Remarks on Lines of Reversibility for Poincaré's Centre Problem

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Synopsis

We characterize lines of reversibility for the centre-problem by using their slope as parameter. As a result of our method we formulate in a rather concise way the conditions for reversibility for cubic systems with nonvanishing quadratic terms.

1 Introduction

Consider a system of differential equations of the form

$$\left.\begin{aligned} \dot{x} &= y + q(x,y) \\ \dot{y} &= -x - p(x,y) \end{aligned}\right\} \tag{1.1}$$

where p, q are polynomials whose terms of lowest order are of degree at least two. A well-known sufficient condition, due to Poincaré, for the origin to be a centre is that the system be reversible with respect to a line L, which passes through the origin, i.e. that the system be invariant under a reflection in the line L, and under a simultaneous reversal of the independent variable t. Thus system (1.1) is reversible with respect to the line $x = 0$ if and only if it is invariant under the transformation $(x, y, t) \to (-x, y, -t)$, i.e. if and only if $q(-x, y) = q(x, y)$ and $p(-x, y) = -p(x, y)$. Thus q contains only even powers of x and p only odd ones. Reversibility with respect to $y = 0$ is thus equivalent to $q(x, -y) = -q(x, y)$, $p(x, -y) = p(x, y)$, i.e. q contains only odd powers of y and p only even ones. As for a general line L one can apply a rotation which transforms L into the line $x = 0$ or $y = 0$ and a criterion for reversibility may then readily be attained [1]. Collins, by using tensor-calculus, derives in [2] a necessary and sufficient condition for the existence of such a line without involving its unknown equation.

Here we discuss another method which neither uses purely orthogonal transformations nor tensor calculus. By a suitable change of variables we reduce the problem of finding a line of reversibility $y = -\frac{1}{m}x$ with $m \in \mathbb{R} - \{0\}$ to the question whether the system is reversible to $y = 0$. This access therefore leaves out the coordinate-axes as possible lines of reversibility. As mentioned above it is however easy to decide if one of the coordinate axes is a line of reversibility. If we write

$$\left.\begin{aligned} p &= p_2 + p_3 + \ldots + p_N \\ q &= q_2 + q_3 + \ldots + q_N \end{aligned}\right\} \tag{1.2}$$

with homogeneous polynomials p_1, q_i of degree i it turns out that possible lines of reversibility are already determined by the quadratic case $\dot{x} = y + q_2$, $\dot{y} = -x - p_2$ provided $(p_2, q_2) \neq (0, 0)$. The values of m found there have to be inserted into polynomial equations corresponding to $p_2, q_2, \ldots, p_N, q_N$. These equations then provide the necessary and sufficient conditions for (1.1) to have a line of reversibility. The coefficients of the polynomial equations linearly depend on the coefficients of $p_3, q_3, \ldots, p_N, q_N$ respectively. We use this method to discuss the case

$$\left.\begin{aligned} p &= p_2 + p_3 \\ q &= q_2 + q_3 \end{aligned} \quad \textit{for } (p_2, q_2) \neq (0,0)\right\} \tag{1.3}$$

and to bring the conditions for the existence of a line of reversibility into a manageable form.

2 Reversibility in Polynomial Systems

In what follows we frequently use instead of (1.1) the single equation

$$y' = -\frac{x + p(x,y)}{y + q(x,y)} \tag{2.1}$$

as done in [3] or [4]. For $m \neq 0$ we employ the linear transformation of variables.

$$\begin{aligned} \xi &= y - mx, \\ \eta &= y + \tfrac{1}{m}x \end{aligned} \tag{2.2}$$

or

$$\begin{aligned} x &= \tfrac{m}{m^2+1}(\eta - \xi) = \varphi(\xi,\eta), \\ y &= \tfrac{m^2}{m^2+1}\eta + \tfrac{1}{m^2+1}\xi = \psi(\xi,\eta). \end{aligned} \tag{2.3}$$

We set $\Phi(\xi,\eta) = (\varphi(\xi,\eta), \psi(\xi,\eta))^T$ with $.^T$ for transposition. Φ^{-1} consists of a rotation and a stretching of the x, y-coordinates. So does Φ but in opposite order. Thus reversibility with respect to a line is a property which is invariant under Φ^{-1} and Φ. Then (2.1) becomes

$$\eta' = -\frac{\xi + (q - mp)\circ\Phi}{m^2\eta + m(mq + p)\circ\Phi}. \tag{2.4}$$

Now we arrive at

Theorem 2.1: (2.1) *has a centre at* $(0,0)$ *with line of reversibility* $y = -\frac{1}{m}x (m \neq 0)$ *if and only if each*

$$\left.\begin{aligned} &(q_i - mp_i)\circ\Phi \text{ contains only even powers of } \eta \text{ and each} \\ &(mq_i + p_i)\circ\Phi \text{ contains only odd powers of} \eta,\ 2 \leq i \leq N. \end{aligned}\right\} \tag{2.5}$$

Proof:
If (2.4) satisfies (2.5), then (2.4) has a centre at $(0,0)$ with line of reversibility $\eta = 0$. Thus (2.1) has a centre at $(0,0)$ with line of reversibility $y = -\frac{1}{m}x$. If conversely (2.1) has a centre at $(0,0)$ with line of reversibility $y = -\frac{1}{m}x$, then (2.4) has so with line of reversibility $\eta = 0$. Consequently (2.5) is satisfied. □

(2.5) can be transformed into a more explicit form.

Theorem 2.2: (2.5) *is equivalent to* $N-1$ *matrix equations*

$$\mathcal{L}_{i+1}(p_i,q_i)\begin{pmatrix} m \\ m^2 \\ \vdots \\ m^{i+1} \end{pmatrix} = \begin{pmatrix} b_1(p_i,q_i) \\ b_2(p_i,q_i) \\ \vdots \\ b_{i+1}(p_i,q_i) \end{pmatrix},\ 2 \leq i \leq N, \tag{2.6}$$

where $\mathcal{L}_{i+1}(p_i,q_i)$, $(b_1(p_i,q_i),\ldots,b_{m+1}(p_i,q_i))^T$ *are* $(i+1)\times(i+1)$, $(i+1)\times 1$ *matrices respectively whose coefficients linearly depend on the coefficients of* p_i, q_i.

Proof:
We have to evaluate $(q_i - mp_i)\circ\Phi$, $(mq_i + p_i)\circ\Phi$. These expressions are of the form

$$\frac{1}{(m^2+1)^i}\sum_{k,l,k+l=i}(q_{ikl} - mp_{ikl})\sum_{j=0}^{k}\binom{k}{j}m^k\eta^{k-j}(-\xi)^j \cdot \sum_{q=0}^{l}\binom{l}{q}m^{2(l-q)}\eta^{l-q}\xi^q =$$

$$= \frac{m^i}{(m^2+1)^i}\sum_{\lambda=0}^{i}\eta^{i-\lambda}\sum_{k,l,k+l=i}(q_{ikl} - mp_{ikl})m^l \cdot \sum_{\substack{j,q,j+q=\lambda \\ j\leq\min(k,\lambda) \\ q\leq\min(l,\lambda)}} m^{-2q}(-\xi)^j\xi^q\binom{k}{j}\binom{l}{q}$$

if we set $\lambda = j + q$ and if p_{ikl}, q_{ikl} denote the coefficients of p_i, q_i respectively. If $i - \lambda$ is odd, the coefficient of $\eta^{i-\lambda}$ has to vanish. If i is odd the values

$$\lambda = i-1,\ i-3, \ldots, 0$$

furnish the powers in question. For $\lambda = 0$ the largest occurring power of m is $2i+1$, the smallest one i. For $\lambda = 2$ we obtain $2i-1$ as largest one and $i-2$ as smallest one and so on. Dividing by $m^i, m^{i-2}, \ldots$ and multiplying by $(m^2+1)^i$ we end up with $(i+1)/2$ polynomials in m of degree $i+1$ which have to vanish. If i is even the values

$$\lambda = i-1, i-3, \ldots, 1$$

furnish the powers in question. For $\lambda = 1$ the largest occurring power of m is $2i$, the smallest one is $i-1$. Observe that these values are assumed for $l = i-1$, $q = 0$ and $l = 1$, $q = 1$. For $\lambda = 3$ we obtain $2i-2$ as largest one and $i-3$ as smallest one and so on. Dividing by m^{i-1}, $m^{i-3}, \ldots$ and multiplying by $(m^2+1)^i$ we arrive at $i/2$ polynomials in m of degree $i+1$ which have to vanish. As for $(mq_i + p_i) \circ \Phi$ the coefficients of even powers of η have to vanish. The calculations are very similar to the preceding ones. If i is odd we again obtain $(i+1)/2$ polynomials in m of degree $i+1$ which have to vanish; if i is even we arrive at $i/2+1$ polynomials in m which have to vanish. $\square$

The systems (2.6) have to be considered as necessary and sufficient conditions on the coefficients of p_i, q_i for the existence of a line of reversibility different from the coordinate-axes. This can be seen as follows. If (p_j, q_j) is the first pair where p_j, q_j do not vanish identically we can find the possible values of m from (2.6) for $i = j$ in terms of the coefficients of p_j, q_j. These then have to be inserted into (2.6) for $i = j, \ldots, N$. For instance let us assume that in

$$\mathcal{L}_{j+1}(p_j, q_j) = (l_{ik})_{i,k=1,\ldots,j+1}$$

the matrix

$$(l_{ik})_{i,k=2,\ldots,j+1} \text{ has rank } j$$

then we can possibly obtain the value of m from the first row of (2.6, $i = j$). At least this is so if $\mathcal{L}_{j+1}(p_j, q_j)$ has rank $j+1$. This value of m if $\neq 0$ then has to be inserted into the remaining equations in (2.6). It is an expression in the coefficients of p_j, q_j. Thus we obtain the necessary and sufficient conditions for the existence of a line of reversibility different from the coordinate-axes. In the example to follow in the next section we will see that in more detail.

3 Cubic Systems with Nonvanishing Quadratic Parts

Let us consider

$$y' = -\frac{x + p_2 + p_3}{y + q_2 + q_3}$$

with

$$\begin{aligned}
p_2 &= \widehat{a}x^2 + (2\widehat{b} + \alpha)xy + \widehat{c}y^2, \\
q_2 &= \widehat{b}x^2 + (2\widehat{c} + \beta)xy + \widehat{d}y^2, \\
p_3 &= ax^3 + bx^2y + cxy^2 + dy^3, \\
q_3 &= Ax^3 + Bx^2y + Cxy^2 + Dy^3.
\end{aligned}$$

Here we adopted the usual notation for the quadratic parts p_2, q_2 (cf. [3, 4, 5]). The conditions (2.6, $i = 2, 3$) read as follows.

$$m^3(2\widehat{b} + \alpha) + m^2(-(4\widehat{c} + \beta) + 2\widehat{a}) + m(-(4\widehat{b} + \alpha) + 2\widehat{d}) = -(2\widehat{c} + \beta), \tag{3.1}$$

$$m^3\widehat{b} + m^2(-(2\widehat{c} + \beta) + \widehat{a}) + m(-(2\widehat{b} + \alpha) + \widehat{d}) = -\widehat{c}, \tag{3.2}$$

$$m^3\widehat{d} + m^2((2\widehat{c} + \beta) + \widehat{c}) + m((2\widehat{b} + \alpha) + \widehat{\beta}) = -\widehat{a}, \tag{3.3}$$

$$-m^4 d + m^3(D - c) + m^2(C - b) + m(B - a) = -A \tag{3.4}$$

$$-m^4b + m^3(B - (3a - 2c)) + m^2(3A - 2C - (3d - 2b)) + m(3D - 2B - c) = -C, \tag{3.5}$$

$$-m^4C + m^3(3D - 2B - c) + m^2(2C - 3A - (2b - 3d)) + m(B - (3a - 2c)) = -b, \tag{3.6}$$

$$-m^4A + m^3(B - a) + m^2(-C + b) + m(D - c) = -d. \tag{3.7}$$

(3.1, 3.2, 3.3) stem from $i = 2$, (3.4, 3.5, 3.6, 3.7) from $i = 3$. We start with $i = 2$. Then (3.1, 3.2, 3.3) are equivalent to

$$m^3\alpha + m^2\beta + m\alpha + \beta = 0 \tag{3.8}$$

$$m^3(\widehat{b} + \widehat{d}) + m^2(\widehat{a} + \widehat{c}) + m(\widehat{b} + \widehat{d}) + \widehat{a} + \widehat{c} = 0 \tag{3.9}$$

$$m^3\widehat{d} + m^2((2\widehat{c} + \beta) + \widehat{c}) + m((2\widehat{b} + \alpha) + \widehat{\beta}) + \widehat{a} = 0 \tag{3.10}$$

For further treatment we introduce the vector

$$\mathfrak{a} = (\widehat{a} + \widehat{c}, \widehat{b} + \widehat{d}, \alpha, \beta) \in \mathbb{R}^4$$

If $\mathfrak{a}$ has only nonvanishing components (3.8, 3.9) admit within $\mathbb{R} - \{0\}$ only the solutions $-\frac{\beta}{\alpha}$, $-\frac{\widehat{a}+\widehat{c}}{\widehat{b}+\widehat{d}}$ respectively. Thus we obtain as necessary and sufficient conditions for the solvability of (3.1, 3.2, 3.3) the relations

$$\beta(\widehat{b} + \widehat{d}) = \alpha(\widehat{a} + \widehat{c}), \tag{3.11}$$

$$-\beta^3\widehat{d} + \alpha\beta^2(3\widehat{c} + \beta) - \alpha^2\beta(3\widehat{b} + \alpha) + \alpha^3\widehat{a} = 0. \tag{3.12}$$

(3.11, 3.12) coincide with condition II in [5, p. 13].

Inserting $m = -\frac{\beta}{\alpha}$ into (3.4, ..., 3.7) we obtain together with (3.11, 3.12) the necessary and sufficient conditions for the existence of a line of reversibility, different from the coordinate-axes.

We briefly discuss the other possibilities for $\mathfrak{a}$. If $\mathfrak{a} \neq 0$ there are only two cases where we may have a line of reversibility different from the coordinate-axes, namely

$$\begin{array}{llll} \widehat{a} + \widehat{c} \neq 0, & \widehat{b} + \widehat{d} \neq 0, & \alpha = 0, & \beta = 0, \\ \widehat{a} + \widehat{c} = 0, & \widehat{b} + \widehat{d} = 0, & \alpha \neq 0, & \beta \neq 0; \end{array}$$

then $m = -\frac{\widehat{a}+\widehat{c}}{\widehat{b}+\widehat{d}}$ in the first case and then necessary and sufficient conditions for the existence of a line of reversibility as above are

$$-(\widehat{a} + \widehat{c})^3\widehat{d} + 3(\widehat{b} + \widehat{d})(\widehat{a} + \widehat{c})^2\widehat{c} - 3(\widehat{b} + \widehat{d})^2(\widehat{a} + \widehat{c})\widehat{b} + (\widehat{b} + \widehat{d})^3\widehat{a} = 0,$$

$$(3.4, \ldots, 3.7) \textit{ with } m = -\frac{\widehat{a}+\widehat{c}}{\widehat{b}+\widehat{d}}.$$

In the second case we have $m = -\frac{\beta}{\alpha}$ and an analogous result. It remains to deal with $\mathfrak{a} = 0$. In this case we are left with

$$m^3\widehat{d} + 3m^2\widehat{c} - 3m\widehat{d} - \widehat{c} = 0.$$

If $\widehat{d} \neq 0$ we obtain three distinct real solutions m_1, m_2, m_3 since the discriminant is < 0. If $\widehat{c} \neq 0$ these solutions do not vanish and $y = -\frac{1}{m_i}x$ is a line of reversibility if and only if (3.4, ..., 3.7) are satisfied with $m = m_i$. Since m_1, m_2, m_3 can be computed by means of Cardano's formula we arrive thus at the necessary and sufficient conditions for the existence of a line of reversibility different from the coordinate axes. If $\widehat{d} \neq 0$, $\widehat{c} = 0$ one of m_i vanishes, say m_3. For $m_1 = \sqrt{3}$, $m_2 = -\sqrt{3}$ the conclusion before holds. The case $\widehat{d} = 0$, $\widehat{c} \neq 0$ furnishes two roots, namely $m_1 = \frac{1}{\sqrt{3}}$, $m_2 = -\frac{1}{\sqrt{3}}$ and we can proceed as before. The case $\widehat{d} = \widehat{c} = 0$ implies $(p_2, q_2) = (0, 0)$ since $\mathfrak{a} = 0$. It therefore contradicts our assumption.

References

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