mu;\varepsilon)d\rho(\mu) \leq \rho(\lambda_1-\delta)-\rho(\lambda_1-\eta),$$

$$\int_{\lambda_1-\eta}^{\lambda_1-0} k(\mu;\varepsilon)d\rho(\mu) \leq \rho(\lambda_1-0)-\rho(\lambda_1-\eta).$$

The integrals $\int_{\lambda_1+0}^{\lambda_1+\eta} k(\mu;\varepsilon)d\rho(\mu)$, $\int_{\lambda_2-\eta}^{\lambda_2-0} k(\mu;\varepsilon)d\rho(\mu)$,

$\int_{\lambda_2+0}^{\lambda_2+\eta} k(\mu;\varepsilon)d\rho(\mu)$ can be treated analogously. Summing up all these four integrals we get a term $S_2(\eta;\varepsilon)$ whose absolute value can be estimated by quantity $\tilde{S}_2(\eta)$ with $\tilde{S}_2(\eta)\to 0$, $\eta\to 0$. Thus

$$D' = S_1(\eta;\varepsilon)+S_2(\eta;\varepsilon)+T(\eta).$$

Let $\gamma>0$. Fixing an $\eta_o>0$ such that $|T(\eta)|<\gamma/3$, $|\tilde{S}_2(\eta)|<\gamma/3$ and then an $\varepsilon_o>0$ with $|S_1(\eta;\varepsilon_o)|<\gamma/3$ we obtain

$$|D'| < \gamma,$$

provided ε is sufficiently small, $\varepsilon\leq\varepsilon_o$. This proves our theorem. □

We deal a little bit more with functions $\rho:\mathbb{R}\to\mathbb{C}$ having bounded variation on $(-\infty,+\infty)$. It immediately follows from the decomposition (II.2.10) that

$$\rho(-\infty) = \lim_{\lambda\to-\infty} \rho(\lambda)$$

exists. Also the existence of $\rho(\lambda+0)$, $\rho(\lambda-0)$, which has been mentioned already, can be concluded from (II.2.10). Moreover it is shown in [N, p. 245] that ρ is discontinuous in at most countably many λ. For the proof of the theorem to follow we refer to [N, p. 250].

Theorem II.3.2 (Helly's selection principle): Let $\rho_n:\mathbb{R}\to\mathbb{C}$, $n=1,2,\ldots$, be a sequence of functions having bounded variation on $(-\infty,+\infty)$. We assume that

$$|\rho_n(\lambda)| \leq M,$$

$$V(\rho_n) = \int_{-\infty}^{+\infty} |d\rho_n(\lambda)| \leq M, \; n\in\mathbb{N}.$$

Then there is a subsequence $\{\rho_{n_j}\}$ of $\{\rho_n\}$ and a function $\rho:\mathbb{R}\to\mathbb{C}$ having bounded variation $(-\infty,+\infty)$ such that

$$|\rho(\lambda)| \leq M,$$

$$\int_{-\infty}^{+\infty} |d\rho(\lambda)| \leq M,$$

$$\rho_{n_j}(\lambda)\to\rho(\lambda), \; j\to\infty, \; \lambda\in\mathbb{R}.$$

The next theorem describes a further property of functions of bounded variation.

Theorem II.3.3 (Helly's convergence theorem): Let $f:\mathbb{R}\to\mathbb{C}$ be continuous, let

$$f(\lambda)\to 0,\quad \lambda\to\pm\infty.$$

Let $\rho_n:\mathbb{R}\to\mathbb{C}$, $n=1,2,\dots$ be a sequence of functions having bounded variation on $(-\infty,+\infty)$. We assume that

$$\rho_n(\lambda)\to\rho(\lambda),\quad n\to\infty,\quad \lambda\in\mathbb{R},$$

$$\int_{-\infty}^{+\infty} |d\rho_n(\lambda)| \leq M.$$

Then also ρ is of bounded variation on $(-\infty,+\infty)$ and $\int_{-\infty}^{+\infty} |d\rho(\lambda)| \leq M$. Moreover

$$\int_{-\infty}^{+\infty} f(\lambda)d\rho_n(\lambda) \to \int_{-\infty}^{+\infty} f(\lambda)d\rho(\lambda),\quad n\to\infty.$$

Proof: The first part of our theorem is an easy consequence of the definition of the total variation of a function. As for the second part we choose $\lambda_1,\dots,\lambda_{m+1}$, $m\in\mathbb{N}$, such that

$$-\infty<\lambda_1<\lambda_2<\dots<\lambda_{m+1}<+\infty,$$

$$|f(\lambda)| \leq \varepsilon,\quad \lambda\leq\lambda_1,\quad \lambda\geq\lambda_{m+1},$$

$$|f(\lambda)-f(\lambda_j)| \leq \varepsilon,\quad \lambda_j\leq\lambda\leq\lambda_{j+1},\quad j=1,\dots,m,$$

where ε is any given positive number. Then

$$\delta = \left|\int_{-\infty}^{+\infty} f(\lambda)d\rho(\lambda) - \sum_{j=1}^{m} f(\lambda_j)\cdot(\rho(\lambda_{j+1})-\rho(\lambda_j))\right|,$$

$$\leq \varepsilon \int_{-\infty}^{+\infty} |d\rho(\lambda)| \leq \varepsilon M,$$

$$\delta_n = \left|\int_{-\infty}^{+\infty} f(\lambda)d\rho_n(\lambda) - \sum_{j=1}^{m} f(\lambda_j)\cdot(\rho_n(\lambda_{j+1})-\rho_n(\lambda_j))\right|,$$

$$\leq \varepsilon \int_{-\infty}^{+\infty} |d\rho_n(\lambda)| \leq \varepsilon M.$$

From this it follows that

$$\left|\int_{-\infty}^{+\infty} f(\lambda)d\rho(\lambda) - \int_{-\infty}^{+\infty} f(\lambda)d\rho_n(\lambda)\right| \leq$$

$$\leq \left|\sum_{j=1}^{m} f(\lambda_j)[(\rho(\lambda_{j+1})-\rho(\lambda_j))-(\rho_n(\lambda_{j+1})-\rho_n(\lambda_j))]\right| + 2\varepsilon M.$$

If n is sufficiently large, say $n \geq n_o$, the absolute value of the last sum becomes $< \varepsilon M$. Our theorem is proved. □

Definition II.3.1: Let $M > 0$. Then $\Gamma(M)$ denotes the set of all functions $\rho: \mathbb{R} \to \mathbb{C}$ having bounded variation on $(-\infty,+\infty)$ and the following additional properties:

$$\int_{-\infty}^{+\infty} |d\rho(\lambda)| \leq M,$$

$$\rho(-\infty) = 0,$$

$$\rho(\lambda+0) = \lim_{\varepsilon\to 0} \rho(\lambda+\varepsilon) = \rho(\lambda).$$

Proposition II.3.1: Let $\rho: \mathbb{R} \to \mathbb{C}$ have bounded variation on $(-\infty,+\infty)$. Let

$$M \geq \int_{-\infty}^{+\infty} |d\rho(\lambda)|.$$

Then the function $\rho^*: \mathbb{R} \to \mathbb{C}$, defined by $\rho^*(\lambda) = \rho(\lambda+0) - \rho(-\infty)$ is contained in $\Gamma(M)$. If $f: \mathbb{R} \to \mathbb{C}$ is continuous and if $f(\lambda) \to 0$, $\lambda \to \pm\infty$, then

$$\int_{-\infty}^{+\infty} f(\lambda)d\rho(\lambda) = \int_{-\infty}^{+\infty} f(\lambda)d\rho^*(\lambda).$$

Proof: First we have to show that $\int_{-\infty}^{+\infty} |d\rho^*(\lambda)| \leq M$. Let us take $\lambda_1,\ldots,\lambda_{m+1}$, $m \in \mathbb{N}$, such that

$$\lambda_1+\varepsilon < \lambda_2+\varepsilon < \ldots < \lambda_{m+1}+\varepsilon,$$

where ε is any positive number. Then

$$\sum_{j=1}^{m} |\rho^*(\lambda_{j+1})-\rho^*(\lambda_j)| =$$

$$= \lim_{\varepsilon\to 0} \sum_{j=1}^{m} |\rho(\lambda_{j+1}+\varepsilon)-\rho(\lambda_j+\varepsilon)| \leq M.$$

Thus ρ^* has bounded variation $(-\infty,+\infty)$ and $\int_{-\infty}^{+\infty} |d\rho^*(\lambda)| \leq M$. Consequently $\rho^*(\lambda+0)$ is well defined for any $\lambda \in \mathbb{R}$. Since ρ is discontinuous in at most countably many points we can choose a sequence $\{a_\nu\}$ with

$$a_\nu > 0,\ \nu \in \mathbb{N},\ a_\nu \to 0,\ \nu \to \infty,$$

$$\rho \text{ is continuous in } a_\nu,\ \nu \in \mathbb{N}.$$

Then $\rho^*(\lambda+0) = \lim_{\nu\to\infty} \rho^*(\lambda+a_\nu) = \lim_{\nu\to\infty} \rho(\lambda+a_\nu+0) - \rho(-\infty) =$
$= \lim_{\nu\to\infty} \rho(\lambda+a_\nu) - \rho(-\infty) = \rho^*(\lambda)$. A similar argument shows that $\rho^*(-\infty) = 0$. Thus $\rho^* \in \Gamma(M)$. Let $\varepsilon' > 0$. $m, \lambda_j, \varepsilon_o$ are chosen in such a way that $|f(\lambda)| \leq \varepsilon'$, $\lambda \leq \lambda_1$, $|f(\lambda)| \leq \varepsilon'$, $\lambda \geq \lambda_{m+1}$, $|f(\lambda)-f(\lambda_j)| \leq \varepsilon'$, $\lambda_j \leq \lambda \leq \lambda_{j+1}$, $\lambda_j+\varepsilon \leq \lambda \leq \lambda_{j+1}+\varepsilon$, $0 < \varepsilon \leq \varepsilon_o$. Then

$$\left|\int_{-\infty}^{+\infty} f(\lambda)d\rho^*(\lambda) - \int_{-\infty}^{+\infty} f(\lambda)d\rho(\lambda)\right| =$$

$$= \Big|\int_{-\infty}^{+\infty} f(\lambda)d\rho^*(\lambda) - \sum_{j=1}^{m} f(\lambda_j)\cdot(\rho^*(\lambda_{j+1})-\rho^*(\lambda_j))$$

$$+ \sum_{j=1}^{m} [f(\lambda_j)(\rho^*(\lambda_{j+1})-\rho^*(\lambda_j)) - f(\lambda_j+\varepsilon)(\rho(\lambda_{j+1}+\varepsilon)-\rho(\lambda_j+\varepsilon))] +$$

$$+ \sum_{j=1}^{m} f(\lambda_j+\varepsilon)(\rho(\lambda_{j+1}+\varepsilon)-\rho(\lambda_j+\varepsilon)) - \int_{-\infty}^{+\infty} f(\lambda)d\rho(\lambda)\Big|$$

$$\leqq 2\varepsilon' M + \Big|\sum_{j=1}^{m} [f(\lambda_j)(\rho^*(\lambda_{j+1})-\rho^*(\lambda_j)) -$$

$$- f(\lambda_j+\varepsilon)(\rho(\lambda_{j+1}+\varepsilon)-\rho(\lambda_j+\varepsilon))]\Big|,$$

$0 < \varepsilon \leqq \varepsilon_o$. Making ε sufficiently small we arrive at the assertion. □

For practical reasons we give

Definition II.3.2: By $\Gamma^*(M)$ we denote the set of all holomorphic functions F on $\operatorname{Im} z \neq 0$ which admit a representation

$$F(z) = \int_{-\infty}^{+\infty} \frac{d\rho(\lambda)}{\lambda - z}, \quad \operatorname{Im} z \neq 0,$$

where ρ is some element from $\Gamma(M)$.

We now show that ρ is determined uniquely by F.

Theorem II.3.4: Let $\rho_1, \rho_2 \in \Gamma(M)$ and let

$$\int_{-\infty}^{+\infty} \frac{d\rho_1(\lambda)}{\lambda - z} = \int_{-\infty}^{+\infty} \frac{d\rho_2(\lambda)}{\lambda - z}, \quad \operatorname{Im} z \neq 0.$$

<u>Then</u> $\rho_1(\lambda) = \rho_2(\lambda)$, $\lambda \in \mathbb{R}$.

<u>Proof:</u> For $\mu, \lambda \in \mathbb{R}$, $\mu < \lambda$, we get by Theorem II.3.1 (Stieltjes inversion formula)

$$\frac{1}{2}(\rho_2(\lambda+0)+\rho_2(\lambda-0)) - \frac{1}{2}(\rho_2(\mu+0)+\rho_2(\mu-0)) =$$

$$= \frac{1}{2}(\rho_1(\lambda+0)+\rho_1(\lambda-0)) - \frac{1}{2}(\rho_1(\mu+0)+\rho_1(\mu-0)).$$

Since ρ_1, ρ_2 are discontinuous at most countably many points, the set E, where ρ_1 <u>and</u> ρ_2 are continuous, is dense in $\mathbb{R}$. If $\lambda, \mu \in E$ we get $\rho_2(\lambda)-\rho_2(\mu) = \rho_1(\lambda)-\rho_1(\mu)$. Letting μ tend to $-\infty$ we obtain $\rho_1(\lambda) = \rho_2(\lambda)$, $\lambda \in E$. The same argument as in the first part of the proof of Proposition II.3.1 now shows that $\rho_1(\lambda) = \rho_2(\lambda)$ for any $\lambda \in \mathbb{R}$. □

$\Gamma^*(M)$ also has a closedness property, namely

<u>Theorem II.3.5: Let</u> $\{F_k\}$ <u>be a sequence contained in</u> $\Gamma^*(M)$. <u>Then there is a subsequence</u> $\{F_{k_n}\}$ of $\{F_k\}$ <u>with</u>

$$F_{k_n}(z) \to F(z), \; n \to \infty, \; \text{Im } z \neq 0, \; F \in \Gamma^*(M).$$

<u>Proof:</u> We have

$$F_k(z) = \int_{-\infty}^{+\infty} \frac{d\rho_k(\lambda)}{\lambda - z}, \; k \in \mathbb{N}, \; \text{Im } z \neq 0,$$

where $\rho_k \in \Gamma(M)$. Since $\rho_k(-\infty) = 0$ we easily obtain that $|\rho_k(\lambda)| \leq M$. Helly's selection principle shows that a subsequence $\{\rho_{k_n}\}$ of $\{\rho_k\}$ such that

$$\rho_{k_n}(\lambda) \to \rho(\lambda),\ n \to \infty,\ \lambda \in \mathbb{R},$$

$$\int_{-\infty}^{+\infty} |d\rho(\lambda)| \leq M,$$

$$|\rho(\lambda)| \leq M.$$

Helly's convergence theorem furnishes

$$F_{k_n}(z) \to \int_{-\infty}^{+\infty} \frac{d\rho(\lambda)}{\lambda - z},\ n \to \infty,\ \operatorname{Im} z \neq 0.$$

Setting $\rho^*(\lambda) = \rho(\lambda+0) - \rho(-\infty)$ we obtain an element from $\Gamma(M)$ and

$$F(z) := \int_{-\infty}^{+\infty} \frac{d\rho(\lambda)}{\lambda - z} = \int_{-\infty}^{+\infty} \frac{d\rho^*(\lambda)}{\lambda - z};$$

here we have applied Proposition II.3.1. □

§ 4. Integral Representation of the Resolvent

Our aim in the present paragraph is to prove a representation formula for the resolvent $(A-z)^{-1}$, $\operatorname{Im} z \neq 0$, of a selfadjoint operator A in a Hilbert space $\mathcal{H}$. This formula is of the following type:

$$(R_z f,g) = \int_{-\infty}^{+\infty} \frac{1}{\lambda-z} d\rho(\lambda;f,g)$$

where $\rho(.;f,g)$ is a function having bounded variation $(-\infty,+\infty)$. From now on we assume that $\mathcal{H}$ is separable. In view of the applications we have in mind this is no serious restriction since in most cases the underlying Hilbert space $\mathcal{H}$ is $L^2(\Omega)$, Ω an open subset of $\mathbb{R}^n$. We need the following

Proposition II.4.1: Let H be hermitian in $\mathcal{H}$ with domain of definition $\mathcal{D}(H)$. Then there exists a subspace $\mathcal{D}' \subset \mathcal{D}(H)$ and a sequence $\{H_n\}$ of bounded hermitian operators with the following properties:

(a) $\mathcal{D}(H_n) = \mathcal{D}'$.
$\mathcal{D}'$ is dense in $\mathcal{D}(H)$ with respect to the graph-norm of H, i.e. for any $f \in \mathcal{D}(H)$ there exists a sequence $\{f_n\}$ in $\mathcal{D}'$ with $\|f-f_n\| + \|Hf_n-Hf\| \to 0$ as $n \to \infty$.

(b) For any $f \in \mathcal{D}'$ we have $H_n f \to Hf$, $n \to \infty$.

(c) There exist, for any n, numbers a_n, b_n, $a_n < b_n$, and a spectral family $\{E_n(\lambda) \mid \lambda \in \mathbb{R}\}$ such that

$$H_n = \int_{a_n}^{b_n} \lambda \, dE_n(\lambda).$$

<u>Moreover, there are numbers</u> $\lambda_1^{(n)},\dots,\lambda_{k_n}^{(n)}$ <u>such that</u> $a_n < \lambda_1^{(n)} < \lambda_2^{(n)} < \dots < \lambda_{k_n}^{(n)} < b_n$, <u>and the</u> $E_n(\lambda)$ <u>are constant for</u> $\lambda < \lambda_1^{(n)}$, $\lambda > \lambda_{k_n}^{(n)}$, $\lambda_j^{(n)} < \lambda < \lambda_{j+1}^{(n)}$, $j = 1,\dots,k_n-1$.

<u>Proof:</u> Since $\mathcal{R}(\overline{H}\pm i)$ is a closed subspace of $\mathcal{H}$, it is easy to see that $\mathcal{R}(\overline{H}\pm i)$ are also separable. From this it follows that $\mathcal{R}(H\pm i)$ are separable (cf. Proposition I.3.2). Let e.g. $\{g_k\}$ be a sequence which is dense in $\mathcal{R}(H+i)$, let $f_k \in \mathcal{D}(H)$ and $g_k = (H+i)f_k$. Let $\mathfrak{M}_n$ be the subspace of $\mathcal{H}$ which is spanned by $f_1,\dots,f_n$; it consists of all elements

$$f = \sum_{\tilde{k}=1}^{n} c_{\tilde{k}} f_{\tilde{k}}, \quad c_1,\dots,c_n \in \mathbb{C},$$

its dimension being $\leq n$. We set

$$\mathcal{D}' = \bigcup_{n=1}^{\infty} \mathfrak{M}_n.$$

Since $\mathfrak{M}_n \subset \mathfrak{M}_{n+1}$, $n \in \mathbb{N}$, $\mathcal{D}'$ is itself a subspace of $\mathcal{H}$. For any $f \in \mathcal{D}'$ there is an $n(f) \in \mathbb{N}$, and there are complex numbers $c_1,\dots,c_{n(f)}$ such that

$$f = \sum_{\tilde{k}=1}^{n(f)} c_{\tilde{k}} f_{\tilde{k}}.$$

Our first assertion is that $\mathcal{D}'$ is dense in $\mathcal{D}(H)$ with respect to the graph-norm of H. If $f \in \mathcal{D}(H)$, $g = (H+i)f$, there exists a subsequence $\{g_{k_\nu}\}$ of $\{g_k\}$ with $g_{k_\nu} \to g$, $\nu \to \infty$.

Then $\|(H+i)(f_{k_\nu}-f)\|^2 \to 0$, $\nu \to \infty$, $\|H(f_{k_\nu}-f)\|^2 + \|f_{k_\nu}-f\|^2 \to 0$, $\nu \to \infty$.

Since $f_{k_\nu} \in \mathfrak{M}_{k_\nu} \subset \mathcal{D}'$, the first assertion is proved. Let E_n be the orthogonal projection from H onto $\mathfrak{M}_n$; observe that $\mathfrak{M}_n$ is closed since it is finite-dimensional. Let $H_n = E_n H E_n$; in particular H_n is even defined on H. If $f \in \mathcal{D}'$, then $f \in \mathfrak{M}_p$, $p = n(f)$. Therefore $E_n f = f$, $n \geq n(f)$, and $H_n f = E_n H E_n f = E_n H f$, $n \geq n(f)$. The sequence $\{\|E_n Hf\|\}$ is uniformly bounded. On the subspace $\mathcal{D}'$, which is dense in H, we easily get $(E_n Hf, g) \to (Hf, g)$ $(g \in \mathcal{D}')$ as n tends to ∞. Thus $E_n Hf \rightharpoonup Hf$, $n \to \infty$. Since E_n is a projection we conclude $\|E_n Hf\|^2 = (E_n Hf, E_n Hf) = (E_n Hf, Hf) \to \|Hf\|^2$, $n \to \infty$. Thus the second assertion is also proved. It is obvious that $(H_n f, g) = (f, H_n g)$, $f, g \in \mathcal{D}'$ moreover

$$H_n(\mathfrak{M}_n) \subset \mathfrak{M}_n.$$

Therefore there is an orthonormal basis $\varphi_1^{(n)}, \ldots, \varphi_{p_n}^{(n)}$ of $\mathfrak{M}_n$, $p_n = \dim \mathfrak{M}_n$, with

$$H_n \varphi_j^{(n)} = \mu_j^{(n)} \varphi_j^{(n)}, \quad j = 1, \ldots, p_n,$$

and the $\mu_j^{(n)}$ are real numbers (the eigenvalues of the restriction of H_n to $\mathfrak{M}_n$). Let $e_{\mu_j^{(n)}}$ be the multiplicity of $\mu_j^{(n)}$. Let $E_j^{(n)} f = (f, \varphi_j^{(n)}) \varphi_j^{(n)}$. Then

$$H_n f = \sum_{j=1}^{p_n} \mu_j^{(n)} (f, \varphi_j^{(n)}) \varphi_j^{(n)},$$

$$= \sum_{j=1}^{p_n} \mu_j^{(n)} E_j^{(n)} f, \quad f \in \mathfrak{M}_n.$$

We set $\mu_o^{(n)} := 0$; we have

$$E_n = \sum_{j=1}^{p_n} E_j^{(n)},$$

$$E_0^{(n)} = I - E_n.$$

If $f = f_1+f_2$, $f_1 \in {}_n$, $f_2 \in {}^{\perp}_n$, $f \in H$, then

$$H_n f = E_n H E_n (f_1+f_2),$$

$$= E_n H E_n f_1 = H_n f_1$$

$$= \sum_{j=1}^{p_n} \mu_j^{(n)} E_j^{(n)} f_1,$$

$$= \sum_{j=0}^{p_n} \mu_j^{(n)} E_j^{(n)} f.$$

Now we set

$$E_n(\lambda) = \sum_{j,\mu_j^{(n)} \leq \lambda} E_j^{(n)}, \ \lambda \in \mathbb{R}.$$

If $\lambda \geq b_n > \max\{\mu_j^{(n)} \mid 0 \leq j \leq p_n\}$, then $E_n(\lambda) = I$. If the last sum is void, we set by definition $E_n(\lambda) = 0$. Thus $E_n(\lambda) = 0$, $\lambda \leq a_n < \min\{\mu_j^{(n)} \mid 0 \leq j \leq p_n\}$. It is easy to see that $\lim\limits_{\substack{\varepsilon>0 \\ \varepsilon\to 0}} E_n(\lambda+\varepsilon) f = E_n(\lambda) f$ and that $E_n(\lambda)^* = E_n(\lambda)$, $E_n(\lambda)E_n(\mu) = E_n(\min\{\lambda,\mu\})$, $\lambda,\mu \in \mathbb{R}$. Thus the set $\{E_n(\lambda) \mid \lambda \in \mathbb{R}\}$ is a spectral family (cf. II.2, pp.30-32). Finally we evaluate $\int_{a_n}^{b_n} \lambda \, dE_n(\lambda)$. The integral was already defined in II.2 (Definition II.2.3). We choose a partition ${}_m = \{\lambda_1^{(m)}, \ldots, \lambda_{m+1}^{(m)}\}$ of $[a_n, b_n]$, i.e. $a_n = \lambda_1^{(m)} < \lambda_2^{(m)} < \ldots < \lambda_{m+1}^{(m)} = b_n$, $m \in \mathbb{N}$. We assume that $\delta(\mathfrak{Z}_m) = \max\limits_{1 \leq j \leq m} |\lambda_{j+1}^{(m)} - \lambda_j^{(m)}| \to 0$, $m \to \infty$. Moreover we assume that each $\mu_j^{(n)}$ is contained in one and only one $(\lambda_\kappa^{(m)}, \lambda_{\kappa+1}^{(m)})$, $m \in \mathbb{N}$. Then

$$\int_{a_n}^{b_n} \lambda \, dE_n(\lambda) = \lim_{m\to\infty} \sum_{\substack{\kappa,(\lambda_\kappa^{(m)},\lambda_{\kappa+1}^{(m)}) \text{ con-}\\ \text{tains one } \mu_j^{(n)}}} \mu_j^{(n)} (E_n(\lambda_{\kappa+1})-E(\lambda_\kappa)).$$

If $\mu_1^{(n)},\ldots,\mu_{l_n}^{(n)}$ are the pairwise distinct eigenvalues, then

$$\int_{a_n}^{b_n} \lambda \, dE_n(\lambda) = \lim_{m\to\infty} \Big(\sum_{l=1}^{l_n} \mu_l^{(n)} \sum_{j=e_{\mu_1^{(n)}}+\ldots+e_{\mu_{l-1}^{(n)}}+1}^{e_{\mu_1^{(n)}}+\ldots+e_{\mu_{l-1}^{(n)}}+e_{\mu_l^{(n)}}} E_j^{(n)} + \mu_0^{(n)} E_0^{(n)} \Big)$$

$$= H_n.$$

Our proposition is proved. □

Let us make the following remark: Let H be a selfadjoint operator in $\mathcal{H}$ with domain of definition $\mathcal{D}(H)$. If H' is the restriction of H to $\mathcal{D}'$, where $\mathcal{D}'$ is taken from Proposition II.4.1, then H' is essentially selfadjoint. This is seen as follows: Let $g \in \mathcal{H}$, let $f \in \mathcal{D}(H)$ with $(H+i)f = g$. Then we take a sequence $\{f_n\}$ from $\mathcal{D}'$ such that

$$f_n \to f,\ H'f_n \to Hf,\ n \to \infty.$$

Thus $(H'+i)f_n \to g = (H+i)f$, and $\mathcal{R}(H'+i)$ is dense in $\mathcal{H}$; the same argument shows that $\mathcal{R}(H'-i)$ is dense in $\mathcal{H}$.

<u>Proposition II.4.2:</u> <u>We ass that</u> H <u>is a selfadjoint operator in</u> $\mathcal{H}$ <u>with domain of definition</u> $\mathcal{D}(H)$. <u>Let</u> $\{H_n\}$ <u>be a sequence of selfadjoint operators in</u> $\mathcal{H}$ <u>with domains of definition</u> $\mathcal{D}(H_n)$ <u>such that:</u>

(a) <u>There is a subspace</u> $\mathcal{D}' \subset \mathcal{H}$ <u>with</u> $\mathcal{D}' \subset \mathcal{D}(H)$, $\mathcal{D}' \subset \mathcal{D}(H_n)$ <u>such that the restriction</u> H' <u>of</u> H <u>to</u> $\mathcal{D}'$ <u>is essentially selfadjoint.</u>

(b) <u>For</u> $f \in \mathcal{D}'$ <u>we have</u> $H_n f \to Hf$, $n \to \infty$.

<u>Then</u>

$$R_z^{(n)} f \to R_z f,\ f \in \mathcal{H},\ \text{Im } z \neq 0,$$

<u>where</u> $R_z^{(n)} = (H_n - z)^{-1}$.

<u>Proof:</u> Let $\mathcal{H}_z' = \{g | g = (H-z)f \text{ for some } f \in \mathcal{D}'\}$, Im $z \neq 0$. Since H' is essentially selfadjoint, the space $\mathcal{H}_z'$ is dense in $\mathcal{H}$. Namely, as in the proof of Proposition I.3.2 we get $\overline{\mathcal{R}(H'-z)} = \mathcal{R}(\overline{H}-z)$, Im $z \neq 0$. Since $\overline{H}$ is selfadjoint we have by Theorem II.1.1 the relation $\mathcal{R}(\overline{H}-z) = \mathcal{H}$. Let $g \in \mathcal{H}_z'$. Then

$$R_z^{(n)} g - R_z g = (H_n - z)^{-1} g - (H-z)^{-1} g,$$

$$= (H_n - z)^{-1} (H-z)(H-z)^{-1} g - (H_n - z)^{-1} (H_n - z)(H-z)^{-1} g,$$

$$= (H_n - z)^{-1} (H - H_n)(H-z)^{-1} g,$$

$$\| R_z^{(n)} g - R_z g \| \leq \frac{1}{|\text{Im } z|} \| (H - H_n)(H-z)^{-1} g \| \to 0,\ n \to \infty.$$

Since the operators $R_z^{(n)} - R_z$ have uniformly bounded (with respect to n) norms and since $\mathcal{H}_z'$ is dense in $\mathcal{H}$ we arrive at the assertion. □

<u>Proposition II.4.3:</u> <u>Let</u> H <u>be selfadjoint in</u> $\mathcal{H}$ <u>with domain of definition</u> $\mathcal{D}(H)$. <u>Then the following integral representation holds for</u> R_z, Im $z \neq 0$:

$$(R_z f,g) = \int_{-\infty}^{+\infty} \frac{d\rho(\lambda;f,g)}{\lambda-z},$$

where $\rho(.;f,g)$ is some function from $\Gamma(\|f\|\|g\|)$.

Proof: Let $\{H_n\}$ be the approximating sequence of bounded hermitian operators which has been constructed in Proposition II.4.1. Let $\{E_n(\lambda)|\lambda \in \mathbb{R}\}$ be the spectral family which has been constructed in Proposition II.4.1. It follows from Proposition II.4.1 that the restriction of H to the space $\mathcal{D}'$ in Proposition II.4.1 is essentially selfadjoint. Proposition II.4.2 now furnishes

$$R_z^{(n)} f \to R_z f, \quad n \to \infty.$$

We claim that

$$\text{(II.4.1)} \quad R_z^{(n)} = \int_{a_n}^{b_n} \frac{dE_n(\lambda)}{\lambda-z}.$$

As in the proof of Proposition II.4.1 we obtain (using the same notations)

$$\int_{a_n}^{b_n} \frac{dE_n(\lambda)}{\lambda-z} = \sum_{l=1}^{l_n} (\mu_l^{(n)}-z)^{-1} \cdot \sum_{j=e_{\mu_1^{(n)}}+\ldots+e_{\mu_{l-1}^{(n)}}+1}^{e_{\mu_1^{(n)}}+\ldots+e_{\mu_{l-1}^{(n)}}+e_{\mu_l^{(n)}}} E_j^{(n)} + (\mu_0^{(n)}-z)^{-1} E_0^{(n)},$$

$$(H_n-z)\int_{a_n}^{b_n} \frac{dE_n(\lambda)}{\lambda-z} = \int_{a_n}^{b_n} \frac{dE_n(\lambda)}{\lambda-z}(H_n-z)$$

$$= \int_{a_n}^{b_n} \frac{dE_n(\lambda)}{\lambda-z} \int_{a_n}^{b_n} (\lambda-z)\,dE_n(\lambda)$$

since

$$H_n - z = \sum_{l=1}^{l_n} (\mu_l^{(n)} - z) \cdot \sum_{j=e_{\mu_1^{(n)}}+\dots+e_{\mu_{l-1}^{(n)}}+1}^{e_{\mu_1^{(n)}}+\dots+e_{\mu_{l-1}^{(n)}}+e_{\mu_l^{(n)}}} E_j^{(n)} + (\mu_0^{(n)} - z) E_0^{(n)},$$

$$= \int_{a_n}^{b_n} (\lambda - z) dE_n(\lambda).$$

Inserting the finite sums for $\int_{a_n}^{b_n} \frac{dE_n(\lambda)}{\lambda - z}$, $\int_{a_n}^{b_n} (\lambda - z) dE_n(\lambda)$ and taking into consideration that $E_j^{(n)} E_{j'}^{(n)} = 0$, $j \neq j'$, we arrive at (II.4.1). We have

$$(R_z^{(n)} f, g) = \int_{a_n}^{b_n} \frac{1}{\lambda - z} d(E_n(\lambda) f, g) = \int_{-\infty}^{+\infty} \frac{1}{\lambda - z} d(E_n(\lambda) f, g).$$

by Theorem II.2.4. The function $\rho_n(\lambda; f, g) = (E_n(\lambda) f, g)$ has bounded variation on $(-\infty, +\infty)$ with $\int_{-\infty}^{+\infty} |d(E(\lambda) f, g| \leq \|f\| \cdot \|g\|$; this was proved in Theorem II.2.1. The defining properties of a spectral family imply that $\rho_n \in \Gamma(\|g\| \|f\|)$ and consequently $(R^{(n)} f, g) \in \Gamma^*(\|g\| \|f\|)$. By Theorem II.3.5 there is a subsequence $\{(R^{n_j} f, g)\}$ of $\{(R^{(n)} f, g)\}$ such that

$$(R_z^{n_j} f, g) \to \phi(z), \quad j \to \infty, \quad \text{Im } z \neq 0,$$

with $\phi \in \Gamma^*(\|g\| \|f\|)$. Since $R_z^{(n)} f \to R_z f$, $n \to \infty$, we get $\phi(z) = (R_z f, g)$, which completes the proof. □

§ 5. Fundamental Properties of the Function $\rho(.;f,g)$

Our aim here is to show that $\rho(\lambda;f,g) = (E(\lambda)f,g)$ with a spectral family $\{E(\lambda)\,|\,\lambda \in \mathbb{R}\}$.

Proposition II.5.1: Let $f,g:\mathbb{R}\to\mathbb{C}$ be continuous functions with

$$\lim_{\lambda\to\pm\infty} f(\lambda) = \lim_{\lambda\to\pm\infty} g(\lambda) = 0.$$

Let $\rho \in \Gamma(M)$. Then the function

$$G(\lambda) = \int_{-\infty}^{\lambda} g(\mu)d\rho(\mu)$$

is from $\Gamma(M')$ for some suitable $M' \geqq 0$, and we have

$$\text{(II.5.1)} \quad \int_{-\infty}^{+\infty} f(\lambda)g(\lambda)d\rho(\lambda) = \int_{-\infty}^{+\infty} f(\lambda)dG(\lambda).$$

Proof: First we have to show that G is continuous from the right: Let $\varepsilon > 0$; then

$$G(\lambda+\varepsilon)-G(\lambda) = \int_{\lambda}^{\lambda+\varepsilon} g(\mu)d\rho(\mu).$$

If $\lambda = \mu_1 < \mu_2 < \ldots < \mu_{n+1} = \lambda+\varepsilon$ we get

$$\left|\sum_{j=1}^{n} g(\mu_j)(\rho(\mu_{j+1})-\rho(\mu_j))\right| \leqq$$

$$\leqq \left|\sum_{j=1}^{n} (g(\mu_j)-g(\lambda))(\rho(\mu_{j+1})-\rho(\mu_j))\right| +$$

$$+ \left|\sum_{j=1}^{n} g(\lambda)(\rho(\mu_{j+1})-\rho(\mu_j))\right|,$$

$$\leq \sup_{\lambda \leq s, t \leq \lambda+\varepsilon} |g(s)-g(t)| \cdot \int_{-\infty}^{+\infty} |d\rho(\mu)| +$$

$$+ \left| \sum_{j=1}^{n} g(\lambda) \left(\rho(\mu_{j+1})-\rho(\mu_j)\right)\right| .$$

The last sum is estimated by

$$\sup_{\lambda \leq s \leq \lambda+1} |g(s)| \cdot |\rho(\lambda+\varepsilon) - \rho(\lambda+0)|$$

as $g(\lambda)$ doesn't depend on j. The preceding calculations show that G is continuous from the right. Let us take $n+1$ points $\lambda_1, \ldots, \lambda_{n+1} \in \mathbb{R}$ with $\lambda_1 < \lambda_2 < \ldots < \lambda_{n+1}$. Then

$$\sum_{i=1}^{n} |G(\lambda_{i+1})-G(\lambda_i)| =$$

$$= \sum_{i=1}^{n} \left|\int_{\lambda_i}^{\lambda_{i+1}} g(\lambda) d\rho(\lambda)\right| ,$$

$$\leq \sup_{\lambda \in \mathbb{R}} |g(\lambda)| \sum_{i=1}^{\infty} \int_{\lambda_i}^{\lambda_{i+1}} |d\rho(\lambda)| ,$$

$$\leq \sup_{\lambda \in \mathbb{R}} |g(\lambda)| \int_{-\infty}^{+\infty} |d\rho(\lambda)| ,$$

$$\int_{-\infty}^{+\infty} |dG(\lambda)| \leq \sup_{\lambda \in \mathbb{R}} |g(\lambda)| \int_{-\infty}^{+\infty} |d\rho(\lambda)| =: M' .$$

The preceding calculations also show that $G(\lambda) \to 0$ if $\lambda \to -\infty$. Thus $G \in \Gamma(M')$. Now we have to prove (II.5.1). If $\lambda_1, \ldots, \lambda_{n+1}$ are as before, then for any $\eta > 0$

$$\left|\sum_{i=1}^{n} f(\lambda_i)(G(\lambda_{i+1})-G(\lambda_i)) - \right.$$

$$\left. - \sum_{i=1}^{n} f(\lambda_i)g(\lambda_i)(\rho(\lambda_{i+1})-\rho(\lambda_i))\right|$$

$$= \left|\sum_{i=1}^{n} f(\lambda_i) \int_{\lambda_i}^{\lambda_{i+1}} (g(\lambda)-g(\lambda_i))d\rho(\lambda)\right|$$

$$\leq \eta\cdot\sup_{\lambda\in\mathbb{R}} |f(\lambda)| \int_{-\infty}^{+\infty} |d\rho(\lambda)|,$$

$$\leq \eta\cdot\sup_{\lambda\in\mathbb{R}} |f(\lambda)|\cdot M,$$

provided $|g(\lambda)-g(\lambda_i)| \leq \eta$, $\lambda_i \leq \lambda \leq \lambda_{i+1}$. On the other hand we may also assume that

$$\left|\int_{-\infty}^{+\infty} f(\lambda)dG(\lambda) - \sum_{i=1}^{n} f(\lambda_i)(G(\lambda_{i+1})-G(\lambda_i))\right| \leq \eta,$$

since $\lim_{\lambda\to\pm\infty} f(\lambda) = 0$ and, that

$$\left|\sum_{i=1}^{n} f(\lambda_i)g(\lambda_i)(\rho(\lambda_{i+1})-\rho(\lambda_i)) - \int_{-\infty}^{+\infty} f(\lambda)g(\lambda)d\rho(\lambda)\right| \leq \eta$$

since $\lim_{\lambda\to\pm\infty} f(\lambda)g(\lambda) = 0$. Thus we end up with

$$\left|\int_{-\infty}^{+\infty} f(\lambda)dG(\lambda) - \int_{-\infty}^{+\infty} f(\lambda)g(\lambda)d\rho(\lambda)\right| \leq 2\eta + \eta\cdot\sup_{\lambda\in\mathbb{R}}|f(\lambda)|\cdot M.$$

The proposition is proved. □

Proposition II.5.2: Let ρ be as in Proposition II.4.3. We have for all $\lambda \in \mathbb{R}$ the relation

$$\rho(\lambda;f,g) = (E(\lambda)f,g),$$

where the $E(\lambda)$, $\lambda \in \mathbb{R}$, are everywhere in H defined bounded operators having the following properties:

$$E(\lambda) \text{ is hermitian, } \lambda \in \mathbb{R},\ \|E(\lambda)\| \leq 1.$$

Proof: We have for $c_i \in \mathbb{C}$, $f_i \in H$, $g_i \in H$, $i = 1,2$, $f,g \in H$:

$$\int_{-\infty}^{+\infty} \frac{d\rho(\lambda;c_1f_1+c_2f_2,g)}{\lambda-z} = (R_z(c_1f_1+c_2f_2),g),$$

$$= c_1(R_zf_1,g) + c_2(R_zf_2,g)$$

$$= \int_{-\infty}^{+\infty} \frac{1}{\lambda-z} d(c_1\rho(\lambda;f_1,g)+c_2\rho(\lambda;f_2,g)),$$

$\operatorname{Im} z \neq 0$. Set $M = [\max\{\|c_1f_1+c_2f_2\|, |c_1|\|f_1\|, |c_2|\|f_2\|\}]\cdot\|g\|$. Then $\rho(.;c_1f_1+c_2f_2,g)$, $c_1\rho(.;f_1,g) + c_2\rho(.;f_2,g) \in \Gamma(M)$, and Theorem II.3.4 then shows

$$\rho(\lambda;c_1f_1+c_2f_2,g) = c_1\rho(\lambda;f_1,g) + c_2\rho(\lambda;f_2,g).$$

In the same way it is shown that

$$\rho(\lambda;f,c_1g_1+c_2g_2) = \overline{c_1}\rho(\lambda;f,g_1) + \overline{c_2}\rho(\lambda;f,g_2).$$

Since $\rho \in \Gamma(\|f\|\|g\|)$ we get the inequality

(II.5.2) $|\rho(\lambda;f,g)| \leq \|f\|\|g\|$, $\lambda \in \mathbb{R}$.

We have $(R_z f,g) = (f,R_{\bar z} g) = \overline{(R_{\bar z} g,f)}$ since $R_z^* = R_{\bar z}$. Thus

$$\int_{-\infty}^{+\infty} \frac{d\rho(\lambda;f,g)}{\lambda-z} = \overline{\int_{-\infty}^{+\infty} \frac{d\rho(\lambda;g,f)}{\lambda-\bar z}},$$

$$= \int_{-\infty}^{+\infty} \frac{d\overline{\rho(\lambda;g,f)}}{\lambda-\bar z},$$

and Theorem II.3.4 furnishes

$$\rho(\lambda;f,g) = \overline{\rho(\lambda;g,f)}, \quad \lambda \in \mathbb{R}.$$

For each $\lambda \in \mathbb{R}$ thus $\rho(\lambda;.,.)$ is an hermitian sesquilinear form satisfying (II.5.2). Thus for each $\lambda \in \mathbb{R}$ there is one and only one everywhere defined hermitian bounded operator $E(\lambda)$ with

$$\rho(\lambda;f,g) = (E(\lambda)f,g).$$

From (II.5.2) it follows that $\|E(\lambda)\| \leq 1$. □

Proposition II.5.3: There is one and only one $\rho(.;f,g) \in \Gamma(\|f\|\,\|g\|)$ such that

$$(R_z f,g) = \int_{-\infty}^{+\infty} \frac{d\rho(\lambda;f,g)}{\lambda-z}, \quad f,g \in H.$$

The operators $E(\lambda)$, $\lambda \in \mathbb{R}$, constructed in the preceding proposition, form a spectral family.

Proof: By Theorem II.1.3 we have

$$\frac{R_{z_1}-R_{z_2}}{z_1-z_2} = R_{z_1}R_{z_2}, \quad z_1 \neq z_2, \ \mathrm{Im}\, z_1, \mathrm{Im}\, z_2 \neq 0.$$

$$\left(\int_{-\infty}^{+\infty} \frac{d(E(\lambda)f,g)}{\lambda-z_1} - \int_{-\infty}^{+\infty} \frac{d(E(\lambda)f,g)}{\lambda-z_2}\right)\frac{1}{z_1-z_2} =$$

$$= \int_{-\infty}^{+\infty} \frac{d(E(\lambda)f,g)}{(\lambda-z_1)(\lambda-z_2)} = \left(\frac{R_{z_1}-R_{z_2}}{z_1-z_2}f,g\right),$$

$$= (R_{z_1}R_{z_2}f,g) = (R_{z_2}f,R_{\bar{z}_1}g),$$

$$= \int_{-\infty}^{+\infty} \frac{d\rho(\lambda;f,R_{\bar{z}_1}g)}{\lambda-z_2},$$

$$= \int_{-\infty}^{+\infty} \frac{d(E(\lambda)f,R_{\bar{z}_1}g)}{\lambda-z_2};$$

We set $\sigma_{z_1}(\lambda) = \int_{-\infty}^{\lambda} \frac{d(E(\mu)f,g)}{\mu-z_1}$. Then

$$\int_{-\infty}^{+\infty} \frac{d\sigma_{z_1}(\lambda)}{\lambda-z_2} = \int_{-\infty}^{+\infty} \frac{d(E(\lambda)f,g)}{(\lambda-z_1)(\lambda-z_2)},$$

$$= \int_{-\infty}^{+\infty} \frac{d(E(\lambda)f,R_{\bar{z}_1}g)}{\lambda-z_2}$$

by Proposition II.5.1 and the preceding calculations. Theorem II.3.4 yields

$$\sigma_{z_1}(\lambda) = (E(\lambda)f,R_{\bar{z}_1}g),$$

$$\text{(II.5.3)} \qquad = \int_{-\infty}^{\lambda} \frac{d(E(\mu)f,g)}{\mu-z_1}.$$

We can replace z_1 by z with $\operatorname{Im} z \neq 0$ and get

$$(E(\lambda)f, R_{\bar z}g) = \overline{(R_{\bar z}g, E(\lambda)f)},$$

$$= \overline{\int_{-\infty}^{+\infty} \frac{d(E(\mu)g, E(\lambda)f}{\mu - \bar z}},$$

$$= \int_{-\infty}^{+\infty} \frac{d\overline{(E(\mu)g, E(\lambda)f)}}{\mu - z},$$

$$= \int_{-\infty}^{+\infty} \frac{d(E(\mu)E(\lambda)f, g)}{\mu - z},$$

$$= \int_{-\infty}^{\lambda} \frac{d(E(\mu)f, g)}{\mu - z},$$

where we have used Proposition II.5.2 and equation (II.5.3). Let us set

$$\tau_\lambda(\mu) = \begin{cases} (E(\mu)f, g), & \mu \leqq \lambda, \\ (E(\lambda)f, g), & \mu \geqq \lambda. \end{cases}$$

Then

$$\int_{-\infty}^{+\infty} \frac{d\tau_\lambda(\mu)}{\mu - z} = \int_{-\infty}^{\lambda} \frac{d(E(\mu)f, g)}{\mu - z}$$

$$= \int_{-\infty}^{+\infty} \frac{d(E(\mu)E(\lambda)f, g)}{\mu - z}, \quad \mathrm{Im}\, z \neq 0.$$

Again Theorem II.3.4 yields

$$\tau_\lambda(\mu) = (E(\mu)E(\lambda)f, g),$$

$$E(\mu)E(\lambda)f = E(\min(\mu, \lambda)f, \quad f, g \in H.$$

In particular each $E(\lambda)$ is a projection in H. Since $(E(.)f, g) \in F(\|f\|\,\|g\|)$ we have

$$(E(\lambda+\varepsilon)f,g) \to (E(\lambda)f,g), \ \varepsilon > 0, \ \varepsilon \to 0.$$

However

$$\|E(\lambda+\varepsilon)f-E(\lambda)f\|^2 = (E(\lambda+\varepsilon)f,f) + (E(\lambda)f,f) - 2(E(\lambda)f,f)$$
$$= (E(\lambda+\varepsilon)f,f) - (E(\lambda)f,f)$$

and consequently

$$\lim_{\substack{\varepsilon>0,\\ \varepsilon\to 0}} E(\lambda+\varepsilon)f = E(\lambda)f,$$

$\lambda \in \mathbb{R}$. If $\lambda \to -\infty$, then $(E(\lambda)f,f) \to 0$, since $(E(.)f,f) \in \Gamma(\|f\|\,\|g\|)$. Using $(E(\lambda)f,f) = \|E(\lambda)f\|^2$ we get

$$\lim_{\lambda\to-\infty} E(\lambda)f = 0, \ f \in H.$$

Now we consider the case $\lambda \to +\infty$. The function $(E(\lambda)f,f)$ is bounded and monotonically non decreasing from $\mathbb{R}$ into the nonnegative reals. Thus

$$\lim_{\lambda\to+\infty} (E(\lambda)f,f) \text{ exists.}$$

Assume that $\lambda < \mu$. Then

$$\|E(\mu)f-E(\lambda)f\|^2 = (E(\mu)f,f) - (E(\lambda)f,f)$$
$$\le \sup_{\lambda<\mu} |(E(\mu)f,f)-(E(\lambda)f,f)| =: \varepsilon(\lambda),$$

and $\varepsilon(\lambda)$ tends to 0 if $\lambda \to +\infty$. Thus there is, for each $f \in H$, an element $L(f) \in H$ such that $E(\lambda)f \to L(f)$, $\lambda \to \infty$. Set

$$g = f - L(f) = f - \lim_{\lambda\to+\infty} E(\lambda)f.$$

For $\mu \in \mathbb{R}$ we get

$$E(\mu)g = E(\mu)f - \lim_{\lambda\to+\infty} E(\mu)E(\lambda)f = 0.$$

If $\operatorname{Im} z \neq 0$, $h \in H$, then consequently

$$(R_z g,h) = \int_{-\infty}^{+\infty} \frac{d(E(\mu)g,h)}{\mu - z} = 0,$$

$$((H-z)^{-1}g,h) = 0.$$

Since $R(H-\bar{z}) = H$ we can find an $u \in D(H)$ with $h = (H-\bar{z})u$. Thus

$$((H-z)^{-1}g,(H-\bar{z})u) = 0, \quad u \in D(H),$$

$$(g,u) = 0, \quad u \in D(H).$$

Since $D(H)$ is dense we obtain $g = 0$ and $f = L(f)$. The first part of Proposition II.5.3 is an easy consequence of Theorem II.3.4. □

We are now in a position to prove the main result of the present paragraph, namely

Theorem II.5.1: Let H be a selfadjoint operator in H with domain of definition $D(H)$. Then there is one and only one spectral family $\{E(\lambda) | \lambda \in \mathbb{R}\}$ such that

$$(H-z)^{-1} = \int_{-\infty}^{+\infty} \frac{dE(\lambda)}{\lambda - z},$$

where the last integral is convergent with respect to the norm of $L(H,H)$.

Proof: The convergence of $\int_{-\infty}^{+\infty} \frac{dE(\lambda)}{\lambda-z} := \lim_{\substack{b\to+\infty,\\ a\to-\infty}} \int_a^b \frac{dE(\lambda)}{\lambda-z}$ is an easy consequence of Theorem II.2.4 (if $\{E(\lambda)\,|\,\lambda \in \mathbb{R}\}$ is any spectral family). Now let us take the spectral family just constructed in Proposition II.5.3. Then

$$(R_z f,g) = \int_{-\infty}^{+\infty} \frac{d(E(\lambda)f,g)}{\lambda-z}$$

$$= \left(\int_{-\infty}^{+\infty} \frac{dE(\lambda)}{\lambda-z} f,g\right), \quad f,g \in \mathcal{H},$$

and we obtain

$$R_z f = \int_{-\infty}^{+\infty} \frac{dE(\lambda)}{\lambda-z} f.$$

If there is any other spectral family $\{\tilde{E}(\lambda)\,|\,\lambda \in \mathbb{R}\}$ such that

$$R_z = \int_{-\infty}^{+\infty} \frac{d\tilde{E}(\lambda)}{\lambda-z}$$

we get in turn

$$\int_{-\infty}^{+\infty} \frac{d(E(\lambda)f,g)}{\lambda-z} = \int_{-\infty}^{+\infty} \frac{d(\tilde{E}(\lambda)f,g)}{\lambda-z}.$$

Theorem II.3.4 furnishes $E(\lambda)f = \tilde{E}(\lambda)f$, $\lambda \in \mathbb{R}$. □

§ 6. The Spectral Theorem for Selfadjoint Operators

The novelty now is that we consider integrals $\int_{-\infty}^{+\infty} \varphi(\lambda) dE(\lambda) f$ for unbounded continuous functions $\varphi: \mathbb{R} \to \mathbb{C}$. In this paragraph however we only take a very simple one, namely $\varphi(\lambda) = \lambda$.

<u>Proposition II.6.1:</u> <u>Let</u> $\{E(\lambda) \mid \lambda \in \mathbb{R}\}$ <u>be a spectral family</u>. <u>Let</u> $f \in \mathcal{H}$. <u>Then</u>

$$\lim_{\substack{a\to-\infty,\\ b\to+\infty}} \int_a^b \lambda dE(\lambda) f =: \int_{-\infty}^{+\infty} \lambda dE(\lambda) f$$

<u>exists if and only if</u>

$$\lim_{\substack{a\to-\infty,\\ b\to+\infty}} \int_a^b \lambda^2 d(E(\lambda)f,f) =: \int_{-\infty}^{+\infty} \lambda^2 d(E(\lambda)f,f)$$

<u>exists.</u>

<u>Proof:</u> Let us first assume that

$$\lim_{\substack{a\to-\infty,\\ b\to+\infty}} \int_a^b \lambda dE(\lambda) f$$

exists. For the definition of the integrals $\int_a^b \lambda dE(\lambda) f$ the reader may confer Definition II.2.3. We have then

$$\left\| \int_a^b \lambda \, dE(\lambda) f \right\|^2 \leq M.$$

We again refer to Definition II.2.3 and take the Riemannian sums T_n for $\varphi(\lambda) = \lambda$. This gives

$$\left(\sum_{i=1}^{k_n} \lambda_i^{(n)} E(\Delta_i^{(n)})f, \sum_{i=1}^{k_n} \lambda_i^{(n)} E(\Delta_i^{(n)})f\right)$$

$$= \sum_{i,j=1}^{k_n} \lambda_i^{(n)} \lambda_j^{(n)} (E(\Delta_j^{(n)}) E(\Delta_i^{(n)}) f, f)$$

$$= \sum_{i=1}^{k_n} \lambda_i^{(n)^2} (E(\Delta_i^{(n)}) f, f);$$

thus

$$\int_a^b \lambda^2 d(E(\lambda)f,f) \leq M,$$

and $\lim_{\substack{a\to-\infty,\\ b\to+\infty}} \int_a^b \lambda^2 d(E(\lambda)f,f)$ exists. Secondly we assume that $\int_{-\infty}^{+\infty} \lambda^2 d(E(\lambda)f,f)$ exists. Let $-\infty < c < a < b < d < +\infty$,

$$\delta = \left\| \int_c^d \lambda\, dE(\lambda)f - \int_a^b \lambda\, dE(\lambda)f \right\|^2,$$

$$= \left\| \int_c^a \lambda\, dE(\lambda)f + \int_b^d \lambda\, dE(\lambda)f \right\|^2.$$

Taking the Riemannian sums as before we obtain

$$\delta = \int_c^a \lambda^2 d(E(\lambda)f,f) + \int_b^d \lambda^2 d(E(\lambda)f,f),$$

$$\leq \int_{-\infty}^a \lambda^2 d(E(\lambda)f,f) + \int_b^{+\infty} \lambda^2 d(E(\lambda)f,f).$$

The latter integrals tend to 0 if $a \to -\infty$, $b \to +\infty$. Our proposition is proved. □

Theorem II.6.1: Let $\{E(\lambda) \mid \lambda \in \mathbb{R}\}$ be a spectral family. Let

$$\mathcal{D} = \{f \mid f \in \mathcal{H}, \int_{-\infty}^{+\infty} \lambda^2 d(E(\lambda)f,f) < +\infty\}.$$

Then $\mathcal{D}$ is a dense linear subspace of $\mathcal{H}$. The operator H, defined by

$$Hf = \lim_{\substack{a\to-\infty,\\ b\to+\infty}} \int_a^b \lambda dE(\lambda)f, \quad f \in \mathcal{D} = \mathcal{D}(H),$$

is selfadjoint.

Proof: Proposition II.6.1 shows that H is well defined and that $\mathcal{D}$ is a linear subspace of $\mathcal{H}$. Clearly $H:\mathcal{D} \to \mathcal{H}$ is linear. First we show that H is hermitian. Set

$$H_{ab}f = \int_a^b \lambda dE(\lambda)f, \quad f \in \mathcal{H}.$$

Then $(H_{ab}f,g) = \int_a^b \lambda d(E(\lambda)f,g) = \int_a^b \lambda d(f,E(\lambda)g) = (f,H_{ab}g)$. Thus H_{ab} is a bounded everywhere defined hermitian operator. If $g,f \in \mathcal{D}$ we obtain

$$(Hf,g) = \lim_{\substack{a\to-\infty,\\ b\to+\infty}} (H_{ab}f,g),$$

$$= \lim_{\substack{a\to-\infty,\\ b\to+\infty}} (f,H_{ab}g) = (f,Hg).$$

Now we have to show that $\mathcal{D}$ is dense in $\mathcal{H}$. Let $-\infty < a < b < +\infty$. Set $\Delta = [a,b]$. Let $f \in \mathcal{H}$. First we prove that $g = E(\Delta)f$ is in $\mathcal{D}$. We have $E(\lambda)g = E(\lambda)(E(b)-E(a))f$ and consequently

$$E(\lambda)g = \begin{cases} 0, & \lambda \leq a \\ (E(\lambda)-E(a))f, & a \leq \lambda \leq b, \\ (E(b)-E(a))f, & \lambda \geq b. \end{cases}$$

Thus for $c < a < b < d$

$$\int_c^d \lambda \, dE(\lambda)g = \int_a^b \lambda \, dE(\lambda)f = \int_a^b \lambda \, dE(\lambda)g$$

$$= \int_{-\infty}^{+\infty} \lambda \, dE(\lambda)g,$$

and Proposition II.6.1 shows that $g \in \mathcal{D}$. Since $\lim_{a\to-\infty,\, b\to+\infty} (E(b)-E(a))f = f$ it follows that $\mathcal{D}$ is dense in $\mathcal{H}$. We will write now $\mathcal{D}(H)$ instead of $\mathcal{D}$ and so far we know already that H is hermitian. Let $z \in \mathbb{C}$, Im $z \neq 0$. Let again $c < a < b < d$. We take a decomposition of $[c,d]$ of the following form:

$$c = \mu_1 < \mu_2 < \dots < \mu_{k+1} = a < \mu_{k+2} < \dots < \mu_{\tilde{n}+1} =$$

$$b < \mu_{\tilde{n}+2} < \dots < \mu_{\tilde{\tilde{n}}+1} = d.$$

Then $\int_a^b (\lambda-z)dE(\lambda)(\int_c^d \frac{1}{\lambda-z} dE(\lambda)h)$ is the limit of the Riemannian sums

$$\sum_{j=k}^{\tilde{n}} (\tilde{\mu}_j-z)(E(\mu_{j+1})-E(\mu_j))\cdot[\sum_{j=1}^{\tilde{\tilde{n}}} \frac{1}{\tilde{\mu}_j-z}(E(\mu_{j+1})-E(\mu_j))h],$$

$$= \sum_{j=1}^{\tilde{n}} (E(\mu_{j+1})-E(\mu_j))h,$$

$$= (E(b)-E(a))h, h \in \mathcal{H},$$

provided $\max_{1\leq j\leq \tilde{\tilde{n}}} |\mu_{j+1}-\mu_j|$ tends to 0. This furnishes

$$\int_a^b (\lambda-z)dE(\lambda)\left(\int_{-\infty}^{+\infty} \frac{1}{\lambda-z}dE(\lambda)h\right) = (E(b)-E(a))h.$$

If $f\in\mathcal{H}$, $g = \int_{-\infty}^{+\infty} \frac{1}{\lambda-z}dE(\lambda)f$ we get

$$\int_a^b (\lambda-z)dE(\lambda)g = (E(b)-E(a))f,$$

$$\lim_{\substack{a\to-\infty,\\ b\to+\infty}} \int_a^b (\lambda-z)dE(\lambda)g = f = \lim_{\substack{a\to-\infty,\\ b\to+\infty}} \left[\int_a^b \lambda\, dE(\lambda)g-z(E(b)-E(a))g\right].$$

In particular $\lim_{\substack{a\to-\infty,\\ b\to+\infty}} \int_a^b \lambda\, dE(\lambda)g$ exists and consequently $g\in\mathcal{D}(H)$, $(H-z)g=f$. Thus $\mathcal{R}(H-z)=\mathcal{H}$, $\operatorname{Im} z\neq 0$, and H is selfadjoint. □

The theorem to follow is the spectral theorem for selfadjoint operators.

Theorem II.6.2: Let H be selfadjoint in $\mathcal{H}$ with domain of definition $\mathcal{D}(H)$. Then there is one and only one spectral family $\{E(\lambda)\,|\,\lambda\in\mathbb{R}\}$ such that

$$\text{(II.6.1)}\quad \mathcal{D}(H) = \{f\,|\,f\in\mathcal{H},\ \int_{-\infty}^{+\infty} \lambda^2 d(E(\lambda)f,f) < +\infty\},$$

$$\text{(II.6.2)}\quad Hf = \int_{-\infty}^{+\infty} \lambda\, dE(\lambda)f,\ f\in\mathcal{D}(H).$$

Proof: Let us take the spectral family $\{E(\lambda)\,|\,\lambda\in\mathbb{R}\}$ from Theorem II.5.1. Let again

$-\infty < c < a < b < d < +\infty.$

Let us take a decomposition of [c,d] as in the proof of Theorem II.6.1. Then we get for the Riemannian sums

$$\text{(II.6.3)}\quad \sum_{j=1}^{\tilde{\tilde{n}}} \frac{1}{\tilde{\mu}_j - z}(E(\mu_{j+1})-E(\mu_j))\cdot[\sum_{j=k}^{\tilde{n}} (\tilde{\mu}_j - z)(E(\mu_{j+1})-E(\mu_j))f]$$

$$= (E(b)-E(a))f,\ f \in \mathcal{H},\ \mathrm{Im}\, z \neq 0.$$

Thus

$$(H-z)^{-1}g = E(\Delta)f,\ \Delta = [a,b],\ g = \int_a^b (\lambda - z)dE(\lambda)f,$$

$$E(\Delta)f \in \mathcal{D}(H),\ (H-z)E(\Delta)f = g,$$

$$HE(\Delta)f - zE(\Delta)f = \int_a^b (\lambda - z)dE(\lambda)f$$

$$= \int_a^b \lambda dE(\lambda)f - zE(\Delta)f,$$

$$HE(\Delta)f = \int_a^b \lambda dE(\lambda)f,\ f \in \mathcal{H}.$$

If $g \in \mathcal{D}(H)$ we obtain

$$(HE(\Delta)f,g) = \int_a^b \lambda d(E(\lambda)f,g),$$

$$= \int_a^b \lambda d(f,E(\lambda)g),$$

$$= (f,\int_a^b \lambda dE(\lambda)g),$$

$$E(\Delta)Hg = \int_a^b \lambda dE(\lambda)g.$$

Let us take $\Delta = \Delta_n = [-n,+n]$, $n \in \mathbb{N}$. Since $\lim_{n\to\infty} E(\Delta_n)Hg = Hg$ we obtain that

$$\lim_{n\to\infty} \int_{-n}^{+n} \lambda dE(\lambda)g = Hg.$$

The general case $a \to -\infty$, $b \to +\infty$ is treated in the same way. According to Proposition II.6.1 this implies that

$$\int_{-\infty}^{+\infty} \lambda^2 d(E(\lambda)g,g) < +\infty.$$

Let in turn now $f \in \mathcal{H}$ and

$$\int_{-\infty}^{+\infty} \lambda^2 d(E(\lambda)f,f) < +\infty.$$

As just proved we have then $E(\Delta_n)f \in \mathcal{D}(H)$,

$$HE(\Delta_n)f = \int_{-n}^{+n} \lambda dE(\lambda)f.$$

Since by Proposition II.6.1 $\lim_{n\to\infty} \int_{-n}^{+n} \lambda dE(\lambda)f = \int_{-\infty}^{+\infty} \lambda dE(\lambda)f$ and since $E(\Delta_n)f \to f$, $n \to \infty$, the closedness of H implies $f \in \mathcal{D}(H)$ and

$$Hf = \int_{-\infty}^{+\infty} \lambda dE(\lambda)f.$$

In the last part of the proof we have to show that the spectral family $\{E(\lambda) \mid \lambda \in \mathbb{R}\}$ is determined uniquely. Taking again the Riemannian sums (II.6.3) we obtain

$$\int_{-\infty}^{+\infty} \frac{dE(\lambda)}{\lambda - z} \int_a^b (\lambda - z) dE(\lambda)f = (E(b)-E(a))f,$$

$$f \in \mathcal{H}, \ \mathrm{Im}\, z \neq 0.$$

If $f \in \mathcal{D}(H)$, then $\int_{-\infty}^{+\infty} (\lambda - z)dE(\lambda)f$ exists and

$$\int_{-\infty}^{+\infty} \frac{dE(\lambda)}{\lambda - z} g = f \text{ with } g = \int_{-\infty}^{+\infty} (\lambda - z) dE(\lambda) f,$$

i.e. $g = (H-z)f$. In particular

$$f = (H-z)^{-1} g = \int_{-\infty}^{+\infty} \frac{dE(\lambda) g}{\lambda - z}.$$

This formula, however, holds for any spectral family $\{\tilde{E}(\lambda) \mid \lambda \in \mathbb{R}\}$ having the properties (II.6.1), (II.6.2) in the present theorem. Theorem II.5.1 completes the proof. □

§ 7. The Spectrum of a Selfadjoint Operator

We remind the reader to the Definition II.1.1 of the spectrum of an operator in a Hilbert space. If this operator is selfadjoint we know already that its spectrum is contained in $\mathbb{R}$. In the sequel we want to give a more precise description of the spectrum $S(H)$ of a selfadjoint operator H in $\mathcal{H}$ with domain of definition $\mathcal{D}(H)$. $\mathcal{H}$ is assumed to be separable and to have infinite dimension. Obviously $S(H) \neq \emptyset$.

Definition II.7.1: Let H be selfadjoint in $\mathcal{H}$ with domain of definition $\mathcal{D}(H)$. Let $\{E(\lambda) \mid \lambda \in \mathbb{R}\}$ the uniquely determined spectral family which belongs to H according to Theorem II.6.2. Let $\Delta = [a,b]$ for some a,b with $-\infty < a < b < +\infty$. Then

$$\mathfrak{M}(\Delta) = E(\Delta)\mathcal{H} = (E(b)-E(a))\mathcal{H}$$

is called the spectral space belonging to Δ.

If $\Delta \subseteq \Delta'$ then $\mathfrak{M}(\Delta) \subseteq \mathfrak{M}(\Delta')$.

Theorem II.7.1: Let H be selfadjoint in $\mathcal{H}$ with domain of definition $\mathcal{D}(H)$. Let $\lambda_o \in \mathbb{R}$. Then $\lambda_o \in S(H)$ if and only if

$$+\infty \geq \dim \mathfrak{M}(\Delta) > 0$$

for any $\Delta = [a,b]$ with $\lambda_o \in (a,b)$.

Proof: First we assume that $+\infty \geq \dim \quad (\Delta) > 0$ for any Δ with $\lambda_o \in \mathring{\Delta}$. Then there is a $\varphi \in \quad (\Delta)$ with $\|\varphi\| = 1$. It is easily seen that

$$(H-\lambda_o)\varphi = \int_a^b (\lambda-\lambda_o)dE(\lambda)\varphi$$

(cf. the proof of Theorem II.6.2). If we choose $\Delta = [\lambda_o-\varepsilon,\lambda_o+\varepsilon]$ for some $\varepsilon > 0$, we get

$$\|(H-\lambda_o)\varphi\|^2 = \int_{\lambda_o-\varepsilon}^{\lambda_o+\varepsilon} (\lambda-\lambda_o)^2 d(E(\lambda)\varphi,\varphi)$$

$$\leq \varepsilon^2.$$

Thus, for any $\varepsilon > 0$, there is a $\varphi_\varepsilon \in \mathcal{D}(H)$ with $\|\varphi_\varepsilon\| = 1$ and $\|(H-\lambda_o)\varphi_\varepsilon\| \leq \varepsilon$. The assumption $\lambda_o \in \Sigma(H)$ then contradicts Theorem II.1.2 and it follows: $\lambda_o \in S(H)$. In the second part of the proof we assume that

$$\dim \quad ([\lambda_o-\varepsilon,\lambda_o+\varepsilon]) = 0$$

for some $\varepsilon > 0$. Let $f \in \mathcal{D}(H)$. Then

$$(H-\lambda_o)f = \int_{-\infty}^{+\infty} (\lambda-\lambda_o) dE(\lambda)f,$$

$$\|(H-\lambda_o)f\|^2 = \int_{-\infty}^{+\infty} (\lambda-\lambda_o)^2 d(E(\lambda)f,f).$$

Let $\lambda_o-\varepsilon \leq \lambda_1 < \lambda_2 \leq \lambda_o+\varepsilon$. Let us assume that $\|E(\lambda_2)f_o\|^2 - \|E(\lambda_1)f_o\|^2 > 0$ for some $f_o \in \mathcal{H}$. This means that $\|(E(\lambda_2)-E(\lambda_1))f_o\|^2 > 0$ and that $g = (E(\lambda_2)-E(\lambda_1))f_o \neq 0$. Since $g \in \quad ([\lambda_o-\varepsilon,\lambda_o+\varepsilon])$ this is a contradiction to our assumption. Thus we have

$$\|(H-\lambda_o)f\|^2 = \int_{-\infty}^{\lambda_o-\varepsilon} (\lambda-\lambda_o)^2 d(E(\lambda)f,f) + \int_{\lambda_o+\varepsilon}^{+\infty} (\lambda-\lambda_o)^2 d(E(\lambda)f,f),$$

$$\geq \varepsilon^2 \left(\int_{-\infty}^{\lambda_o-\varepsilon} d(E(\lambda)f,f) + \int_{\lambda_o+\varepsilon}^{+\infty} d(E(\lambda)f,f)\right),$$

$$= \varepsilon^2 \int_{-\infty}^{+\infty} d(E(\lambda)f,f),$$

$$= \varepsilon^2 \| f \|^2.$$

From Theorem II.1.2 it follows that $\lambda_o \in \Sigma(H)$. Our theorem is proved. □

Definition II.7.2: Let T be a linear operator in a Hilbert space $\mathcal{H}$ with domain of definition $\mathcal{D}(T)$. A complex number λ_o is called eigenvalue of T if there is a $\varphi \neq 0$, $\varphi \in \mathcal{D}(T)$, with $T\varphi = \lambda_o \varphi$.

The eigenvalues of a selfadjoint operator are characterized as follows:

Theorem II.7.2: Let H be selfadjoint in $\mathcal{H}$ with domain of definition $\mathcal{D}(\mathcal{H})$. All eigenvalues of H are real. λ_o $\mathbb{R}$ is an eigenvalue of H if and only if $E(.)x$ is not for every $x \in \mathcal{H}$ continuous from the left in λ_o.

Theorem II.7.2 can be reformulated in the following way. As it was pointed out in the beginning of § 3, the limit

$$\lim_{\substack{\varepsilon \to 0, \\ \varepsilon > 0}} (E(\lambda-\varepsilon)x,y), \quad x,y \in \mathcal{H},$$

exists. It follows that

$$\lim_{\substack{\varepsilon \to 0, \\ \varepsilon > 0}} E(\lambda-\varepsilon)x, \quad x \in \mathcal{H},$$

exists. We call it $E(\lambda-0)x$. Thus $E(\lambda-0) \in L(\mathcal{H},\mathcal{H})$, $\| E(\lambda-0) \|^2 \leq 1$, $E^2(\lambda-0) = E(\lambda-0)$, and $E(\lambda-0)$ is hermitian. The criterion in Theorem II.7.2 can now be written as

$$E(\lambda_o) - E(\lambda_o - 0) \neq 0.$$

Proof of Theorem II.7.2: Let first $E(\lambda_o)-E(\lambda_o-0) \neq 0$. Then there is an $f_o \in \mathcal{H}$ with

$$(E(\lambda_o)-E(\lambda_o-0))f_o = g_o \neq 0.$$

Let $\varepsilon', \varepsilon'' > 0$. We consider $(E(\lambda_o+\varepsilon'')-E(\lambda_o-\varepsilon'))g_o$. If $0 < \delta \leqq \varepsilon', \varepsilon''$ we have

$$(E(\lambda_o+\varepsilon'')-E(\lambda_o-\varepsilon'))(E(\lambda_o+\delta)-E(\lambda_o-\delta)) =$$

$$= E(\lambda_o+\delta) - E(\lambda_o-\delta).$$

Thus

$$(E(\lambda_o+\varepsilon'')-E(\lambda_o-\varepsilon'))g_o$$

$$= (E(\lambda_o+\varepsilon'')-E(\lambda_o-\varepsilon'))\cdot\lim_{\substack{\delta\to 0,\\ \delta>0}} (E(\lambda_o+\delta)-E(\lambda_o-\delta))f_o,$$

$$= \lim_{\substack{\delta\to 0,\\ \delta>0}} (E(\lambda_o+\delta)-E(\lambda_o-\delta))f_o$$

$$= g_o,$$

$$(H-\lambda_o)g_o = \int_{-\infty}^{+\infty} (\lambda-\lambda_o)dE(\lambda)g_o,$$

$$= \int_{\lambda_o-\varepsilon'}^{\lambda_o+\varepsilon''} (\lambda-\lambda_o)dE(\lambda)g_o,$$

$$\|(H-\lambda_o)g_o\|^2 = \int_{\lambda_o-\varepsilon'}^{\lambda_o+\varepsilon''} (\lambda-\lambda_o)^2 d(E(\lambda)g_o,g_o)$$

$$\leqq \max(\varepsilon'^2,\varepsilon''^2)\|g_o\|^2,$$

$$Hg_o = \lambda_o g_o.$$

Since $g_o \neq 0$ we have shown that λ_o is an eigenvalue. Secondly let us assume that λ_o is an eigenvalue of H and that φ_o is an element of $\mathcal{D}(H)$ with $\varphi_o \neq 0$, $H\varphi_o = \lambda\varphi_o$. Then

$$0 = \|(H-\lambda_o)\varphi_o\|^2 = \int_{-\infty}^{+\infty} (\lambda-\lambda_o)^2 d(E(\lambda)\varphi_o,\varphi_o).$$

If $\Delta = [a,b]$ is chosen in such a way that $\lambda_o \notin \Delta$, then

$$0 \geqq \int_a^b (\lambda-\lambda_o)^2 d(E(\lambda)\varphi_o,\varphi_o),$$

$$\geqq \mathrm{dist}^2(\lambda_o,\Delta) \int_a^b d(E(\lambda)\varphi_o,\varphi_o),$$

$$= \mathrm{dist}^2(\lambda_o,\Delta)\|E(\Delta)\varphi_o\|^2.$$

Varying Δ we see that $E(\lambda)\varphi_o$ is constant if either $\lambda > \lambda_o$ or $\lambda < \lambda_o$. Since $\lim_{\lambda\to-\infty} E(\lambda)\varphi_o = 0$ we get $E(\lambda)\varphi_o = 0$, $\lambda < \lambda_o$. Since $\lim_{\lambda\to+\infty} E(\lambda)\varphi_o = \varphi_o$ and since $E(\lambda)\varphi_o$ is continuous from the right we arrive at $E(\lambda)\varphi_o = \varphi_o$, $\lambda \geqq \lambda_o$. In particular

$$E(\lambda_o)-E(\lambda_o-0) \neq 0,$$

and our theorem is proved. □

<u>Theorem II.7.3:</u> <u>Let</u> H <u>be selfadjoint in</u> $\mathcal{H}$ <u>with domain of definition</u> $\mathcal{D}(H)$. <u>Let</u> $\Delta = [a,b]$ <u>with</u> $-\infty < a < b < +\infty$. <u>Let the dimension of</u> $\mathfrak{M}(\Delta) = E(\Delta)\mathcal{H}$ <u>be a finite number, say</u> m <u>with</u> $m > 0$. <u>Then there are</u> m <u>pairwise orthonormal eigenvectors</u> $\varphi_1,\dots,\varphi_m$ <u>to</u> H <u>with eigenvalues</u> $\lambda_1,\dots,\lambda_m$, <u>i.e.</u> $H\varphi_i = \lambda_i\varphi_i$, $1 \leqq i \leqq m$. <u>Moreover, the</u> $\varphi_1,\dots,\varphi_m$ <u>span</u> $\mathfrak{M}(\Delta)$, <u>and for the eigenvalues</u> $\lambda_1,\dots,\lambda_m$ <u>the inequality</u>

$$a < \lambda_i \leqq b, \quad i = 1,\dots,m,$$

<u>holds</u>.

Proof: Let $f \in \mathfrak{M}(\Delta)$. Then

$$Hf = HE(\Delta)f = \int_a^b \lambda\, dE(\lambda)f$$

$$= E(\Delta)Hf$$

(cf. the proof of Theorem II.6.2). Thus $H\mathfrak{M}(\Delta) \subset \mathfrak{M}(\Delta)$ and $\mathfrak{M}(\Delta)$ is an "invariant subspace under H". The restriction of H to $\mathfrak{M}(\Delta)$ can thus be considered as a (bounded) linear hermitian mapping from $\mathfrak{M}(\Delta)$ into itself. Consequently $\mathfrak{M}(\Delta)$ has an orthonormal basis $\{\varphi_1,\dots,\varphi_m\}$ with

$$H\varphi_i = \lambda_i\varphi_i, \quad 1 \leq i \leq m,$$

and real numbers $\lambda_1,\dots,\lambda_m$. We have

$$(H\varphi_i,\varphi_i) = \lambda_i = \int_a^b \lambda\, d(E(\lambda)\varphi_i,\varphi_i),$$

$$a \leq \lambda_i \leq b.$$

Let us assume that one of the numbers $\lambda_1,\dots,\lambda_m$ equals a, say λ_1. Then $(E(\lambda_1)-E(\lambda_1-0))\varphi_1 = (E(a)-E(a-0))\varphi_1 \neq 0$ as was shown in the second part of the proof of Theorem II.7.2. On the other hand, since $\varphi_1 = (E(b)-E(a))\varphi_1$, we get

$$(E(a)-E(a-0))\varphi_1 = 0,$$

which is a contradiction. The theorem is proved. □

In the definition to follow we decompose the spectrum of a selfadjoint operator.

Definition II.7.3: Let H be a selfadjoint operator in $\mathcal{H}$ with domain of definition $\mathcal{D}(H)$. Let $\{E(\lambda)\,|\,\lambda \in \mathbb{R}\}$ be the spectral family belonging H. A real number λ_o belongs to the essential spectrum $S_e(H)$ of H if and only if the subspace $\mathfrak{M}(\Delta) = E(\Delta)\mathcal{H}$ has infinite dimension for any compact interval Δ with $\lambda_o \in \overset{o}{\Delta}$.

Definition II.7.4: Let H be as in the preceding definition. A real number λ_o belongs to the discrete part $S_d(H)$ of the spectrum of H if and only if there is a compact interval Δ with $\lambda_o \in \overset{o}{\Delta}$ and

$$0 < \dim \mathfrak{M}(\Delta) < +\infty,$$

$$1 \leq \dim \mathfrak{M}([\lambda_o-\varepsilon,\lambda_o+\varepsilon])$$

for all $\varepsilon > 0$ with $[\lambda_o-\varepsilon,\lambda_o+\varepsilon] \subseteq \Delta$.

Definition II.7.5: We say that $+\infty$ belongs to the essential spectrum of a selfadjoint operator H in $\mathcal{H}$ as in the preceding definitions if and only if the spaces

$$(I-E(N))\mathcal{H}, \quad N \in \mathbb{N},$$

have infinite dimension. We say that $-\infty$ belongs to the essential spectrum of H if and only if the spaces

$$E(-N)\mathcal{H}, \quad N \in \mathbb{N}$$

have infinite dimension.

Theorem II.7.1 shows that $S_d(H) \subset S(H)$. Then

$$S(H) \cup \{\pm\infty\} = (S_e(H) \cup \{\pm\infty\}) \cup S_d(H),$$

$$S_e(H) \cap S_d(H) = \emptyset.$$

Theorem II.7.4: Let H be a selfadjoint operator in $\mathcal{H}$ with domain of definition $\mathcal{D}(H)$. Let $\lambda_o \in S_d(H)$. Then λ_o is an eigenvalue of H and an isolated point of S(H), i.e. there is an $\varepsilon > 0$ such that $[\lambda_o-\varepsilon, \lambda_o+\varepsilon] \cap S(H) = \{\lambda_o\}$.

Proof: Let $\lambda_o \in S_d(H)$. Set

$$V = \bigcap_{\substack{n \geq n_o,\\ n \in \mathbb{N}}} \mathfrak{M}([\lambda_o-\tfrac{1}{n}, \lambda_o+\tfrac{1}{n}])$$

with n_o sufficiently large. The sequence $\{\dim \;\; ([\lambda-\frac{1}{n}, \lambda_o+\frac{1}{n}])\}$ assumes the value $\min\{\dim \mathfrak{M}([\lambda_o-\frac{1}{n}, \lambda_o+\frac{1}{n}]) \mid n \geqq n_o\} \geqq 1$ infinitely many times. Thus V has finite dimension $\geqq 1$, and in particular

$$V = \mathfrak{M}([\lambda_o-\tfrac{1}{n}, \lambda_o+\tfrac{1}{n}]), \; n \geqq n_1.$$

Consequently there is a $g_o \in V-\{0\}$ and there are $f_n \in \mathcal{H}$, $n \geqq n_1$, with

$$(E(\lambda_o+\tfrac{1}{n})-E(\lambda_o-\tfrac{1}{n}))f_n = g_o, \; n \geqq n_1.$$

If $\varepsilon'', \varepsilon' > 0$ we have

$$(E(\lambda_o+\varepsilon'')-E(\lambda_o-\varepsilon'))(E(\lambda_o+\tfrac{1}{n})-E(\lambda_o-\tfrac{1}{n}))f_n =$$

$$= (E(\lambda_o+\varepsilon'')-E(\lambda_o-\varepsilon'))g_o,$$

$$= (E(\lambda_o+\tfrac{1}{n})-E(\lambda_o-\tfrac{1}{n}))f_n$$

$$= g_o, \; n \geqq n_2(\varepsilon'', \varepsilon').$$

Now it is easily seen as in the proof of Theorem II.7.2 that $Hg_o = \lambda_o g_o$. Thus λ_o is an eigenvalue of H. Let us take the interval Δ of Definition II.7.4 with $\lambda_o \in \overset{o}{\Delta}$. We set $\Delta = [a,b]$.

Let $\mu \in S(H) \cap (a,b)$. As before it is possible to prove that μ is an eigenvalue of H. Evidently μ is also an eigenvalue of the restriction of H to $\mathfrak{M}(\Delta)$: The latter is an "invariant subspace under H" as was shown in the proof of Theorem II.7.3. The eigenvalues of this restriction consist of $\lambda_1,\ldots,\lambda_m$ and have been constructed in Theorem II.7.3. Thus $\mu \in \{\lambda_1,\ldots,\lambda_m\}$, $a < \mu < b$, and our theorem is proved. □

The proof of Theorem II.7.4 not only shows that any $\lambda_o \in S_d(H)$ is an eigenvalue of H and an isolated point of S(H) but also that λ_o has finite multiplicity, i.e. the vector space $\tilde{V} = \{g \mid g \in \mathcal{D}(H),\ Hg = \lambda_o g\}$ is a finite dimensional (and therefore closed) subspace of $\mathcal{H}$: As in the proof of Theorem II.7.2 we can show that

$$E(\Delta)g = g,\ g \in \tilde{V},$$

provided λ_o is contained in the open kernel $\overset{o}{\Delta}$ of the compact interval Δ. Thus

$$\tilde{V} \subset E(\Delta)\mathcal{H}.$$

Next we prove the criterion of H. Weyl concerning the essential spectrum of H.

<u>Theorem II.7.5:</u> <u>Let</u> H <u>be a selfadjoint operator in</u> $\mathcal{H}$ <u>with domain of definition</u> $\mathcal{D}(H)$. <u>A real number</u> λ_o <u>belongs to the essential spectrum</u> $S_e(H)$ <u>of</u> H <u>if and only if there is a sequence</u> $\{\varphi_i\}$ <u>of elements</u> $\varphi_i \in \mathcal{D}(H)$ <u>with the following properties</u>:

$$\|\varphi_i\| = 1, \; i \in \mathbb{N},$$

$$\varphi_i \rightharpoonup 0, \; i \to \infty,$$

$$(H-\lambda_o)\varphi_i \to 0, \; i \to \infty.$$

<u>Proof:</u> First we assume that $\lambda_o \in S_e(H)$. Set

$$\Delta_n = [\lambda_o - \tfrac{1}{n}, \lambda_o + \tfrac{1}{n}], \; n \in \mathbb{N}.$$

Then dim $\mathfrak{M}(\Delta_n) = +\infty$. Therefore there is a sequence $\{\varphi_i\}$ with

$$\varphi_i \in \mathfrak{M}(\Delta_i), \; i \in \mathbb{N},$$

$$\|\varphi_i\| = 1, \; i \in \mathbb{N},$$

$$(\varphi_i, \varphi_k) = 0, \; i \neq k,$$

$$\varphi_i \rightharpoonup 0, \; i \to \infty.$$ [1]

Then

$$\|(H-\lambda_o)\varphi_i\|^2 = \int_{-\infty}^{+\infty} (\lambda-\lambda_o)^2 d(E(\lambda)\varphi_i, \varphi_i),$$

$$= \int_{\lambda_o-\frac{1}{i}}^{\lambda_o+\frac{1}{i}} (\lambda-\lambda_o)^2 d(E(\lambda)\varphi_i, \varphi_i),$$

$$\leq \frac{1}{i^2},$$

$$(H-\lambda_o)\varphi_i \to 0, \; i \to \infty.$$

---

[1] c.f. the remark after this proof.

Secondly we assume that the criterion of the present theorem is fulfilled. If $\lambda_o \notin S_e(H)$ then there is an $\varepsilon > 0$ such that

$$\dim \mathfrak{M}([\lambda_o - \varepsilon, \lambda_o + \varepsilon]) < +\infty.$$

We have

$$\|(H-\lambda_o)\varphi_i\|^2 \geq$$

$$\geq \int_{-\infty}^{\lambda_o - \varepsilon} (\lambda-\lambda_o)^2 d(E(\lambda)\varphi_i,\varphi_i) + \int_{\lambda_o+\varepsilon}^{+\infty} (\lambda-\lambda_o)^2 d(E(\lambda)\varphi_i,\varphi_i)$$

$$\geq \varepsilon^2 \left(\int_{-\infty}^{\lambda_o - \varepsilon} d(E(\lambda)\varphi_i,\varphi_i) + \int_{\lambda_o+\varepsilon}^{+\infty} d(E(\lambda)\varphi_i,\varphi_i)\right)$$

$$= \varepsilon^2 (\|E(\lambda_o-\varepsilon)\varphi_i\|^2 + \|(I-E(\lambda_o+\varepsilon))\varphi_i\|^2),$$

$$= \varepsilon^2 (\|E(\lambda_o-\varepsilon)\varphi_i\|^2 + \|\varphi_i\|^2 - \|E(\lambda_o+\varepsilon)\varphi_i\|^2),$$

$$= \varepsilon^2 (1-(\|E(\lambda_o+\varepsilon)\varphi_i\|^2 - \|E(\lambda_o-\varepsilon)\varphi_i\|^2)),$$

$$= \varepsilon^2 (1-\|(E(\lambda_o+\varepsilon)-E(\lambda_o-\varepsilon))\varphi_i\|^2).$$

The operator $E(\lambda_o+\varepsilon)-E(\lambda_o-\varepsilon)$ is the projection of H onto a finite dimensional subspace of H and therefore completely continuous. From $\varphi_i \to 0$, $i \to \infty$ we thus infer that

$$(E(\lambda_o+\varepsilon)-E(\lambda_o-\varepsilon))\varphi_i \to 0, \ i \to \infty.$$

If $i \geq i_o$ we obtain

$$\|(H-\lambda_o)\varphi_i\| \geq \frac{\varepsilon}{2}$$

which is a contradiction to our assumption. Our theorem is proved. □

We want to make a remark concerning the construction of the orthonormal sequence in the first part of the preceding proof. If we set $\Delta_n^{(1)} = [\lambda_o - \frac{1}{n}, \lambda_o - \frac{1}{n+1}]$, $\Delta_n^{(2)} = [\lambda_o + \frac{1}{n+1}, \lambda_o + \frac{1}{n}]$, then

$$\Delta_n - \Delta_{n+1} = \Delta_n^{(1)} \cup \Delta_n^{(2)} .$$

We distinguish two cases. First we assume that there is a sequence $\{n_j\}$ of indices with $\mathfrak{M}(\Delta_{n_j}^{(1)}) \neq \{0\}$ or $\mathfrak{M}(\Delta_{n_j}^{(2)}) \neq \{0\}$. Then we choose a $\varphi_{n_j} \neq 0$ in $\mathfrak{M}(\Delta_{n_j}^{(1)})$ or in $\mathfrak{M}(\Delta_{n_j}^{(2)})$. We can assume that $\|\varphi_{n_j}\| = 1$. Since $\mathfrak{M}(\Delta_{n_j}^{(i)})$ and $\mathfrak{M}(\Delta_{n_k}^{(l)})$ are pairwise orthogonal if $i \neq l$ or $j \neq k$ it follows that the φ_{n_j} are pairwise orthogonal. Bessel's inequality

$$\|f\|^2 \geq \sum_{j=1}^{\infty} |(f, \varphi_{n_j})|^2$$

shows that $\varphi_{n_j} \rightharpoonup 0$, $j \to \infty$. Secondly we have the possibility that

$$\mathfrak{M}(\Delta_n^{(1)}) = \mathfrak{M}(\Delta_n^{(2)}) = \{0\}, \; n \geq n_o .$$

Then $\mathfrak{M}(\Delta_{n_o}) = \mathfrak{M}(\Delta_{n_o+1}) = \dots$. We choose a complete orthonormal system in $\mathfrak{M}(\Delta_{n_o})$, say $\{\varphi_{n_o}, \varphi_{n_o+1}, \dots\}$. Then $\varphi_i \in \mathfrak{M}(\Delta_i)$, $i = n_o, n_o+1, \dots$, and again Bessel's inequality gives $\varphi_i \rightharpoonup 0$, $i \to \infty$.

<u>Proposition II.7.1:</u> <u>Let</u> H <u>be selfadjoint in</u> $\mathcal{H}$ <u>with domains of definition</u> $\mathcal{D}(H)$. <u>Let</u> $\lambda_o \in \mathbb{R}$ <u>and let</u> λ_o <u>be an accumulation point of</u> $S(H)$. <u>Then</u> $\lambda_o \in S_e(H)$.

Proof: We choose a sequence $\{\lambda_n\}$ of pairwise distinct numbers $\lambda_n \in S(H)$ with $\lambda_n \to \lambda_o$, $n \to \infty$. To each λ_n we assign an interval $\Delta_n = [a_n, b_n]$ such that $\lambda_n \in \overset{o}{\Delta}_n$, $\overset{o}{\Delta}_n \cap \overset{o}{\Delta}_m = \emptyset$, $n \neq m$. Then any interval $\Delta = [a,b]$ with $\lambda_o \in \overset{o}{\Delta}$ contains infinitely many intervals Δ_n, say $\Delta_{n_1}, \Delta_{n_2}, \ldots$. We have

$$\mathfrak{M}(\Delta) \supset \bigcup_{j=1}^{\infty} \mathfrak{M}(\Delta_{n_j})$$

with $\dim \mathfrak{M}(\Delta_{n_j}) \geq 1$,

$$\mathfrak{M}(\Delta_{n_j}) \perp \mathfrak{M}(\Delta_{n_k}), \; j \neq k.$$

Thus $\dim \mathfrak{M}(\Delta) = +\infty$. □

Proposition II.7.2: Let H be selfadjoint in $\mathcal{H}$ with domain of definition $\mathcal{D}(H)$. Let $\Delta = [a,b] \subset \Sigma(H)$. Then

$$E(b) = E(\lambda) = E(a), \; a \leq \lambda \leq b.$$

Proof: According to Theorem II.7.1 for any $\lambda \in [a,b]$ there is an $\Delta_\lambda = [a_\lambda, b_\lambda]$ with

$$\lambda \in (a_\lambda, b_\lambda),$$

$$\mathfrak{M}(\Delta_\lambda) = \{0\}.$$

Since

$$[a,b] \subset \bigcup_{\lambda \in [a,b]} (a_\lambda, b_\lambda),$$

the compactness of $[a,b]$ implies

$$[a,b] \subset \bigcup_{j=1}^{N} (a_{\lambda_j}, b_{\lambda_j}).$$

Without loss of generality we can assume that $a \in (a_{\lambda_1}, b_{\lambda_1})$. Since $\mathfrak{M}(\Delta_{\lambda_1}) = \{0\}$ we have

$$E(\lambda) = E(a_{\lambda_1}) = E(b_{\lambda_1}), \quad a_{\lambda_1} \leq \lambda \leq b_{\lambda_1}.$$

If $a_{\lambda_1} < a_{\lambda_2} < b_{\lambda_1} < b_{\lambda_2}$ we also get

$$\begin{aligned} E(\lambda) &= E(a_{\lambda_1}) = E(b_{\lambda_1}), \\ &= E(a_{\lambda_2}) = E(b_{\lambda_2}), \quad a_{\lambda_1} \leq \lambda \leq b_{\lambda_2}, \end{aligned}$$

and so on. □

<u>Proposition II.7.3:</u> <u>Let</u> H <u>be a selfadjoint operator in</u> $\mathcal{H}$ <u>with domain of definition</u> $\mathcal{D}(H)$. <u>Let</u> $\lambda_o \in \mathbb{R}$ <u>be an eigenvalue of</u> H. <u>Then</u>

$$\{\varphi \mid \varphi \in \mathcal{D}(H),\ H\varphi = \lambda_o \varphi\} = (E(\lambda_o) - E(\lambda_o - 0))\mathcal{H}.$$

<u>Proof:</u> The second part of the proof of Theorem II.7.2 shows that

$$\{\varphi \mid \varphi \in \mathcal{D}(H),\ H\varphi = \lambda_o \varphi\} \subset (E(\lambda_o) - E(\lambda_o - 0))\mathcal{H}.$$

If on the other hand

$$g = (E(\lambda_o) - E(\lambda_o - 0))f,$$

where f is any element from $\mathcal{H}$, then the first part of the proof of Theorem II.7.2 yields $Hg = \lambda_o g$. □

<u>Proposition II.7.4:</u> <u>Let $\mathcal{H}$ be a bounded hermitian operator in</u> $\mathcal{H}$ <u>with domain of definition</u> $\mathcal{D}(H) = \mathcal{H}$. <u>Then</u> H <u>is selfadjoint</u>, <u>and if</u>

$N > \|H\|$,

then

$$H = \int_{-N}^{+N} \lambda dE(\lambda),$$

where $\{E(\lambda) | \lambda \in \mathbb{R}\}$ is the accompanying spectral family and where the integral is taken in $L(\mathcal{H},\mathcal{H})$.

Proof: We have

$$\|(H+\lambda)f\| \geqq |\lambda| \|f\| - N\|f\|,$$
$$\geqq (|\lambda| - N)\|f\|.$$

Clearly H is selfadjoint. Thus, by Theorem II.1.2, we see that $\{\lambda | \lambda > N\} \subset \Sigma(H)$. The continuity from the right of $E(\lambda)x$, $x \in \mathcal{H}$, furnishes, together with Proposition II.7.2, that $E(\lambda) = I$, $\lambda \geqq N$. Again by Theorem II.1.2 it follows that $\{\lambda | \lambda < -N\} \subset \Sigma(H)$ and, by Proposition II.7.2, that $E(\lambda) = 0$, $\lambda < -N$. Replacing N by $N-\varepsilon$ with $N-\varepsilon > \|H\|$ and a sufficiently small $\varepsilon > 0$ the previous arguments show that $E(\lambda) = 0$, $\lambda \leqq -(N-\frac{\varepsilon}{2})$, $E(\lambda) = I$, $\lambda \geqq N-\varepsilon$. This proves the proposition in question. □

Now we can characterize compact hermitian operators in terms of its spectra.

Theorem II.7.6: Let H be an hermitian bounded operator in $\mathcal{H}$ with domain of definition $\mathcal{D}(H) = \mathcal{H}$. Then H is compact if and only if

$$S_e(H) = \{0\}.$$

Proof: Let us assume that H is compact, let $\lambda_o \in S_e(H) \cap \mathbb{R}$. According to Theorem II.7.5 there is a sequence $\{\varphi_i\}$ of elements of H with $\|\varphi_i\| = 1$, $\varphi_i \rightharpoonup 0$, $i \to \infty$, $\|(H-\lambda_o)\varphi_i\| \to 0$, $i \to \infty$. Since H is compact we have $\|H\varphi_i\| \to 0$, $i \to \infty$. This implies $\lambda_o = 0$. The proof of Proposition II.7.4 shows that $\pm\infty \notin S_e(H)$. Now we assume that $\{0\} = S_e(H)$. Let $N > \|H\|$. Then by Proposition II.7.1 we have

(II.7.1) $\dim \mathfrak{M}([-N,-\frac{1}{n}]) < +\infty$,

(II.7.2) $\dim \mathfrak{M}([\frac{1}{n},N]) < +\infty$, $n > \frac{1}{N}$, $n \in \mathbb{N}$.

Let us set $\Delta_n^- = [-N,\frac{1}{n}]$, $\Delta_n^+ = [\frac{1}{n},N]$. The representation

$$Hf = \int_{-N}^{+N} \lambda dE(\lambda)$$

from Proposition II.7.4 infers

$$HE(\Delta_n^+)f = \int_{1/n}^{N} \lambda dE(\lambda)f$$

$$= E(\Delta_n^+)Hf,$$

$$HE(\Delta_n^-)f = \int_{-N}^{-1/n} \lambda dE(\lambda)f$$

$$= E(\Delta_n^-)Hf.$$

Thus

$$\|(E(\Delta_n^+)+E(\Delta_n^-))Hf-Hf\| = \|\int_{-1/n}^{1/n} \lambda dE(\lambda)f\|$$

$$\leq \frac{1}{n}\|f\|,$$

$$\lim_{n\to\infty} \|H-(E(\Delta_n^+)+E(\Delta_n^-))H\| = 0.$$

(II.7.1) and (II.7.2) show that $E(\Delta_n^+)$, $E(\Delta_n^-)$ are compact. Thus H is compact. Finally $S_e(H) \cap \mathbb{R} \neq \emptyset$. Otherwise $\dim \mathcal{H} < +\infty$. □

Corollary to Theorem II.7.6: Let H be a bounded everywhere defined hermitian operator in $\mathcal{H}$. Let H be compact. Then $S_d(H)$ consists of precisely countably many eigenvalues $\lambda_1, \lambda_2, \ldots$ with

$$|\lambda_1| \geqq |\lambda_2| \geqq \ldots > 0$$

and there is a complete orthonormal system $\{\varphi_1, \varphi_2, \ldots, \psi_1, \psi_2, \ldots\}$ in $\mathcal{H}$ such that

$$H\varphi_n = \lambda_n \varphi_n, \; H\psi_n = 0, \; n \in \mathbb{N}.$$

Here the set $\{\psi_1, \psi_2, \ldots\}$ is a complete orthonormal system in the closed subspace $\mathcal{N} = \{z \mid Hz = 0\}$ of $\mathcal{H}$, provided $+\infty \geqq \dim \mathcal{N} \geqq 1$. If $\dim \mathcal{N} = 0$ then the $\{\varphi_1, \varphi_2, \ldots\}$ form a complete orthonormal system in $\mathcal{H}$.

Proof: Let $N > \|H\|$. In $[-N, -\frac{1}{m}]$, $[\frac{1}{m}, N]$, $m \in \mathbb{N}$, $m > \frac{1}{N}$, there are at most finitely many points of S(H) (by Proposition II.7.1), say $\lambda_1, \ldots, \lambda_{k_m}$. We can order them:

$$|\lambda_1| \geqq |\lambda_2| \geqq \ldots \geqq |\lambda_{k_m}|.$$

It's the same with $[-N, -\frac{1}{m+1}]$, $[\frac{1}{m+1}, N]$. The points of S(H) in these two intervals then are $\lambda_1, \ldots, \lambda_{k_m}, \lambda_{k_m+1}, \ldots, \lambda_{k_{m+1}}$ with

$$|\lambda_1| \geqq |\lambda_2| \geqq \ldots \geqq |\lambda_{k_m}| > |\lambda_{k_m+1}| \geqq \ldots \geqq |\lambda_{k_{m+1}}|.$$

In this way we proceed. The points $\lambda_1, \lambda_2, \ldots$ are isolated points of S(H). Thus by Proposition II.7.2

$$E(\lambda_j) = E(\lambda_j + \varepsilon),$$

$$E(\lambda_j - 0) = E(\lambda_j - \varepsilon), \; j = 1, 2, \ldots,$$

provided ε is sufficiently small and > 0. Since $+\infty >$ $\dim([\lambda_j-\varepsilon,\lambda_j+\varepsilon]) \geq 1$ by Theorem II.7.1 we obtain $E(\lambda_j)-E(\lambda_j-0) \neq 0$. By Theorem II.7.2 it is seen that λ_j is an eigenvalue of H. The spaces $(E(\lambda_j)-E(\lambda_j-0))\mathcal{H}$ are pairwise orthogonal and have finite dimension. By Proposition II.7.4 there is finite orthonormal system $\{\varphi_j^{(1)},\ldots,\varphi_j^{(e_j)}\}$ which spans $(E(\lambda_j)-E(\lambda_j-0))\mathcal{H}$ and fulfills $H\varphi_j^{(\mu)} = \lambda_j\varphi_j^{(\mu)}$, $\mu = 1,\ldots,e_j$. We have

$$Hf = \lim_{m\to\infty} \left(\int_{-N}^{-\frac{1}{m}} \lambda dE(\lambda)f + \int_{\frac{1}{m}}^{N} \lambda dE(\lambda)f\right)$$

$$= \lim_{m\to\infty} \sum_{j=1}^{k_m} \lambda_j \sum_{\mu=1}^{e_j} ((E(\lambda_j)-E(\lambda_j-0))f,\varphi_j^{(\mu)})\varphi_j^{(\mu)}$$

$$= \lim_{m\to\infty} \sum_{j=1}^{k_m} \lambda_j \sum_{\mu=1}^{e_j} (f,\varphi_j^{(\mu)})\varphi_j^{(\mu)}.$$

Let us change enumeration as follows: Instead of $\{\lambda_1,\ldots,\lambda_{k_m},$ $\lambda_{k_m+1},\ldots,\lambda_{k_{m+1}},\ldots\}$ we are going to write $\{\lambda_1,\lambda_2,\ldots\}$ and each λ_j appears as often as e_j times. Consequently we write $\{\varphi_1,\varphi_2,\ldots\}$ instead of $\{\varphi_1^{(1)},\ldots,\varphi_1^{(e_1)},\varphi_2^{(1)},\ldots,\varphi_2^{(e_2)},\ldots\}$. Then our last formula reads

$$Hf = \sum_{j=1}^{\infty} \lambda_j(f,\varphi_j)\varphi_j,$$

$$= \sum_{j=1}^{\infty} (Hf,\varphi_j)\varphi_j.$$

Now take a sequence $\{Hf_n\}$ with $Hf_n \to g$, $n\to\infty$. Set $c_j^{(n)} = (f_n,\varphi_j)$.

Then by Parseval's inequality

$$\|H(f_n-f_1)\|^2 = \sum_{j=1}^{\infty} |\lambda_j|^2 |c_j^{(n)}-c_j^{(1)}|^2.$$

Thus $c_j^{(n)} \to d_j$, $n \to \infty$. Since

$$\|Hf_n\|^2 \geqq \sum_{j=1}^{K} |\lambda_j|^2 |c_j^{(n)}|^2,$$

$K \in \mathbb{N}$, we obtain that

$$\sum_{j=1}^{\infty} |\lambda_j|^2 |d_j|^2 < \infty.$$

Setting

$$\tilde{g} = \sum_{j=1}^{\infty} \lambda_j d_j \varphi_j$$

we thus see that $Hf_n \to \tilde{g}$, $n \to \infty$. Consequently $g = \tilde{g}$. Since

$$\lambda_j d_j = \lim_{n\to\infty} \lambda_j c_j^{(n)},$$

$$= \lim_{n\to\infty} (Hf_n, \varphi_j) = (g, \varphi_j)$$

we end up with the expansion

$$g = \sum_{j=1}^{\infty} (g,\varphi_j)\varphi_j$$

if g is in the closure of $\mathcal{R}(H)$ in $\mathcal{H}$.

Finally we choose a complete orthonormal system $\{\psi_1,\psi_2,\ldots\}$ in the closed subspace $\mathcal{N}$ of $\mathcal{H}$. Since for any $\tilde{g} \in \mathcal{H}$ there exist a $z \in \mathcal{N}$ and a $g \in \overline{\mathcal{R}(H)}$ with $\tilde{g} = z+g$ we have shown that $\{\varphi_1,\varphi_2,\ldots,\psi_1,\psi_2,\ldots\}$ is a complete orthonormal system in $\mathcal{H}$. So far we have tacitly assumed that $+\infty \geqq \dim \mathcal{N} \geqq 1$. If $\dim \mathcal{N} = 0$

then the $\{\varphi_1, \varphi_2, \ldots\}$ already form a complete orthonormal system in H. □

For the sake of completeness we briefly touch the behaviour of the spectrum under a compact perturbation of a selfadjoint operator.

Definition II.7.6: Let A and B be two linear operators in H with $\mathcal{D}(A) = \mathcal{D}(B)$. Then the operator B is called compact with respect to A (A-compact) if and only if the following holds: Let (u_k) be a sequence with $u_k \in \mathcal{D}(A)$, $k \in \mathbb{N}$, $\|u_k\| + \|Au_k\| \leq D$ for $k \in \mathbb{N}$ and some $D > 0$. Then there is a subsequence (u_{k_l}) of (u_k) such that

$$\lim_{l\to\infty} Bu_{k_l} \text{ exists.}$$

Theorem II.7.7: Let A be selfadjoint in H with domain of definition $\mathcal{D}(A)$. Let B be an hermitian operator in H with $\mathcal{D}(B) = \mathcal{D}(A)$, which is A-compact. We define C by setting

$$Cu = Au + Bu, \quad u \in \mathcal{D}(A).$$

Let C be selfadjoint. Then

$$S_e(A) \cap \mathbb{R} \subset S_e(C) \cap \mathbb{R}.$$

Proof: Let $\lambda_o \in S_e(A) \cap \mathbb{R}$. According to Theorem II.7.5 there is a sequence $\{\varphi_k\}$ with

$$\|\varphi_k\| = 1, \quad k \in \mathbb{N},$$

$$\varphi_k \in \mathcal{D}(A), \quad k \in \mathbb{N},$$

$\varphi_k \rightharpoonup 0,\ k \to \infty,$

$\|(A-\lambda_o)\varphi_k\| \to 0,\ k \to \infty.$

Firstly we see that $\|\varphi_k\| + \|A\varphi_k\| \le D$, $k \in \mathbb{N}$, for some $D > 0$. Since B is A-compact there is a subsequence (φ_{k_j}) of (φ_k) with $B\varphi_{k_j} \to f$, $j \to \infty$. Let $g \in \mathcal{D}(A)$. Then $(f,g) = \lim_{j\to\infty} (B\varphi_{k_j}, g) = \lim_{j\to\infty} (\varphi_{k_j}, Bg) = 0$. This implies $f = 0$. Now

$$\|C\varphi_{k_j} - \lambda_o\varphi_{k_j}\| \le \|A\varphi_{k_j} - \lambda_o\varphi_{k_j}\| + \|B\varphi_{k_j}\|,$$

and the right hand side tends to 0 if $j \to \infty$. Thus $\lambda_o \in S_e(C)$. □

If A is selfadjoint in $\mathcal{H}$ and has domain of definition $\mathcal{D}(A)$ and if B is bounded hermitian in $\mathcal{H}$ with $\mathcal{D}(B) = \mathcal{H}$, then it is easily seen that the operator C defined by $Cu = Au + Bu$, $u \in \mathcal{D}(C) = \mathcal{D}(A)$, is selfadjoint in $\mathcal{H}$. If $V = B$ is even compact then Theorem II.7.7 shows that

$$\begin{aligned} S_e(A+V) \cap \mathbb{R} &\subset S_e((A+V)-V) \cap \mathbb{R}, \\ &= S_e(A) \cap \mathbb{R}. \end{aligned}$$

On the other hand

$$S_e(A) \cap \mathbb{R} \subset S_e(A+V) \cap \mathbb{R},$$

where we have used again Theorem II.7.7. Thus we end up with

$$\text{(II.7.3)} \quad S_e(A+V) \cap \mathbb{R} = S_e(A) \cap \mathbb{R}$$

for compact V.

We now consider selfadjoint operators having a discrete spectrum. This notion is made more precise in

Definition II.7.7: Let A be selfadjoint in $\mathcal{H}$ with domain of definition $\mathcal{D}(A)$. A is said to have a discrete spectrum if and only if for any compact interval $\Delta = [a,b]$ the inequality

$$\dim \mathfrak{M}(\Delta) < +\infty$$

holds.

Selfadjoint operators with discrete spectrum could be characterized in the following way:

Theorem II.7.8: Let A be selfadjoint in $\mathcal{H}$ with domain of definition $\mathcal{D}(A)$. A has discrete spectrum if and only if $S_e(A) \subset \{-\infty,+\infty\}$. Moreover A has discrete spectrum if and only if $S(A)$ consists of countably many eigenvalues $\lambda_1,\lambda_2,\ldots$ with

$$|\lambda_1| \leq |\lambda_2| \leq \ldots, \lim_{n\to\infty} |\lambda_n| = +\infty,$$

$$1 \leq \dim((E(\lambda_j)-E(\lambda_j-0)) \quad) =: e_j < +\infty.$$

Moreover, then there is an orthonormal system $\{\varphi_1,\varphi_2,\ldots\}$ of elements $\varphi_k \in \mathcal{D}(A)$ such that $A\varphi_k = \lambda_k\varphi_k$, $k \in \mathbb{N}$, and such that the following expansion holds:

$$Af = \sum_{k=1}^{\infty} \lambda_k (f,\varphi_k)\varphi_k, \quad f \in \mathcal{D}(A).$$

Proof: Let A have discrete spectrum. We consider the intervals $\Delta_n = [n,n+1]$, $n \in \mathbb{Z}$. As was pointed out in the proof of Theorem II.7.4 the intersection of $S(A)$ with $(n,n+1]$ consists of at most finitely many eigenvalues $\lambda_1^{(n)},\ldots,\lambda_{k_n}^{(n)}$ with $1 \leq \dim((E(\lambda_j^{(n)})-E(\lambda_j^{(n)}-0))\mathcal{H}) < +\infty$, $j = 1,\ldots,k_n$. We can order them as described in the theorem in question and get that $S(A)$ consists of at most countably many eigenvalues $\lambda_1,\lambda_2,\ldots$ with $|\lambda_1| \leqq |\lambda_2| \leqq \ldots$.

Now let us assume that S(A) is bounded, say $S(A) \subset [-M+\varepsilon, M-\varepsilon]$ for some $\varepsilon > 0$. Then

$$Af = \int_{-M}^{+M} \lambda dE(\lambda) f.$$

$[-M+\varepsilon, M-\varepsilon] \cap S(A)$ contains at most finitely many pairwise distinct eigenvalues, say $\lambda_1, \ldots, \lambda_N$, with $1 \leq e_1, \ldots, e_N < +\infty$. Otherwise $[-M+\varepsilon, M-\varepsilon]$ would contain an accumulation point of eigenvalues. By Proposition II.7.1 this is a point of $S_e(A)$ and by Definition II.7.5 this contradicts our assumption that A has discrete spectrum. As in the proof of the Corollary to Theorem II.7.6 we obtain the expansion

$$Af = \sum_{k=1}^{N} \lambda_k \sum_{\mu=1}^{e_k} (f, \varphi_k^{(\mu)}) \varphi_k^{(\mu)}, \quad f \in \mathcal{D}(A),$$

where the $\varphi_k^{(1)}, \ldots, \varphi_k^{(e_k)}$ are an orthonormal basis of $(E(\lambda_k) - E(\lambda_k - 0))\mathcal{H}$. In particular A admits a bounded hermitian extension to $\mathcal{H}$ and the range of this extension is contained in the finite dimensional subspace being spanned by the $\varphi_k^{(\mu)}$. Since A is self-adjoint this extension coincides with A (cf. Proposition I.3.5) and moreover A is compact. Thus $0 \in S_e(A)$. This contradicts our assumption. Thus S(A) is unbounded and $\lim_{n\to\infty} |\lambda_n| = +\infty$. From the second criterion it immediately follows that $S_e(A) \subset \{-\infty, +\infty\}$. From this inclusion we get in turn that A has discrete spectrum. As for the expansion we have

$$Af = \lim_{M\to\infty} \int_{-M}^{M} \lambda dE(\lambda) f,$$

$$= \lim_{N\to\infty} \sum_{k=1}^{N} \lambda_k \sum_{\mu=1}^{e_k} (f, \varphi_k^{(\mu)}) \varphi_k^{(\mu)}.$$

If in the sequence $\{\lambda_1, \lambda_2, \ldots\}$ each eigenvalue is as often repeated as its multiplicity e_j prescribes we get the expansion of the theorem. □

Theorem II.7.9 (Rellich, Friedrichs): Let A be selfadjoint in H with domain of definition $\mathcal{D}(A)$. A has discrete spectrum if and only if each sequence $(\tilde{\varphi}_k)$ with

$$\tilde{\varphi}_k \in \mathcal{D}(A),\ k \in \mathbb{N},$$

$$\|\tilde{\varphi}_k\|^2 + \|A\tilde{\varphi}_k\|^2 \leq D^2 \text{ for some } D > 0 \text{ and all } k \in \mathbb{N}$$

contains a convergent subsequence.

Proof: Let A have a discrete spectrum. We take the expansion in Theorem II.7.8 and set

$$x_k = (x, \varphi_k),\ x \in H.$$

Then, if $x \in \mathcal{D}(A)$, we get

$$\sum_{k=1}^{\infty} \lambda_k^2 |x_k|^2 = \|Ax\|^2,$$

$$\sum_{k=1}^{\infty} |x_k|^2 \leq \|x\|^2.$$

If $(x^{(p)})$ is a sequence with the properties stated in the theorem we thus get

$$\text{(II.7.4)}\quad \sum_{k=1}^{\infty} (1+\lambda_k^2)|x_k^{(p)}|^2 \leq D^2,\ p \in \mathbb{N};$$

in particular $\|\tilde{x}^{(p)}\|^2 \leq D^2$ for some $D > 0$, where we have set

$$\tilde{x}^{(p)} = \sum_{k=1}^{\infty} x_k^{(p)} \varphi_k.$$

Thus there is a subsequence $(\tilde{x}^{(p_j)})$ of $(\tilde{x}^{(p)})$ such that

$$\sum_{k=1}^{\infty} x_k^{(p_j)} \bar{y}_k \to \sum_{k=1}^{\infty} x_k^* \bar{y}_k$$

if $j \to \infty$; here $\{y_1, y_2, \ldots\}$ is any sequence of complex numbers with

$$\sum_{k=1}^{\infty} |y_k|^2 < +\infty,$$

and $\{x_1^*, x_2^*, \ldots\}$ is some sequence of complex numbers with

$$\sum_{k=1}^{\infty} |x_k^*|^2 < +\infty$$

(cf. the example after Proposition I.3.1). In particular $x_k^{(p_j)} \to x_k^*$, $j \to \infty$. We set

$$x^* = \sum_{k=1}^{\infty} x_k^* \varphi_k$$

and obtain

$$\|\tilde{x}^{(p_j)} - x^*\|^2 \leq \sum_{k=1}^{N} |x_k^{(p_j)} - x_k^*|^2 + 2 \sum_{k=N+1}^{\infty} |x_k^*|^2 + $$

$$+ 2 \sum_{k=N+1}^{\infty} |x_k^{(p_j)}|^2.$$

From (II.7.4) we infer

$$\sum_{k=N+1}^{\infty} |x_k^{(p_j)}|^2 \leq \frac{D^2}{1+\lambda_{N+1}^2}.$$

If ε is any positive number we see now with the aid of Theorem II.7.8 that

$$2 \sum_{k=N+1}^{\infty} (|x_k^*|^2 + |x_k^{(p_j)}|^2) \leq \frac{1}{2}\varepsilon^2$$

if for N is chosen some fixed integer $N(\varepsilon)$. Taking j sufficiently large we get

$$\lim_{j\to\infty} \tilde{x}^{(p_j)} = x^*.$$

Finally we show that $\tilde{x}^{(p)} = x^{(p)}$. It is left to the reader to show that A has discrete spectrum if and only if $A+\gamma I$ has discrete spectrum for any $\gamma \in \mathbb{R}$. In the latter case $S(A)+\gamma := \{\lambda+\gamma \mid \lambda \in S(A)\}$ coincides with $S(A+\gamma I)$. According to Theorem II.7.8 we can choose a $\gamma \in \mathbb{R}$ such that $-\gamma \in \Sigma(A)$, i.e. $A+\gamma I$ admits a bounded everywhere defined inverse. Let $x \in D(A)$, set

$$x^{(N)} = \sum_{k=1}^{N} x_k \varphi_k.$$

Then by Theorem II.7.8

$$(A+\gamma)x^{(N)} = \sum_{k=1}^{N} (\lambda_k+\gamma)x_k\varphi_k \to (A+\gamma)x, \quad N\to\infty.$$

Since also $x^{(N)} \to \tilde{x}$, $N\to\infty$, with

$$\tilde{x} = \sum_{k=1}^{\infty} x_k\varphi_k$$

the closedness of A implies $(A+\gamma I)x = (A+\gamma I)\tilde{x}$. Therefore $x = \tilde{x}$ and the first direction of our proof is finished. As for the second one assume that the criterion of our theorem holds. If A does not have a discrete spectrum then there is a compact interval $\Delta = [a,b]$ such that $\dim \mathfrak{M}(\Delta) = +\infty$. Let (φ_k) be an infinite orthonormal system in $\mathfrak{M}(\Delta)$. $E(\Delta)$ commuting with A we have

$$\|\varphi_k\|^2 + \|A\varphi_k\|^2 = \int_a^b (1+\lambda^2)\, d(E(\lambda)\varphi_k,\varphi_k),$$

$$\leq \sup_{\lambda\in\Delta} (1+\lambda^2)\|\varphi_k\|^2,$$

$$\leq \sup_{\lambda\in\Delta} (1+\lambda^2).$$

Therefore there is a subsequence (φ_{k_j}) of (φ_k) which is convergent. Since by Parseval's inequality

$$\|y\|^2 \geq \sum_{k=1}^{\infty} |(\varphi_k, y)|^2, \quad y \in H,$$

we can conclude that $\varphi_{k_j} \rightharpoonup 0$, $j \to \infty$. Thus $\varphi_{k_j} \to 0$, $j \to \infty$ which is a contradiction. Our theorem is proved. □

For later use we give the following definition:

Definition II.7.7: Let A be a linear operator in a Hilbert space H with domain of definition $\mathcal{D}(A)$. A is said to be bounded from below if and only if there is some $\gamma \in \mathbb{R}$ with

$$(Au, u) \geq \gamma \|u\|^2, \quad u \in \mathcal{D}(A).$$

Problem II.7.1: Let A be selfadjoint and bounded from below. Prove: If it is possible to select from every sequence $\{\tilde{\varphi}_k\}$ with

$$\tilde{\varphi}_k \in \mathcal{D}(A),$$
$$\|\tilde{\varphi}_k\| \leq D,$$
$$(A\tilde{\varphi}_k, \tilde{\varphi}_k) \leq D,$$

for every $k \in \mathbb{N}$ and some $D > 0$, a convergent subsequence then A has a discrete spectrum.

Problem II.7.2: Let A be selfadjoint and bounded from below. Let A have a discrete spectrum. Assume that $\gamma \geq 0$ in Definition II.7.7. Prove: If one takes the expansion in Theorem II.7.8 then $\lambda_k \geq 0$,

$$(Ax, x) = \sum_{k=1}^{\infty} \lambda_k |x_k|^2,$$

$$x = \sum_{k=1}^{\infty} x_k \varphi_k$$

(for the notations cf. the proof of Theorem II.7.9).

Problem II.7.3: Under the assumptions of Problem II.7.2 prove that every sequence $(\tilde{\varphi}_k)$ having the properties stated in Problem II.7.1 contains a convergent subsequence.

Problem II.7.4: Remove the assumption $\gamma \geq 0$ in Problem II.7.3.

§ 8. Functions of a Selfadjoint Operator. The Heinz-Kato Inequality

In this paragraph H is a separable Hilbert space; A is a selfadjoint operator in H with domain of definition $\mathcal{D}(A)$ and $\{E(\lambda) \mid \lambda \in \mathbb{R}\}$ is the spectral family which belongs to A. Then the following proposition holds:

Proposition II.8.1: Let $u:\mathbb{R} \to \mathbb{R}$ be continuous. Let $f \in H$. Then

$$\lim_{\substack{a\to-\infty,\\ b\to+\infty}} \int_a^b u(\lambda)dE(\lambda)f =: \int_{-\infty}^{+\infty} u(\lambda)dE(\lambda)f$$

exists if and only if

$$\lim_{\substack{a\to-\infty,\\ b\to+\infty}} \int_a^b |u(\lambda)|^2 d(E(\ )f,f) =: \int_{-\infty}^{+\infty} |u(\lambda)|^2 d(E(\lambda)f,f)$$

exists.

Proof: The proof is the same as that of Proposition II.6.1 but with λ replaced by $u(\lambda)$ and with λ^2 replaced by $|u(\lambda)|^2 = u^2(\lambda)$. □

It is our aim to define the notion of a function u of a selfadjoint operator. This is done in the theorem to follow:

Theorem II.8.1: Let $u:\mathbb{R} \to \mathbb{R}$ be continuous. Let

$$\mathcal{D}(u(A)) = \{f \mid f \in H, \int_{-\infty}^{+\infty} |u(\lambda)|^2 d(E(\lambda)f,f) < +\infty\}.$$

Then $\mathcal{D}(u(A))$ is a dense linear subspace of H. The operator $u(A)$, defined by

$$u(A)f = \lim_{\substack{a\to-\infty,\\ b\to+\infty}} \int_a^b u(\lambda)dE(\lambda)f, \quad f \in \mathcal{D}(u(A))$$

is selfadjoint.

Proof: The proof is carried through as that of Theorem II.6.1. Again one has to replace λ by $u(\lambda)$. □

Up to now we have only considered realvalued functions. For a complex valued function $w:\mathbb{R}\to\mathbb{C}$ being continuous we define

$$u(\lambda) = \mathrm{Re}\, w(\lambda),$$
$$v(\lambda) = \mathrm{Im}\, w(\lambda), \quad \lambda \in \mathbb{R},$$
$$\mathcal{D}(w(A)) = \mathcal{D}(u(A)) \cap \mathcal{D}(v(A))$$
$$w(A)f = u(A)f + iv(A)f, \quad f \in \mathcal{D}(w(A)).$$

This means that

$$\mathcal{D}(w(A)) = \{f \mid f \in H, \int_{-\infty}^{+\infty} |u(\lambda)|^2 d(E(\lambda)f,f) + \int_{-\infty}^{+\infty} |v(\lambda)|^2 d(E(\lambda)f,f) < \infty\}.$$

Since $|u(\lambda)|^2 + |v(\lambda)|^2 = u^2(\lambda) + v^2(\lambda) = |w(\lambda)|^2$ we see that $\mathcal{D}(w(A)) = \mathcal{D}(|w|(A))$. Therefore $\mathcal{D}(w(A))$ is dense in H

$$w(A)f = \int_{-\infty}^{+\infty} w(\lambda)dE(\lambda)f, \quad f \in \mathcal{D}(w(A)) = \mathcal{D}(|w|(A)).$$

Proposition II.8.2: Let $w:\mathbb{R}\to\mathbb{C}$ be continuous, let $u(\lambda) = \mathrm{Re}\, w(\lambda)$, $v(\lambda) = \mathrm{Im}(\lambda)$, $\lambda \in \mathbb{R}$. Then

$$\lim_{\substack{a\to-\infty,\\ b\to+\infty}} \int_a^b w(\lambda)d(E(\lambda)f,g) =: \int_{-\infty}^{+\infty} w(\lambda)d(E(\lambda)f,g)$$

exists and

$$(w(A)f,g) = \int_{-\infty}^{+\infty} w(\lambda)d(E(\lambda)f,g), \quad f \in \mathcal{D}(w(A)), \quad g \in H,$$

$$\|w(A)f\|^2 = \int_{-\infty}^{+\infty} |w(\lambda)|^2 d(E(\lambda)f,f), \quad f \in \mathcal{D}(w(A)).$$

If $\lim_{\substack{a\to-\infty,\\ b\to+\infty}} \int_a^b w(\lambda)dE(\lambda)f$ exists then $f \in \mathcal{D}(w(A))$, $w(A)f = \lim_{\substack{a\to-\infty,\\ b\to+\infty}} \int_a^b w(\lambda)dE(\lambda)f = \int_{-\infty}^{+\infty} w(\lambda)dE(\lambda)f.$

Proof:

$-\infty < c < a < b < d < +\infty$. Then

$$\left|\int_a^b w(\lambda)d(E(\lambda)f,g) - \int_c^d w(\lambda)d(E(\lambda)f,g)\right|$$

$$\le \left|\int_c^a w(\lambda)d(E(\lambda)f,g)\right| + \left|\int_b^d w(\lambda)d(E(\lambda)f,g)\right|.$$

Taking the Riemannian sums $T_n f$ as in the proof of Theorem II.2.1 we get

$$|(T_n f,g)|^2 = \left|\sum_{i=1}^{k_n} w(\lambda_i^{(n)})\cdot(E(\Delta_i^{(n)})f,E(\Delta_i^{(n)})g)\right|,$$

$$\leq \|g\|^2\cdot\left|\sum_{i=1}^{k_n} |w(\lambda_i^{(n)})|^2\cdot(E(\Delta_i^{(n)})f,f)\right|,$$

$$\left|\int_c^a w(\lambda)d(E(\lambda)f,g)\right| \leq$$

$$\leq \|g\|^2\cdot\int_c^a |w(\lambda)|^2 d(E(\lambda)f,f),$$

if we concentrate on the integral $\int_c^a w(\lambda)d(E(\lambda)f,g)$. The second one is treated analogously. This immediately infers the existence of

$$\int_{-\infty}^{+\infty} w(\lambda)d(E(\lambda)f,g) = \lim_{\substack{a\to-\infty,\\ b\to+\infty}} \int_a^b w(\lambda)d(E(\lambda)f,g)$$

provided $f \in \mathcal{D}(w(A))$, $g \in \mathcal{H}$.

This gives

$$\int_{-\infty}^{+\infty} w(\lambda)d(E(\lambda)f,g) = \lim_{\substack{a\to+\infty,\\ b\to-\infty}} \int_b^a w(\lambda)d(E(\lambda)f,g)$$

$$= (w(A)f,g),$$

$f \in \mathcal{D}(w(A))$, $g \in \mathcal{H}$. Since

$$\|w(A)f\|^2 = \lim_{\substack{a\to-\infty,\\ b\to+\infty}} \left(\int_a^b w(\lambda)dE(\lambda)f, \int_a^b w(\lambda)dE(\lambda)f\right)$$

and since

$$\left\| \int_a^b w(\lambda)dE(\lambda)f \right\|^2 = \lim_{n\to\infty} (T_n f, T_n f),$$

$$= \lim_{n\to\infty} \sum_{i=1}^{k_n} |w(\lambda_i^{(n)})|^2 (E(\Delta_i^{(n)})f,f),$$

$$= \int_a^b |w(\lambda)|^2 d(E(\lambda)f,f), \quad f \in \mathcal{D}(w(A)),$$

the last but one assertion of the present Proposition readily follows. Assume that

$$w(A)f = \lim_{\substack{a\to-\infty,\\ b\to+\infty}} \int_a^b w(\lambda)dE(\lambda)f$$

exists. The preceding calculations show the last assertion.

□

Our next aim is derive rules for the addition and multiplication of operator valued functions. If T_1, T_2 are any two operators in $\mathcal{H}$ with domains of definition $\mathcal{D}(T_1)$, $\mathcal{D}(T_2)$ then we define

$$(T_1+T_2)x = T_1x + T_2x, \quad x \in \mathcal{D}(T_1+T_2) := \mathcal{D}(T_1) \cap \mathcal{D}(T_2),$$

$$(T_1T_2)x = T_1(T_2x), \quad x \in \mathcal{D}(T_1T_2) := \{y \mid y \in \mathcal{D}(T_2), T_2y \in \mathcal{D}(T_1)\}.$$

Now the following theorem holds:

Theorem II.8.2: Let A be selfadjoint in $\mathcal{H}$ with domain of definition $\mathcal{D}(A)$. Let $w_i : \mathbb{R} \to \mathbb{C}$ be continuous functions, $i = 1,2$. Then

$$(w_1+w_2)(A) \supseteq w_1(A) + w_2(A),$$

$$(w_1w_2)(A) \supseteq w_1(A)w_2(A).$$

More precisely, $w_1(A)w_2(A)$ is the restriction of $(w_1w_2)(A)$ to $\mathcal{D}(w_2(A)) \cap \mathcal{D}((w_1w_2)(A))$. We have

$$(w_1w_2)(A) = w_1(A)w_2(A)$$

if and only if $\mathcal{D}(w_2(A)) \supset \mathcal{D}((w_1w_2)(A))$. Moreover

$$\begin{aligned}\mathcal{D}(w_1(A)+w_2(A)) &= \mathcal{D}((w_1+w_2)(A)) \cap \mathcal{D}(w_1(A)),\\ &= \mathcal{D}((w_1+w_2)(A)) \cap \mathcal{D}(w_2(A)).\end{aligned}$$

We have

$$w_1(A) + w_2(A) = (w_1+w_2)(A)$$

if and only if

$$\mathcal{D}((w_1+w_2)(A)) \subset \mathcal{D}(w_1(A)) \text{ or}$$

$$\mathcal{D}((w_1+w_2)(A)) \subset \mathcal{D}(w_2(A)).$$

Proof: If $f \in \mathcal{D}(w_1(A)) \cap \mathcal{D}(w_2(A))$ then by Proposition II.8.2

$$\begin{aligned}(w_1(A)+w_2(A))f &= \lim_{\substack{a\to-\infty,\\ b\to+\infty}} \int_a^b (w_1(\lambda)+w_2(\lambda))dE(\lambda)f\\ &= \int_{-\infty}^{+\infty} (w_1(\lambda)+w_2(\lambda))dE(\lambda)f = (w_1+w_2)(A)f\end{aligned}$$

Thus the first assertion is proved. If

$$\int_{-\infty}^{+\infty} |w_1(\lambda)+w_2(\lambda)|^2 d(E(\lambda)f,f) < +\infty,$$

$$\text{(II.8.3)} \quad \int_{-\infty}^{+\infty} |w_1(\lambda)|^2 d(E(\lambda)f,f) < \infty,$$

then $\int_{-\infty}^{+\infty} (|w_1(\lambda)|^2+|w_2(\lambda)|^2)d(E(\lambda)f,f) < \infty$. The same conclusion holds if (II.8.3) is replaced by $\int_{-\infty}^{+\infty} |w_2(\lambda)|^2 d(E(\lambda)f,f) < \infty$. Thus

$$\mathcal{D}((w_1+w_2)(A)) \cap \mathcal{D}(w_i(A)) \subset \mathcal{D}(w_1(A)+w_2(A)),$$

$i=1,2$. The inclusion the other way round is also trivial. The last assertion is trivial. Let now be $f \in \mathcal{D}(w_2(A))$. As in the proof of Theorem II.6.2 one shows that $E(\lambda)w_2(A)f = w_2(A)E(\lambda)f$; in particular we have $E(\lambda)\mathcal{D}(w_2(A)) \subset \mathcal{D}(w_2(A))$. Thus

$$\|E(\lambda)w_2(A)f\|^2 = \|w_2(A)E(\lambda)f\|^2$$
$$= \int_{-\infty}^{+\infty} |w_2(\mu)|^2 d(E(\lambda)E(\mu)E(\lambda)f, E(\lambda)f)$$

where the last equation is an immediate consequence of the calculations in the end of the proof of Proposition II.8.2. Since $E(\lambda)E(\mu)f = E(\lambda)f$, $\lambda \leq \mu$, $= E(\mu)f$, $\lambda > \mu$, we end with

$$\|E(\lambda)w_2(A)f\|^2 = \int_{-\infty}^{\lambda} |w_2(\mu)|^2 d(E(\mu)f,f).$$

If $w_2(A)f \in \mathcal{D}(w_1(A))$ then

$$+\infty > \int_{-\infty}^{+\infty} |w_1(\lambda)|^2 d(E(\lambda)w_2(A)f, w_2(A)f)$$
$$= \int_{-\infty}^{+\infty} |w_1(\lambda)|^2 d\|E(\lambda)w_2(A)f\|^2$$
$$= \int_{-\infty}^{+\infty} |w_1(\lambda)|^2 dG(\lambda)$$

with $G(\lambda) = \|E(\lambda)w_2(A)f\|^2 = \int_{-\infty}^{\lambda} |w_2(\mu)|^2 d(E(\mu)f,f)$. Next we show that

$$\int_a^b |w_1(\lambda)|^2 dG(\lambda) = \int_a^b |w_1(\lambda)w_2(\lambda)|^2 d(E(\lambda)f,f).$$

The proof is similar to that of Proposition II.5.1. We have $(a=\lambda_1<\lambda_2<\ldots<\lambda_{n+1}=b)$

$$\left|\sum_{j=1}^{n} |w_1(\lambda_j)|^2 (G(\lambda_{j+1})-G(\lambda_j))\right|$$

$$= \sum_{j=1}^{n} |w_1(\lambda_j)|^2 \int_{\lambda_j}^{\lambda_{j+1}} |w_2(\mu)|^2 d(E(\mu)f,f)$$

$$= \sum_{j=1}^{n} |w_1(\lambda_j)|^2 |w_2(\lambda_j)|^2 \cdot ((E(\lambda_{j+1})-E(\lambda_j))f,f) +$$

$$+ \sum_{j=1}^{n} |w_1(\lambda_j)|^2 \int_{\lambda_j}^{\lambda_{j+1}} (|w_2(\mu)|^2-|w_2(\lambda_j)|^2)\cdot d(E(\mu)f,f).$$

If $\max_{1\leq j\leq n} |\lambda_{j+1}-\lambda_j|$ is sufficiently small the last sum can be made arbitrarily small, whereas the first sum converges to

$$\int_a^b |w_1(\lambda)w_2(\lambda)|^2 d(E(\lambda)f,f).$$

Letting a tend to $-\infty$, b to $+\infty$, we see that $\mathcal{D}(w_1(A)w_2(A)) \subset \mathcal{D}((w_1w_2)(A))$. If on the other hand $f \in \mathcal{D}(w_2(A)) \cap \mathcal{D}((w_1w_2)(A))$ then the preceding calculations show that $w_2(A)f \in \mathcal{D}(w_1(A))$. Consequently

$$\mathcal{D}(w_2(A)) \cap \mathcal{D}(w_1w_2(A)) \subseteq \mathcal{D}(w_1(A)w_2(A))$$
$$\subseteq \mathcal{D}(w_2(A)) \cap \mathcal{D}((w_1w_2)(A)).$$

Therefore the equality sign holds. It is also clear now that $\mathcal{D}(w_1w_2(A)) = \mathcal{D}(w_1(A)w_2(A))$ if and only if $\mathcal{D}(w_2(A)) \supset \mathcal{D}((w_1w_2)(A))$. The reader may verify by himself that

$$E(\lambda)w_2(A)f = \int_{-\infty}^{\lambda} w_2(\mu)dE(\mu)f,$$

$$\int_a^b w_1(\lambda)dE(\lambda)w_2(A)f = \int_a^b w_1(\lambda)w_2(\lambda)dE(\lambda)f,$$

$$f \in \mathcal{D}(w_2(A)),$$

and consequently

$$(w_1w_2)(A)f = w_1(A)w_2(A)f,$$

$f \in \mathcal{D}(w_1(A)w_2(A))$. Our Theorem is proved. □

We draw some consequences of Theorem II.8.2. Let $n \in \mathbb{N}$. Set $w_2(\lambda) = w_1^n(\lambda)$. If $f \in \mathcal{D}((w_1w_2)(A)) = \mathcal{D}(w_1^{n+1}(A))$ then

$$\int_{-\infty}^{+\infty} |w_1(\lambda)|^{2(n+1)} d(E(\lambda)f,f) < +\infty.$$

Taking a Riemannian sum we obtain by applying Hölder's inequality

$$\sum_{i=1}^{m} |w_1(\lambda_j)|^{2n}(E(\Delta_j^{(m)})f,f)$$

$$\leq (\sum_{i=1}^{m} (E(\Delta_j^{(m)})f,f))^{\frac{2}{2(n+1)}} \cdot$$

$$\cdot (\sum_{i=1}^{m} |w_1(\lambda_j)|^{2(n+1)} (E(\Delta_j^{(m)})f,f))^{\frac{2n}{2(n+1)}},$$

$-\infty < a = \lambda_1 < \lambda_2 < \ldots < \lambda_{m+1} = b < +\infty$, $\Delta_j^{(m)} = [\lambda_j, \lambda_{j+1}]$. This gives

$$\int_a^b |w_1(\lambda)|^{2n} d(E(\lambda)f,f) \leq \|f\|^{\frac{2}{n+1}} \cdot (\int_a^b |w_1(\lambda)|^{2(n+1)}$$

$$d(E(\lambda)f,f))^{\frac{2n}{2(n+1)}}.$$

By letting a tend to $-\infty$ and b to $+\infty$ we see that $f \in \mathcal{D}(w_2(A))$. Application of Theorem II.8.2 infers $w_1^{n+1}(A) = w_1(A)w_1^n(A)$; more generally it is implied by this that

(II.8.4) $w^n(A) = (w(A))^n$.

This relation also holds for negative integer exponents. It is sufficient to show this for $n=-1$. We assume that

$$w(\lambda) \neq 0, \quad \lambda \in \mathbb{R}.$$

Then $w^{-1}(\lambda)w(\lambda) = 1$, $\lambda \in \mathbb{R}$, and $w^{-1}(A)w(A)$ is the restriction of the identity to $\mathcal{D}(w(A))$; $w(A)w^{-1}(A)$ is the restriction of the identity to $\mathcal{D}(w^{-1}(A))$. This precisely means that $w(A)$ has an inverse, and

(II.8.5) $(w(A))^{-1} = w^{-1}(A)$.

The following is evident: If w is bounded, say $|w(\lambda)| \leq M$, $\lambda \in \mathbb{R}$, then

(II.8.6) $$w(A) \in L(\mathcal{H},\mathcal{H}), \quad \|w(A)\| \leq M.$$

Now we study the adjoint of $w(A)$. Our result is

Theorem II.8.3: *Let* $w: \mathbb{R} \to \mathbb{C}$ *be continuous. Then*

(II.8.7) $(w(A))^* = \bar{w}(A)$, $w(A) = (\bar{w}(A))^*$.

In particular $w(A)$ *is closed. Here* $\bar{w}(A)$ *denotes the operator corresponding to the function* $\bar{w}$ defined by $\bar{w}(\lambda) = \overline{w(\lambda)}$, $\lambda \in \mathbb{R}$.

Proof: We set

$$w(\lambda) = r(\lambda)\tilde{w}(\lambda), \quad \lambda \in \mathbb{R},$$

with $r(\lambda) = |w(\lambda)|+1$ and $|\tilde{w}(\lambda)| = 1$. Then

$$\mathcal{D}(w(A)) = \mathcal{D}(r(A)),$$
$$w(A) = r(A)\tilde{w}(A) = (r\tilde{w})(A) = (\tilde{w}r)(A) = \tilde{w}(A)r(A)$$

by Theorem II.8.2. $r(A)$ is selfadjoint in H with domain of definition $\mathcal{D}(w(A))$. $\tilde{w}(A)$ is in $L(H,H)$ by (II.8.6). It is easy to see that

$$(\tilde{w}(A))^* = \bar{\tilde{w}}(A)$$

if we write

$$\tilde{w}(A) = (\mathrm{Re}\ \tilde{w})(A) + i(\mathrm{Im}\ \tilde{w})(A),$$

where $(\mathrm{Re}\ \tilde{w})(A)$, $(\mathrm{Im}\ \tilde{w})(A)$ are bounded selfadjoint operators in H with domain of definition H. We have

$$(w(A))^* \subseteq r(A)\bar{\tilde{w}}(A) \subseteq \bar{w}(A).$$

Again by Theorem II.8.2 we conclude $\bar{w}(A) = r(A)\bar{\tilde{w}}(A)$. Let $f \in \mathcal{D}(\bar{w}(A))$, $g \in \mathcal{D}(w(A))$. Then it follows

$$(\tilde{w}(A)r(A)g,f) = (g,r(A\ \bar{\tilde{w}}(A)f),$$
$$\bar{w}(A) \subseteq (w(A))^*.$$

□

We employ Theorem II.8.3. Since by (II.8.7) it follows that $(w(A))^* = w(A)$ we obtain the closedness of $w(A)$ with the aid of Theorem I.1.2. □

Now we deal with a case which occurs frequently in the applications; namely we assume that A is bounded from below.

Theorem II.8.4: Let A be a selfadjoint operator in $\mathcal{H}$ with domain of definition $\mathcal{D}(A)$. Let

$$(Au,u) \geq \gamma \|u\|^2, \quad u \in \mathcal{D}(A),$$

for some $\gamma \in \mathbb{R}$. Let $w:\mathbb{R} \to \mathbb{C}$ be any continuous function. Then

$$w(A)f = \lim_{\substack{a\to-\infty,\\ b\to+\infty}} \int_a^b w(\lambda)dE(\lambda)f,$$

$$= \int_{-\infty}^{+\infty} w(\lambda)dE(\lambda)f,$$

$$w(A)f - w(\gamma)E(\gamma)f = \lim_{b\to+\infty} \int_\gamma^b w(\lambda)dE(\lambda)f,$$

$$=: \int_\gamma^{+\infty} w(\lambda)dE(\lambda)f, \quad f \in \mathcal{D}(w(A)).$$

In particular, if $w:[\gamma,+\infty) \to \mathbb{C}$ is any continuous function which is continued anyhow to a continuous function $\hat{w}:\mathbb{R} \to \mathbb{C}$, then

$$\hat{w}(A)f = \int_{-\infty}^{+\infty} \hat{w}(\lambda)dE(\lambda)f,$$

$$= \int_\gamma^{+\infty} w(\lambda)dE(\lambda)f + w(\gamma)E(\gamma)f,$$

$$=: w(A)f,$$

$f \in \mathcal{D}(\hat{w}(A)) = \{f \mid f \in H, \int_{-\infty}^{+\infty} |\hat{w}(\lambda)|^2 d(E(\lambda)f,f) < +\infty\} =$

$= \{f \mid f \in H, \int_{\gamma-0}^{+\infty} |w(\lambda)|^2 d(E(\lambda)f,f) < +\infty\}$. The latter space is denoted by $\mathcal{D}(w(A))$. The value of $\int_{\gamma-0}^{+\infty} |w(\lambda)|^2 d(E(\lambda)f,f) =$

$= \lim_{\substack{\varepsilon\to 0,\\ b\to+\infty}} \int_{\gamma-\varepsilon}^{b} |w(\lambda)|^2 d(E(\lambda)f,f)$ does not depend on the continuation $\hat{w}$ of w and is precisely $|w(\gamma)|^2 \|E(\gamma)f\|^2 + \int_{\gamma}^{+\infty} |w(\lambda)|^2 d(E(\lambda)f,f)$. In particular

$$\|w(A)f\|^2 = \int_{\gamma-0}^{+\infty} |w(\lambda)|^2 d(E(\lambda)f,f),$$

$$\mathcal{D}(w(A)) = \{f \mid f \in H, \int_{0}^{+\infty} |w(\lambda)|^2 d(E(\lambda)f,f) < +\infty\}.$$

Proof: If $\delta \in \mathbb{R}$, $\delta < \gamma$, then

$$((A-\delta)u,u) \geq (\gamma-\delta)\|u\|^2, \quad u \in \mathcal{D}(A).$$

Since $\gamma-\delta > 0$ we get

$$\|(A-\delta)u\| \geq (\gamma-\delta)\|u\|, \quad u \in \mathcal{D}(A).$$

Theorem II.1.2 shows that $\delta \in \Sigma(A)$. By Proposition II.7.2 the operators $E(\lambda)$ are constant for $\lambda < \gamma$, i.e. $E(\lambda) = 0$, $\lambda < \gamma$. Let $-\infty < a < \gamma < b < +\infty$. Consider a partition $a = \lambda_1 < \lambda_2 < \ldots < \lambda_i = \gamma < \lambda_{i+1} < \ldots < \lambda_{n+1} = b$. Then

$$\sum_{j=1}^{n} w(\lambda_j)E(\Delta_j^{(n)})f = w(\lambda_{i-1})E(\gamma)f + \sum_{j=i}^{n} w(\lambda_j)E(\Delta_j^{(n)})f,$$

$$\Delta_j^{(n)} = [\lambda_j, \lambda_{j+1}].$$

Letting $\max_{1\leq j\leq n} |\lambda_{j+1}-\lambda_j|$ tend to 0 we get $\int_a^b w(\lambda)dE(\lambda)f =$ $\int_\gamma^b w(\lambda)dE(\lambda)f + w(\gamma)E(\gamma)f$. Now the first and the second assertion of the present theorem easily follow. As for the third one we get (with the same notations as before)

$$\sum_{j=1}^{n} |w(\lambda_j)|^2 (E(\Delta_j^{(n)})f,f) =$$

$$= |w(\lambda_{i-1})|^2 \cdot (E(\gamma)f,f) + \sum_{j=i}^{n} |w(\lambda_j)|^2 \cdot (E(\Delta_j^{(n)})f,f).$$

As before we obtain

$$\int_a^b |\hat{w}(\lambda)|^2 d(E(\lambda)f,f) = |w(\gamma)|^2 \|E(\gamma)f\|^2 + \int_\gamma^b |w(\lambda)|^2 d(E(\lambda)f,f).$$

Since $\int_{\gamma-\varepsilon}^{b} |\hat{w}(\lambda)|^2 d(E(\lambda)f,f) = \int_\gamma^b |w(\lambda)|^2 d(E(\lambda)f,f) +$

$+ \int_{\gamma-\varepsilon}^{\gamma} (|\hat{w}(\lambda)|^2 - |w(\gamma)|^2) d(E(\lambda)f,f) + |w(\gamma)|^2 \|E(\gamma)f\|^2$, $\varepsilon > 0$, we arrive at

$$\int_{-\infty}^{+\infty} |\hat{w}(\lambda)|^2 d(E(\lambda)f,f) = \int_{\gamma-0}^{+\infty} |w(\lambda)|^2 d(E(\lambda)f,f),$$

and this relation holds in the following sense: If the left hand side is finite then the right hand side; moreover its value is

$$\int_\gamma^{+\infty} |w(\lambda)|^2 d(E(\lambda)f,f) + |w(\gamma)|^2 \|E(\gamma)f\|^2$$

and thus does not depend on the continuation $\hat{w}$ under consideration. If the right hand side is finite, i.e. if

$$\lim_{\substack{\varepsilon\to 0,\\ b\to+\infty}} \int_{\gamma-\varepsilon}^{b} |\hat{w}(\lambda)|^2 d(E(\lambda)f,f)$$

exists for some continuous continuation $\hat{w}$ of w then it exists for any, and its value is precisely

$$\int_{-\infty}^{+\infty} |\hat{w}(\lambda)|^2 d(E(\lambda)f,f) = \int_{\gamma}^{+\infty} |\hat{w}(\lambda)|^2 d(E(\lambda)f,f) + |w(\gamma)|^2 \|E(\lambda)f\|^2.$$

Our theorem is proved. □

If $\gamma \geqq 0$ we can thus define

$$A^{\alpha}f = \int_0^{\infty} \lambda^{\alpha} dE(\lambda)f, \ \mathrm{Re}\ \alpha > 0$$

$$f \in \mathcal{D}(A^{\alpha}) = \{\tilde{f} | \tilde{f} \in \ , \int_0^{\infty} |\lambda|^{2\ \mathrm{Re}\ \alpha} d(E(\lambda)\tilde{f},\tilde{f}) < \infty\}$$

(observe that $w(0) = 0$ if $w(\lambda) = \lambda^{\alpha}$, $\lambda \geqq 0$). Applying Hölder's inequality to Riemannian sums as we did right after the proof of Theorem II.8.2 we get

$$\mathcal{D}(A^{\alpha}) \supseteq \mathcal{D}(A^{\beta}) \text{ if } \mathrm{Re}\ \beta \geqq \mathrm{Re}\ \alpha > 0.$$

Theorem II.8.2 then furnishes

$$A^{\alpha}A^{\beta}f = A^{\alpha+\beta}f, \ f \in \mathcal{D}(A^{\alpha+\beta}), \ \mathrm{Re}\ \alpha, \ \mathrm{Re}\ \beta > 0.$$

If $\gamma > 0$ we can go further. Then

$$A^{\alpha}f = \int_{\gamma'}^{+\infty} \lambda^{\alpha} dE(\lambda)f, \ \mathrm{Re}\ \alpha \geqq 0, \ f \in \mathcal{D}(A^{\alpha}),$$

for any γ' with $0 < \gamma' < \gamma$. Correspondingly $\mathcal{D}(A^{\alpha})$ is precisely the set of those $f \in H$ for which

$$\int_{\gamma'}^{+\infty} |\lambda|^{2\ \mathrm{Re}\ \alpha} d(E(\lambda)f,f) < +\infty.$$

From what was written before it is clear that the values of the preceding integrals do not depend on γ'. A^{α} admits a bounded inverse

$$A^{-\alpha}f = \int_{\gamma'}^{+\infty} \lambda^{-\alpha} dE(\lambda)f, \quad f \in H,$$

and it holds

$$A^{\alpha}A^{\beta}f = A^{\alpha+\beta}f, \quad f \in \mathcal{D}(A^{\gamma}),$$

$$\gamma = \max(\mathrm{Re}\ \alpha,\ \mathrm{Re}\ \beta,\ \mathrm{Re}(\alpha+\beta)).$$

Next we want to compare the fractional powers of any two selfadjoint operators which are bounded from below with some $\gamma > 0$ and have identical domain of definition. This result is due to E. Heinz [H]; we prefer to give a more general version (in view of our applications to the Navier-Stokes equations); this is due to Kato [K].

Definition II.8.1: Let A be a selfadjoint operator in a Hilbert space H with domain of definition $\mathcal{D}(A)$. Let A be bounded from below with $(Au,u) \geq \gamma \|u\|^2$, $u \in \mathcal{D}(A)$, for some $\gamma > 0$. Then A is called strictly positive (or $\geq \gamma > 0$). If $(Au,u) > 0$, $u \in \mathcal{D}(A)$, $u \neq 0$, then A is called positive (> 0). If $(Au,u) \geq 0$, then A is called nonnegative (or $A \geq 0$).

Theorem II.8.5: Let H_1, H_2 be two Hilbert spaces and let A and B be selfadjoint nonnegative operators in H_1 and H_2 respectively with domains of definition $\mathcal{D}(A)$ and $\mathcal{D}(B)$. Let T be a bounded operator from H_1 into H_2 which maps $\mathcal{D}(A)$ into $\mathcal{D}(B)$. Assume that there exists a number M such that

$$\|BTu\| \leq M\|Au\|, \quad u \in \mathcal{D}(A).$$

Then, for each α satisfying $0 < \alpha < 1$, the image of $\mathcal{D}(A^{\alpha})$ under T is included in $\mathcal{D}(B^{\alpha})$ and, if $B \geq \gamma_2 > 0$, $A \geq \gamma_1 > 0$,

$$\|B^\alpha Tu\| \leq M^\alpha \|T\|^{1-\alpha} \|A^\alpha u\|, \quad u \in \mathcal{D}(A^\alpha);$$

otherwise

$$\|B^\alpha Tu\| \leq (M+\|T\|)^\alpha \|T\|^{1-\alpha} \|A^\alpha u\|,$$

$u \in \mathcal{D}(A^\alpha)$.

Proof: First we assume that A,B are strictly positive. Let $u \in \mathcal{D}(A^\alpha)$ and $v \in \mathcal{D}(B)$. The Hilbert space valued function $z \to A^z u$, $u \in \mathcal{D}(A^\alpha)$, is holomorphic in $\mathrm{Re}\, z < 0$ (i.e. complex differentiable with respect to the norm of $\mathcal{H}$, cf. 0.2), and continuous in $\mathrm{Re}\, z \leq \alpha$. $z \to B^{\alpha-z} v$ is certainly holomorphic in $\alpha-1 < \mathrm{Re}\, z < \alpha$ and continuous in $\alpha-1 \leq \mathrm{Re}\, z \leq \alpha$. Hence the function f defined by

$$f(z) = (TA^z u, B^{\alpha-\bar{z}} v)$$

is holomorphic in $\alpha-1 < \mathrm{Re}\, z < \alpha$ and continuous in $\alpha-1 \leq \mathrm{Re}\, z \leq \alpha$. Now we estimate $|f|$ on $\mathrm{Re}\, z = \alpha-1$ and $\mathrm{Re}\, z = \alpha$. Since $A^{\alpha-1+iy}u = A^{-1}A^{\alpha+iy}u \in \mathcal{D}(A)$ if $y \in \mathbb{R}$ we have $TA^{\alpha-1+iy}u \in \mathcal{D}(B)$ and

$$\begin{aligned} |f(\alpha-1+iy)| &= |(TA^{\alpha-1+iy}u, B^{1+iy}v)|, \\ &= |(BTA^{\alpha-1+iy}u, B^{iy}v)|, \\ &\leq \|BTA^{\alpha-1+iy}u\| \, \|B^{iy}v\| \\ &\leq M\|A^{\alpha+iy}u\| \, \|B^{iy}v\| \leq M\|A^\alpha u\| \, \|v\| \end{aligned}$$

by our assumption, Theorem II.8.2 and (II.8.6). Similarly

$$\begin{aligned} |f(\alpha+iy)| &= |(TA^{\alpha+iy}u, B^{iy}v)|, \\ &\leq \|TA^{\alpha+iy}u\| \, \|B^{iy}v\|, \\ &\leq \|T\| \, \|A^\alpha u\| \, \|v\|. \end{aligned}$$

Hence, by the three-lines theorem in the theory of functions, we get

$$\begin{aligned}
|(Tu,B^{\alpha}v)| &= |f(0)|, \\
&\leq \{\sup_{y\in\mathbb{R}} |f(\alpha-1+iy)|\}^{\alpha}\cdot\{\sup_{y\in\mathbb{R}} |f(\alpha+iy)|\}^{1-\alpha}, \\
&\leq M^{\alpha}\|T\|^{1-\alpha}\|A^{\alpha}u\|\,\|v\|.
\end{aligned}$$

Thus we have shown that $Tu\in\mathcal{D}(B^{\alpha})$ and, that the estimate in question holds. If A,B are merely nonnegative, then we consider $B+\varepsilon \geqq \varepsilon > 0$, $A+\varepsilon \geqq \varepsilon > 0$. We have

$$\|(B+\varepsilon)Tu\| \geqq \|BTu\| - \varepsilon\|T\|\,\|u\|,$$

$$\begin{aligned}
\|(B+\varepsilon)Tu\| &\leqq \|BTu\| + \varepsilon\|T\|\,\|u\|, \\
&\leqq M\|Au\| + \varepsilon\|T\|\,\|u\|, \\
&\leqq M\|(A+\varepsilon)u\| + \varepsilon\|T\|\,\|u\|,
\end{aligned}$$

$$\|(B+\varepsilon)Tu\| \leqq (M+\|T\|)\|(A+\varepsilon)u\|,$$

where we have used our assumption and the inequalities

$$\begin{aligned}
\varepsilon\|u\| &\leqq \|(A+\varepsilon)u\|, \\
\|Au\| &\leqq \|(A+\varepsilon)u\|.
\end{aligned}$$

Consequently $T\mathcal{D}((A+\varepsilon)^{\alpha}) \subset \mathcal{D}((B+\varepsilon)^{\alpha})$,

$$\|(B+\varepsilon)^{\alpha}Tu\| \leqq (M+\|T\|)^{\alpha}\|T\|^{1-\alpha}\cdot\|(A+\varepsilon)^{\alpha}u\|.$$

Since $\mathcal{D}((B+\varepsilon)^{\alpha}) = \mathcal{D}(B^{\alpha})$, $\mathcal{D}((A+\varepsilon)^{\alpha}) = \mathcal{D}(A^{\alpha})$ we can let ε tend to and arrive at the result desired in the cases $A \geqq \gamma_1 > 0$, $B \geqq \gamma_2 > 0$; A,B nonnegative but not strictly positive. It is clear that all possible cases are covered by the preceding calculations. □

Corollary to Theorem II.8.5: Let A,b be nonnegative selfadjoint operators in a Hilbert space H with domains of definition $\mathcal{D}(A)$, $\mathcal{D}(B)$. Suppose $\mathcal{D}(A) \subset \mathcal{D}(B)$. For all $\alpha \in (0,1)$, we have $\mathcal{D}(A^\alpha) \subset \mathcal{D}(B^\alpha)$. If in addition, there is a certain number M such that $\|Bu\| \leq M\|Au\|$, $u \in \mathcal{D}(A)$, then, if $B \geq \gamma_2 > 0$, $A \geq \gamma_1 > 0$,

$$\|B^\alpha u\| \leq M^\alpha \|A^\alpha u\|, \quad u \in \mathcal{D}(A^\alpha);$$

otherwise

$$\|B^\alpha u\| \leq (M+1)^\alpha \|A^\alpha u\|, \quad u \in \mathcal{D}(A^\alpha).$$

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