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Estimating $\nabla \mathbf{u}$ in terms of div u, curl u, either $(\nu, \mathbf{u})$ or $\nu \times \mathbf{u}$ and the topology

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In the present paper we prove C^α-estimates for $\nabla \mathbf{u}$ using components of boundary values of $\mathbf{u}$, div $\mathbf{u}$, curl $\mathbf{u}$ and quantities given by components of boundary values of $\mathbf{u}$ as well as boundary values of elements belonging to de Rhams cohomology modules. The vector field $\mathbf{u}$ is defined on a bounded set $\bar{G} \subset \mathbb{R}^3$, meanwhile the cohomology group will be defined with regard to $\mathbb{R}^3 - G$. Our inequalities turn out to be a priori estimates concerning well-known boundary value problems for vector fields. © 1997 by B. G. Teubner Stuttgart–John Wiley & Sons, Ltd.

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1. Introduction

Let us consider a vector field $\mathbf{u}: \bar{G} \to \mathbb{R}^3$. Here G is a bounded open set of $\mathbb{R}^3$ with a smooth boundary ∂G and an outward normal ν. In [7] the second author has studied the problem, whether $\nabla \mathbf{u}$ can always be estimated by div $\mathbf{u}$ and curl $\mathbf{u}$ provided that one of the quantities $(\nu, \mathbf{u})$ or $\nu \times \mathbf{u}$ vanishes on ∂G. The result is that such an estimate is possible for all $\mathbf{u}$ if and only if the first Betti number of G, respectively, the second one vanishes. The underlying space was $L^p(G)$.

In the present paper we want to generalize this result. The set G may have arbitrary finite first or second Betti number. Neither $(\nu, \mathbf{u})$ nor $\nu \times \mathbf{u}$ is required to vanish on ∂G. In this case we expect that for an estimate of $\nabla \mathbf{u}$ in addition to div $\mathbf{u}$ and curl $\mathbf{u}$ at least one of the quantities $(\nu, \mathbf{u})$ or $\nu \times \mathbf{u}$ is needed. Obviously, we also need a quantity which reflects the topological structure of G. If we estimate $\nabla \mathbf{u}$ by div $\mathbf{u}$, curl $\mathbf{u}$ and $\nu \times \mathbf{u}$, this is

$$\sum_{i=1}^{m} |E_i|, \quad m = \text{second Betti number of } G.$$

Here E_i is the flux $\int_{\partial \hat{G}_i} -(\nu, \mathbf{u})\, d\Omega$ of $\mathbf{u}$ with regard to $\partial \hat{G}_i$, and $\hat{G}_i$ in one of the bounded arcwise connected components of $\hat{G} = \mathbb{R}^3 - \bar{G}$, $i = 1, \ldots, m$. If in contrast we want to

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estimate $\nabla\mathbf{u}$ by div $\mathbf{u}$, curl $\mathbf{u}$ and $(v, \mathbf{u})$, there arises due to the topology

$$\sum_{i=1}^{n} |\Gamma^i|, \quad n = \text{first Betti number of } G,$$

as additional quantity needed. Here we define Γ^i as $\int_{\partial G}(-(v\times\mathbf{u}), \hat{\mathbf{z}}^i)\,d\Omega$, $i = 1, \ldots, n$. The first Betti number n is the number of handles of G, and, according to Alexander's duality theorem, it is also the number of handles of $\hat{G}$. Moreover, n simultaneously denotes the dimension of the Neumann fields on $\bar{G}$ as well as on $\bar{\hat{G}}$. The functions $\hat{\mathbf{z}}^1, \ldots, \hat{\mathbf{z}}^n$ form a basis of the Neumann fields on $\bar{\hat{G}}$ (cf. section 3). It can be shown that in particular cases the quantities Γ^i are nothing else but circulations of $\mathbf{u}$ concerning boundary curves around the handles of G (cf. the remark in the end). Nevertheless, we shall always refer to Γ^i as circulations. As underlying space we choose $C^\alpha(\bar{G})$, $\alpha\in(0, 1)$. This is conceptually easier to tackle than to use $L^p(G)$, since we do not have to deal with trace spaces on ∂G. The estimates we are going to prove are

$$\|\nabla\mathbf{u}\|_{C^\alpha(\bar{G})} \leqslant c(\|\operatorname{div}\mathbf{u}\|_{C^\alpha(\bar{G})} + \|\operatorname{curl}\mathbf{u}\|_{C^\alpha(\bar{G})} + \|v\times\mathbf{u}\|_{C^{1+\alpha}(\partial G)} + \sum_{i=1}^{m}|E_i|), \tag{1.1}$$

$$\|\nabla\mathbf{u}\|_{C^\alpha(\bar{G})} \leqslant c(\|\operatorname{div}\mathbf{u}\|_{C^\alpha(\bar{G})} + \|\operatorname{curl}\mathbf{u}\|_{C^\alpha(\bar{G})} + \|(v, \mathbf{u})\|_{C^{1+\alpha}(\partial G)} + \sum_{i=1}^{n}|\Gamma^i|), \quad \mathbf{u}\in C^{1+\alpha}(\bar{G}). \tag{1.2}$$

If $m = 0$ or $n = 0$, the corresponding sums have to be set equal to 0. Thus, from (1.1, 1.2) there arise the estimates in [7] in the C^α-case with a given bounded domain† as well as the estimate

$$\|\mathbf{u}\|_{C^{1+\alpha}(\bar{G})} \leqslant c(\|\operatorname{div}\mathbf{u}\|_{C^\alpha(\bar{G})} + \|\operatorname{curl}\mathbf{u}\|_{C^\alpha(\bar{G})})$$

for $u|_{\partial G} = 0$.

It may be attractive to compare our conclusion with the general results about differential forms on compact Riemannian manifolds $\mathcal{M}$ with boundary, as treated in chapter 7 of the book [4] by Morrey and in the recent monograph [5] by Schwarz. According to the theorems 7.7.4, 7.7.7 and 7.7.8 in [4], for any differential form ω on $\mathcal{M}$ of class $C^{1+\alpha}$ there exist differential form γ, ε and h such that

$$\omega = \gamma + \varepsilon + h, \quad \text{where } \gamma\in\operatorname{Im}\delta,\ \varepsilon\in\operatorname{Im} d \text{ and } dh = \delta h = 0. \tag{1.3}$$

Following the reasoning in [5, Lemma 2.4.10], the inequalities

$$\|\gamma\|_{C^{1+\alpha}} \leqslant c\|d\omega\|_{C^\alpha} \text{ and } \|\varepsilon\|_{C^{1+\alpha}} \leqslant c\|\delta\omega\|_{C^\alpha} \tag{1.4}$$

are provable. These estimates cannot be found in [4] and need an additional effort. We decompose h into its L^2-projection onto the space

$$\mathcal{H}^- := \overline{\{h\in C^\infty(\mathcal{M})\,|\,dh = \delta h = 0, \text{ with tangential part } \tau h = 0\}}^{\|\cdot\|_{L^2}},$$

†In [7] also the case of an unbounded domain was treated with L^p-spaces, $1 < p < 3$.

called h^-, and its orthogonal complement called h_δ. Provided that suitable regularity properties exist, we obtain by basic results of tensor analysis the estimates

$$\| h^- \|_{C^{1+\alpha}} \leqslant c \| \nu h^- \|_{C^{1+\alpha}(\partial \mathscr{M})} \text{ and } \| h_\delta \|_{C^{1+\alpha}} \leqslant c \| h_\delta \|_{C^{1+\alpha}(\partial \mathscr{M})}, \tag{1.5}$$

where νh^- stands for the normal part of h^-. Now we turn to our particular situation, i.e. to $\mathscr{M} = \bar{G} \subset \mathbb{R}^3$ and the estimate (1.1). By $\partial G_1, \ldots, \partial G_m$ we denote a basis of the second homology group concerning $\bar{G}$. The de Rham isomorphism theorem yields the existence of a basis $h_1, \ldots, h_m$ of $\mathscr{H}^-$ dual to $\partial G_1, \ldots, \partial G_m$. For the accompanying vector fields h_i, we therefore obtain

$$\int_{\partial G_j} h_i \, \mathrm{d}\boldsymbol{\Omega} = \delta_i^j. \tag{1.6}$$

Now we insert this into our previous estimates. We need an estimate for the coefficients λ_j in

$$h^- = \lambda_1 h_1 + \cdots + \lambda_m h_m$$

by the *fluxes of* $\mathbf{u}$ and *at least an estimate*

$$\| \tau h \|_{C^{1+\alpha}(\partial G)} \leqslant c \| \nu \times \mathbf{u} \|_{C^{1+\alpha}(\partial G)}.$$

As will be taken from [5, p. 88], the latter inequality remains to be seen. These differences to our conception are not surprising, since our decomposition is different from (1.3). It is neither orthogonal nor can a harmonic field be isolated in an obvious way. On the other hand, compared with the abstract access (1.3), it provides a more concrete analytical insight.

To begin with, we want to make some general remarks and present some fundamental results:

$$G = \bigcup_{i=1}^{\hat{m}} G_i =$$

bounded open set of $\mathbb{R}^3$ with arcwise connected components G_i.
Here $\hat{m}$ denotes the second Betti number of $\hat{G}$.
Each ∂G_i has a finite number of closed surfaces as arcwise connected components. They are assumed to be of class C^∞.
Furthermore, $\bar{G}_i \cap \bar{G}_j = \emptyset$ if $i \neq j$.

$$\hat{G} = \mathbb{R}^3 - \bar{G} = \bigcup_{i=1}^{m} \hat{G}_i \cup \hat{G}_{m+1}$$

with $\hat{G}_i$ bounded, $\hat{G}_{m+1}$ unbounded. In addition $\bar{\hat{G}}_i \cap \bar{\hat{G}}_j = \emptyset$ if $i \neq j$.
Thus, $\partial \hat{G}_i$ has the same properties as ∂G_i.

Let $\mathbf{u}: \bar{G} \to \mathbb{R}^3$ be of class $C^{1+\alpha}(\bar{G})$ for $\alpha \in (0, 1)$, and $\varepsilon := \operatorname{div} \mathbf{u}$ $\gamma := \operatorname{curl} \mathbf{u}$, $\varepsilon^* := -(\nu, \mathbf{u})$, $\gamma^* := -(\nu \times \mathbf{u})$. The fundamental theorem of vector analysis provides the representation

$$\mathbf{u} = -\operatorname{grad} U + \operatorname{curl} \mathbf{A} \text{ with}$$

$$U = \frac{1}{4\pi} \int_G \frac{1}{r} \varepsilon' \, \mathrm{d}x' + \frac{1}{4\pi} \int_{\partial G} \frac{1}{r} \varepsilon^{*\prime} \, \mathrm{d}\Omega',$$

$$\mathbf{A} = \frac{1}{4\pi}\int_G \frac{1}{r}\gamma' \,\mathrm{d}x' + \frac{1}{4\pi}\int_{\partial G} \frac{1}{r}\gamma^{*\prime}\,\mathrm{d}\Omega',$$

$\operatorname{div} \mathbf{A} = 0$, $r = |x - x'|$ for the volume integrals and

$r = |x - \xi'|$, $\xi' \in \partial G$ for the boundary integrals.

Moreover, $v := \int_{\partial G} \frac{1}{r}\varepsilon^{*\prime}\,\mathrm{d}\Omega'$ solves the Neumann problem

$$\Delta v = 0 \quad \text{in } \bar{G} \text{ with } \frac{\partial v}{\partial \nu} = g,$$

$$\varepsilon^* - K\varepsilon^* = \frac{1}{2\pi} g, \text{ where } (K\varepsilon^*)(\xi) := -\frac{1}{2\pi}\int_{\partial G}\left(\frac{\partial}{\partial \nu}\frac{1}{r}\right)(\xi, \xi')\varepsilon^*(\xi')\,\mathrm{d}\Omega'.$$

Employing the well-known estimates for elliptic equations [1, 8], we obtain

$$\| D^2 v \|_\alpha \leqslant c \| g \|_{C^{1+\alpha}(\partial G)}.$$

Thus, by [2] there arises

$$\| D^2 v \|_\alpha \leqslant c \| \varepsilon^* \|_{C^{1+\alpha}(\partial G)}.$$

Since the components of $\mathbf{w} := \int_{\partial G} \frac{1}{r}\gamma^{*\prime}\,\mathrm{d}\Omega'$ are single layer potentials, we conclude that

$$\| D^2 \mathbf{w} \|_\alpha \leqslant c \| \gamma^* \|_{C^{1+\alpha}(\partial G)}.$$

Using the well-known results for volume potentials, we arrive at

$$\| \nabla \mathbf{u} \|_\alpha \leqslant c(\| \varepsilon \|_\alpha + \| \gamma \|_\alpha + \| \varepsilon^* \|_{C^{1+\alpha}(\partial G)} + \| \gamma^* \|_{C^{1+\alpha}(\partial G)}). \tag{1.7}$$

The objective is now to replace $\| \varepsilon^* \|_{C^{1+\alpha}(\partial G)}$ by fluxes of $\mathbf{u}$ or $\| \gamma^* \|_{C^{1+\alpha}(\partial G)}$ by circulations of $\mathbf{u}$. The number of fluxes employed is m (none if $m = 0$), and the number of circulations is n (none if $n = 0$).

Besides the operator K, which has already been introduced, we need its dual in $L^2(\partial G)$. This is

$$(L\mu)(\xi) := -\frac{1}{2\pi}\int_{\partial G}\left(\frac{\partial}{\partial \nu'}\frac{1}{r}\right)(\xi, \xi')\,\mu(\xi')\,\mathrm{d}\Omega'.$$

The operator L belongs to the Dirichlet problem, $I + L$ is the Dirichlet integral operator for the interior problem, $I - L$ is the corresponding one for the exterior problem.

Moreover, $\mathcal{N}$ denotes the null space of a linear operator, and $\mathcal{R}$ is its range.

2. Estimates using the fluxes of u

We are going to prove

Theorem 2.1 *Let* $\mathbf{u} \in C^{1+\alpha}(\bar{G})$ *for* $\alpha \in (0, 1)$ *and* $m \geqslant 1$. *We set*

$$E_i := \int_{\partial \hat{G}_i} \varepsilon^*\,\mathrm{d}\Omega, \quad 1 \leqslant i \leqslant m \text{ and} \tag{2.1}$$

$\varepsilon :=$ div $\mathbf{u}$, $\gamma :=$ *curl* $\mathbf{u}$ *as well as* $\gamma^* := -(\nu \times \mathbf{u})$. *Then the estimate*

$$\| \nabla \mathbf{u} \|_\alpha \leqslant c\left(\| \varepsilon \|_\alpha + \| \gamma \|_\alpha + \| \gamma^* \|_{C^{1+\alpha}(\partial G)} + \sum_{i=1}^{m} |E_i| \right) \tag{2.2}$$

is valid with some constant $c = c(\alpha, G)$.

Proof. According to [6, pp. 114–116], the quantity ε^* satisfies the integral equation

$$\varepsilon^* + K\varepsilon^* = \frac{1}{2\pi} g \tag{2.3}$$

on ∂G, together with the side conditions (2.1). For g we have

$$g(\xi) = (\nu(\xi), \left(\text{grad} \int_G \frac{\varepsilon'}{r} \mathrm{d}x'\right)(\xi) - \left(\text{curl}\left(\int_G \frac{\gamma'}{r} \mathrm{d}x' + \int_{\partial G} \frac{\gamma^{*\prime}}{r} \mathrm{d}\Omega' \right)\right)(\xi)).$$

In particular, we obtain

$$\| g \|_{C^{1+\alpha}(\partial G)} \leqslant c(\| \varepsilon \|_\alpha + \| \gamma \|_\alpha + \| \gamma^* \|_{C^{1+\alpha}(\partial G)}).$$

The operator $I + K$ is the Neumann integral operator on $\partial G = \partial \hat{G}$ for the exterior Neumann problem on $\hat{G}$. A basis of $\mathcal{N}(I + L)$ is given by (cf. [6, pp. 63–69])

$$\hat{h}^i|_{\partial G}, \text{ with } \hat{h}^i(x) = 1, \quad x \in \bar{\hat{G}}_i, \quad \hat{h}^i(x) = 0, \quad x \in \bar{\hat{G}} - \hat{G}_i, \quad 1 \leqslant i \leqslant m.$$

We find a basis $\{\tilde{\tilde{h}}_1, \ldots, \tilde{\tilde{h}}_m\}$ of $\mathcal{N}(1 + K)$ which is dual to $\{\hat{h}^1|_{\partial G}, \ldots, \hat{h}^m|_{\partial G}\}$ (cf. [6, Theorem 1.2.4]). According to [2, cf. also 8], the operator K boundedly maps $C^\alpha(\partial G)$ into $C^{1+\alpha}(\partial G)$ and $C^{1+\alpha}(\partial G)$ into $C^{2+\alpha}(\partial G)$. (2.3) is now considered in $C^{1+\alpha}(\partial G)$. Let us attend to

$$f + Kf = h, \int_{\partial \hat{G}_i} f \,\mathrm{d}\Omega = E_i, \quad 1 \leqslant i \leqslant m \tag{2.4}$$

if h belongs to the closed subspace $\mathscr{R}(I + K)$ of $C^{1+\alpha}(\partial G)$. The problem (2.4) has one and only one solution f in $C^{1+\alpha}(\partial G)$. The function f allows the decomposition

$$f = f_0 + f_1,$$

with

$$f_0 \equiv f_0(E_1, \ldots, E_m) := \sum_{i=1}^{m} E_i \tilde{\tilde{h}}_i \in \mathcal{N}(I + K),$$

$$f_1 \in C^{1+\alpha}(\partial G), \quad f_1 + Kf_1 = h, \quad \int_{\partial \hat{G}_i} f_1 \,\mathrm{d}\Omega = 0, \quad 1 \leqslant i \leqslant m.$$

Obviously, f_0 and f_1 are uniquely determined. Arguing as usually by contradiction, we obtain that there exists a constant $d > 0$ such that

$$\| (I + K) f_1 \|_{C^{1+\alpha}(\partial G)} \geqslant d \| f_1 \|_{C^{1+\alpha}(\partial G)}.$$

Thus, we infer that

$$\| f \|_{C^{1+\alpha}(\partial G)} \leqslant c \sum_{i=1}^{m} |E_i| + \frac{1}{d} \| h \|_{C^{1+\alpha}(\partial G)}.$$

This completes the proof. □

In the case that $m = 0$, the problem is less complicated since the space $\mathcal{N}(I + K)$ consists only of $\{0\}$. The resulting estimate then is

$$\|\nabla\mathbf{u}\|_\alpha \leqslant c(\|\varepsilon\|_\alpha + \|\gamma\|_\alpha + \|\gamma^*\|_{C^{1+\alpha}(\partial G)}), \tag{2.5}$$

as be clearly seen from the preceding considerations.

3. Estimates using the circulations of u

We are going to prove

Theorem 3.1. *Let* $\mathbf{u} \in C^{1+\alpha}(\bar{G})$ *for* $\alpha \in (0, 1)$. *Furthermore, we confine ourselves to the case that the first Betti number be* $n \geqslant 1$. *Elements of the spaces*

$$Z(G) := \{\mathbf{z} \in C^1(G) \cap C^\rho(\bar{G}),\ \rho \in (0, 1) \mid \operatorname{div}\mathbf{z} = 0,\ \operatorname{curl}\mathbf{z} = 0,\quad (\nu, \mathbf{z}) = 0\},$$

$$Z(\hat{G}) := \left\{\mathbf{z} \in C^1(\hat{G}) \cap C^\rho(\bar{\hat{G}}),\ \rho \in (0, 1) \mid \operatorname{div}\mathbf{z} = 0,\ \operatorname{curl}\mathbf{z} = 0,\right.$$

$$\left.(\nu, \mathbf{z}) = 0,\ |\mathbf{z}(x)| = \mathcal{O}\left(\frac{1}{|x|^2}\right),\ |x| \to \infty\right\}$$

will be called Neumann fields in G and $\hat{G}$, *respectively. We set*

$$\Gamma^i := \int_{\partial G} (\gamma^*, \hat{\mathbf{z}}^i)\, \mathrm{d}\Omega, \quad 1 \leqslant i \leqslant n, \quad \{\hat{\mathbf{z}}^i\}_{1 \leqslant i \leqslant n} \text{ basis of } Z(\hat{G}) \text{ and} \tag{3.1}$$

$\varepsilon := \operatorname{div}\mathbf{u}, \gamma := \operatorname{curl}\mathbf{u}$ *as well as* $\varepsilon^* := -(\nu, \mathbf{u})$. *Then the estimate*

$$\|\nabla\mathbf{u}\|_\alpha \leqslant c\left(\|\varepsilon\|_\alpha + \|\gamma\|_\alpha + \|\varepsilon^*\|_{C^{1+\alpha}(\partial G)} + \sum_{i=1}^{m} |\Gamma^i|\right) \tag{3.2}$$

is valid with some constant $c = c(\alpha, G)$.

Proof. By means of

$$\mathcal{T}_{C^{k+\alpha}}(\partial G) := \{\gamma^* \in C^{k+\alpha}(\partial G) \mid (\nu(\xi), \gamma^*(\xi)) = 0\ \forall\, \xi \in \partial G\},$$

$k \in \mathbb{N}_0$, $\alpha \in [0, 1)$ equipped with the norms $\|\cdot\|_{C^{k+\alpha}(\partial G)}$, we obtain Banach spaces. Moreover, a linear operator

$$R: \mathcal{T}_{C^0}(\partial G) \to \mathcal{T}_{C^0}(\partial G), \quad R\gamma^* := \frac{1}{2\pi}\int_{\partial G}\left(\nu(\cdot) \times \operatorname{curl}\frac{\gamma^*(\xi')}{|\cdot - \xi'|}\right)\mathrm{d}\Omega'$$

will be given. With regard to the set $\mathcal{T}_{C^0}(\partial G)$, we deduce

$$R\gamma^*(\xi) = -\frac{1}{2\pi}\int_{\partial G}\left(\frac{\partial}{\partial\nu}\frac{1}{r}\right)(\xi, \xi')\gamma^*(\xi')\,\mathrm{d}\Omega' +$$

$$+\frac{1}{2\pi}\int_{\partial G}\operatorname{grad}\frac{1}{r}(\xi, \xi')(\nu(\xi), \gamma^*(\xi'))\,\mathrm{d}\Omega'$$

(cf. [6, p. 137]). Henceforth, we choose $\alpha \in (0, 1)$. The vector $\gamma^* := -(\nu \times \mathbf{u})$ belongs to $\mathcal{T}_{C^{1+\alpha}}(\partial G)$ and satisfies the integral equation

$$(I + R)\gamma^*(\xi) = \frac{1}{2\pi}\,\mathbf{f}(\xi) \quad \text{with } \xi \in \partial G \text{ and}$$

$$\mathbf{f}(\xi) := \nu(\xi) \times \left[\left(\operatorname{grad}\left(\int_G \frac{\varepsilon'}{r}\,dx' + \int_{\partial G} \frac{\varepsilon^{*\prime}}{r}\,d\Omega'\right)\right)(\xi) - \left(\operatorname{curl}\int_G \frac{\gamma'}{r}\,dx'\right)(\xi)\right]$$

(cf. [6, p. 126]). As in section 2, we obtain

$$\|\mathbf{f}\|_{C^{1+\alpha}(\partial G)} \leqslant c(\|\varepsilon\|_\alpha + \|\gamma\|_\alpha + \|\varepsilon^*\|_{C^{1+\alpha}(\partial G)}).$$

The Riesz number of R *is* 1 [6, *p.* 152]. Using [6, p. 139, p. 150, p. 141], we therefore realize that there exists a basis $\{\gamma_1^*, \dots, \gamma_n^*\}$ of $\mathcal{N}(I + R)$ with

$$\int_{\partial G} (\gamma_i^*, \hat{\mathbf{z}}^k)\,d\Omega = \delta_i^k$$

(cf. [6, p. 147]). For each $\mathbf{h} \in \mathcal{R}(I + R) \subset C^{1+\alpha}(\partial G)$ we find a uniquely determined solution $\mathbf{g} \in C^{1+\alpha}(\partial G)$ of the problem

$$(I + R)\mathbf{g} = \mathbf{h} \int_{\partial G} (\mathbf{g}, \hat{\mathbf{z}}^i)\,d\Omega = \Gamma^i, \quad 1 \leqslant i \leqslant n. \tag{3.3}$$

Moreover, there exists a unique solution $\mathbf{g}_1 \in C^{1+\alpha}(\partial G)$ of

$$(I + R)\mathbf{g}_1 = \mathbf{h}, \quad \int_{\partial G} (\mathbf{g}_1, \hat{\mathbf{z}}^i)\,d\Omega = 0, \quad 1 \leqslant i \leqslant n. \tag{3.4}$$

We define

$$\mathbf{g}_0 \equiv \mathbf{g}_0(\Gamma^1, \dots, \Gamma^n) := \sum_{i=1}^n \Gamma^i \gamma_i^*.$$

Consequently, $\mathbf{g}_0$ is an element of $\mathcal{N}(I + R)$. Altogether, each solution $\mathbf{g}$ of (3.3) may uniquely be decomposed into $\mathbf{g}_0$ and $\mathbf{g}_1$:

$$\mathbf{g} = \mathbf{g}_0 + \mathbf{g}_1.$$

We take from [2, 8] that $R|_{C^{1+\alpha}(\partial G)}$ is a compact operator in $\mathcal{T}_{C^{1+\alpha}}(\partial G)$. Therefore, $(I + R)|_{C^{1+\alpha}(\partial G)}$ is a Fredholm operator and $\mathcal{R}(I + R)|_{C^{1+\alpha}(\partial G)}$ is closed. Furthermore, $I + R$ maps $\mathcal{T}_{C^{1+\alpha}}(\partial G)\backslash\mathcal{N}(I + R)$ into $\mathcal{T}_{C^{1+\alpha}}(\partial G)$. The restriction to the image of this operator constitutes a homeomorphic mapping. As the solution $\mathbf{g}_1$ of (3.4) belongs to $\mathcal{R}(I + R)|_{C^{1+\alpha}(\partial G)}$, we obtain, according to Banach's open mapping theorem, the estimate

$$d\,\|\mathbf{g}_1\|_{C^{1+\alpha}(\partial G)} \leqslant \|(I + R)\mathbf{g}_1\|_{C^{1+\alpha}(\partial G)}$$

with a constant $d > 0$, and thus

$$\|\mathbf{g}\|_{C^{1+\alpha}(\partial G)} \leqslant c \sum_{i=1}^n |\Gamma^i| + \frac{1}{d}\,\|\mathbf{h}\|_{C^{1+\alpha}(\partial G)}.$$

Hence Theorem 3.1 is proved. □

In the case that $n = 0$, the space $\mathcal{N}(I + R)$ consists only of $\{0\}$. Consequently, we then get

$$\| \nabla \mathbf{u} \|_\alpha \leqslant c(\| \varepsilon \|_\alpha + \| \gamma \|_\alpha + \| \varepsilon^* \|_{C^{1+\alpha}(\partial G)}). \tag{3.5}$$

Remark. Let $\{\hat{c}^1, \dots, \hat{c}^n\}$ be a basis of the first homology group with regard to $\hat{G}$. This basis is given by closed curves around the handles of G. For each $\hat{c}^i$ we denote by $\hat{\tau}$ the corresponding unit tangent vector. Provided

$$(\nu, \operatorname{curl} \mathbf{u}) = 0,$$

we take from [3, p. 72] that

$$\int_{\partial G} (\gamma^*, \hat{\mathbf{z}}^i)\, \mathrm{d}\Omega = \int_{\hat{c}^i} (\hat{\tau}, \mathbf{u})\, \mathrm{d}s, \quad \mathrm{i} = 1, \dots, n. \tag{3.6}$$

Then the Γ^i, $(i = 1, \dots, n)$ are equal to conventional circulations.

References

1. Agmon, S., Douglis, A. and Nirenberg, L., 'Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions I', *Commun. Pure Appl. Math.*, **12,** 623–727 (1959).
2. Heinemann, U., 'Die regularisierende Wirkung der Randintegraloperatoren der klassischen Potentialtheorie in den Räumen hölderstetiger Funktionen', *Diplomarbeit*, Bayreuth, 1992.
3. Kress, R., 'Grundzüge einer Theorie der verallgemeinerten harmonischen Vektorfelder', *Methoden Verfahren Math. Phys.*, **2,** 49–83 (1969).
4. Morrey, C. B., *Multiple Integrals in the Calculus of Variations*, Springer, Berlin, 1966.
5. Schwarz, G., 'Hodge decomposition—a method for solving boundary value problems', Lecture Notes in Mathematics Vol. 1607, Springer, Berlin, 1995.
6. Wahl, W. von, 'Vorlesung über das Außenraumproblem für die instationären Gleichungen von Navier–Stokes, Rudolph–Lipschitz–Vorlesung', *Sonderforschungsbereich* 256, *Nichtlineare Partielle Differentialgleichungen*, **11,** Bonn, 1989.
7. Wahl, W. von, 'Estimating ∇u by div u and curl u', *Math. Methods Appl. Sci.*, **15,** 123–143 (1992).
8. Wiegner, M., 'Schauder estimates for boundary layer potentials', *Math. Methods Appl. Sci.*, **16,** 877–894 (1993).

