Regularity of Weak Solutions of Semilinear Parabolic Systems of Arbitrary Order

Hans-Christoph Grunau
Wolf von Wahl

Abstract

Let u be a weak solution of the initial boundary value problem for the semilinear parabolic system of order $2m$: $u'(t) + Au(t) + f(t, ., u, ..., \nabla^m u) = 0$. Let f satisfy controllable growth conditions. Then u is smooth.

This result is proved by a kind of continuity method, where the time t is the parameter of continuity.

Classification: 35D10, 35K60, 35K50.

1. Introduction

We are interested in the regularity problem for weak solutions u to the initial boundary value problem for systems

$$u'(t) + A(t)u + f(t, ., u, ..., \nabla^m u) = 0, t \in [0, T]. \tag{1}$$

$A(t)$ is a positive uniformly elliptic operator of order $2m$ in a bounded domain $\Omega \subset \mathbb{R}^n$. The data (e.g. the initial value, the function f, the coefficient functions of A, etc.) are assumed to be sufficiently smooth.

A weak solution u is understood to be in the space $L^2((0, T), H_0^{m,2}(\Omega)) \cap L^\infty((0, T), L^2(\Omega))$. On f we impose the so called controllable growth conditions

$$|f(t, ., u, ..., \nabla^m u)| \leq c(1 + \sum_{\nu=0}^{m} |\nabla^\nu u|^{\frac{n+4m}{n+2\nu}}). \tag{2}$$

These are the weakest growth conditions under which a weak solution as above can be dealt with by using testing functions χ with $\chi' \in L^2((0, T), L^2(\Omega))$, $\chi \in L^2((0, T), H_0^{m,2}(\Omega))$.

In [Wa1] single equations were studied. Additionally the sign condition

$$u \cdot f(t, ., u, ..., \nabla^m u) \geq c' u - c'' u^2, \tag{3}$$

was imposed. It was shown that any weak solution is regular. Moreover the condition (3) enabled the author to give a-priori estimates for $\int\limits_0^T \|u'\|_p^p \, ds + \int\limits_0^T \|u\|_{2m,p}^p \, ds$, if $p > n+1$. In some sense, [Wa1] extends a result of Ladyženskaja, Solonnikov and Ural'ceva [LSU, V.2] concerning quasilinear second order equations to arbitrary order.

In the case of elliptic systems, Luckhaus [Lu] discussed the regularity problem for weak solutions completely separately from the problem of finding a-priori estimates. He showed that any weak solution of a semilinear elliptic system is regular if (beside some smoothness assumptions) only controllable growth conditions are imposed on the nonlinear part.

Because of the far reaching analogy between elliptic and parabolic problems it seems reasonable to conjecture that regularity will still hold if the sign condition (3) is omitted, i.e. if no a-priori energy bound is available.

The works of Campanato [Ca] and Marino, Maugeri [MM] are in this direction, but their crucial point is different from that of the present paper. They allow the coefficient functions of the principal part A to depend (nonlinearly) also on $u, ..., \nabla^{m-1} u$, on the other hand they request A to satisfy quite a strong ellipticity condition and the lower order term f to fulfill strictly controlled growth conditions, i.e. in (2) $\frac{n+4m}{n+2\nu}$ is replaced by $\frac{n+4m}{n+2\nu} - \varepsilon$, $\varepsilon > 0$. Then they prove partial regularity of weak solutions. To our knowledge there are no other results concerning the regularity of weak solutions of nonlinear parabolic equations or systems with an elliptic part of arbitrary order. The reason may be that it is difficult to find appropriate testing vectors if the elliptic part is in general form.

Here we employ a different method which is explained in what follows. We show that, in the semilinear case, it is possible to omit the sign condition (3) and to treat systems too. More precisely: A is assumed to be a uniformly elliptic matrix differential operator of order $2m$ in divergence form, elliptic in the sense of Legendre-Hadamard. For simplicity A is supposed to depend not on t. On f only the controllable growth condition (2) is imposed. Then any weak solution of the initial-boundary value problem for the system (1) is smooth.

The central idea of our proof is to reconstruct u as a strong solution. The main problem we have to overcome is the lack of uniqueness for the system (1). Consequently, instead of (1), we consider a modified system, which has the unique weak solution u and which, on the other hand, allows the construction of a global strong solution. The last step is carried out by using a continuity method with time t as parameter of continuity. Doing so we make extensive use of the properties of the given weak solution u. The fundamental tool from linear theory is a maximal L^r-regularity result ([CaV], [Si], see Lemma 4) for abstract evolution equations in Hilbert spaces.

The choice of the modified problem is quite comparable to that taken in [Wa1] in the case of a single equation. In [Wa1], however, the parameter of continuity was selected in a different way. Therefore the results in [Wa1] are essentially restricted to a single equation under the sign condition (3). It is the combination of two particular choices which makes a proof possible in the present case: The choice of the modified problem and the choice of time as parameter of continuity. The latter can be found in [Wa2, chapt. V] already. But to our knowledge this particular combination has not been used previously to prove regularity results.

In section 2 we give the precise formulation of our assumptions and the definition of weak solutions and we present the regularity result. Section 3 is devoted to its proof as sketched above.

2. The Result

Most of our notation is standard; $\|.\|_{k,p}$ denotes the norm in the vector-valued Sobolev space $H^{k,p}(\Omega) := H^{k,p}(\Omega, \mathbb{R}^N)$ and $(.,.)$ the duality product between $L^p(\Omega, \mathbb{R}^N)$ and $L^{p'}(\Omega, \mathbb{R}^N)$.

We will prove the regularity result under the following assumptions:

(A I) $n, m, N \in \mathbb{N},\ n \geq 3$. $\Omega \subset \mathbb{R}^n$ is a bounded smooth domain of class C^{4m} with outer unit normal ν. For simplicity we assume $n > 2m$.

(A II) $A = \sum_{|\alpha|,|\beta| \leq m} (-1)^{|\alpha|} D^\alpha (A_{\alpha\beta}(x) D^\beta)$ is a positive uniformly elliptic matrix differential operator, i.e.: $A_{\alpha\beta} : \bar{\Omega} \to \mathbb{R}^{N \times N}$ are matrices of class $C^m(\bar{\Omega})$, where $\alpha, \beta \in \mathbb{N}_0^n$ are multiindices of length n, $D^\alpha = \prod_{j=1}^n (\frac{\partial}{\partial x_j})^{\alpha_j}$.

There is a constant $M > 0$ such that

$$M|\xi|^{2m}|\zeta|^2 \geq \sum_{i,j=1}^{N} \sum_{|\alpha|=|\beta|=m} A_{\alpha\beta}^{ij}(x)\xi^\alpha \xi^\beta \zeta_i \zeta_j \geq M^{-1}|\xi|^{2m}|\zeta|^2$$

holds for all $x \in \bar{\Omega},\ \xi \in \mathbb{R}^n,\ \zeta \in \mathbb{R}^N$.

Without loss of generality we may assume Gårding's inequality $\sum\limits_{|\alpha|,|\beta|\leq m} (A_{\alpha\beta}D^\beta u, D^\alpha u) \geq C_0 \|u\|_{m,2}^2$ for all $\mathbb{R}^N$-vector functions $u \in H_0^{m,2}(\Omega)$ with a positive constant C_0.

(A III) Let k_j be the number of multiindices α with $|\alpha| = \alpha_1 + ... + \alpha_n = j$.

$$f : \mathbb{R}_0^+ \times \bar{\Omega} \times \mathbb{R}^N \times \mathbb{R}^{N\cdot k_1} \times ... \times \mathbb{R}^{N\cdot k_m} \longrightarrow \mathbb{R}^N$$

is a continuous function, satisfying the growth condition

$$|f(t, x, p_0, ..., p_m)| \leq K(1 + \sum_{\nu=0}^{m} |p_\nu|^{\frac{n+4m}{n+2\nu}}).$$

In order to be able to estimate the nonlinear terms we prove the following lemma.

Lemma 1. Let $w_1, w_2, w_3 \in H_0^{m,2}(\Omega)$. For $\nu = 0, ..., m$ let γ_ν be real numbers satisfying the conditions:

$$\frac{\nu(8m-4\nu)}{n+2\nu} \leq \gamma_\nu \leq \frac{m(8m-4\nu)}{n+2\nu} \text{ and } 0 < \gamma_\nu < 2m.$$

Suppose that there is constant L such that $\|w_1\|_{0,2}, \|w_2\|_{0,2} \leq L$. Then we have

$$\begin{aligned}
&\sum_{\nu=0}^{m} \int_\Omega \left| |\nabla^\nu w_1|^{\frac{n+4m}{n+2\nu}} - |\nabla^\nu w_2|^{\frac{n+4m}{n+2\nu}} \right| \cdot |w_3| \, dx \\
&\leq C \sum_{\nu=0}^{m} \left(\|w_1\|_{m,2}^{\frac{\gamma_\nu}{2m}} + \|w_2\|_{m,2}^{\frac{\gamma_\nu}{2m}} \right) \|w_1 - w_2\|_{m,2} \cdot \|w_3\|_{m,2}^{1-\frac{\gamma_\nu}{2m}} \cdot \|w_3\|_{0,2}^{\frac{\gamma_\nu}{2m}},
\end{aligned}$$

where C only depends on N, n, m, Ω, L and γ_ν $(\nu = 0, ..., m)$.

Proof. We remark that there exist numbers γ_ν as above, e.g. we may put $\gamma_0 = \frac{2m^2}{n}$, $\gamma_\nu = \frac{\nu(8m-4\nu)}{n+2\nu}$ for $\nu = 1, ..., m$. By virtue of $n > 2m$ we have $0 < \gamma_\nu < 2m$.

For $a, b \geq 0$ the mean value theorem gives

$$\left|a^{\frac{n+4m}{n+2\nu}} - b^{\frac{n+4m}{n+2\nu}}\right| \leq \frac{n+4m}{n+2\nu}\left(a^{\frac{4m-2\nu}{n+2\nu}} + b^{\frac{4m-2\nu}{n+2\nu}}\right)|a-b|.$$

From this there arises

$$\begin{aligned}
&\sum_{\nu=0}^{m}\int_\Omega \left||\nabla^\nu w_1|^{\frac{n+4m}{n+2\nu}} - |\nabla^\nu w_2|^{\frac{n+4m}{n+2\nu}}\right| \cdot |w_3|\,dx \\
&\leq C(m,n)\sum_{\nu=0}^{m}\int_\Omega \left(|\nabla^\nu w_1|^{\frac{4m-2\nu}{n+2\nu}} + |\nabla^\nu w_2|^{\frac{4m-2\nu}{n+2\nu}}\right) \cdot |\nabla^\nu(w_1-w_2)| \cdot |w_3|\,dx \\
&\leq C(m,n)\sum_{\nu=0}^{m}\left(\|\nabla^\nu w_1\|_{0,\frac{4m-2\nu}{n+2\nu}\cdot\frac{2n}{4m-\gamma_\nu-2\nu}} + \|\nabla^\nu w_2\|_{0,\frac{4m-2\nu}{n+2\nu}\cdot\frac{2n}{4m-\gamma_\nu-2\nu}}\right)^{\frac{4m-2\nu}{n+2\nu}} \\
&\cdot \|\nabla^\nu(w_1-w_2)\|_{0,\frac{2n}{n-2m+2\nu}} \cdot \|w_3\|_{0,\frac{2n}{n-2m+\gamma_\nu}},
\end{aligned}$$

where we applied Hölder's inequality with exponents $p_{\nu,1} = \frac{2n}{4m-\gamma_\nu-2\nu} \geq \frac{2n}{4m} > 1$, $p_{\nu,2} = \frac{2n}{n-2m+2\nu} \geq 2$, $p_{\nu,3} = \frac{2n}{n-2m+\gamma_\nu} > 2$; $\frac{1}{p_{\nu,1}} + \frac{1}{p_{\nu,2}} + \frac{1}{p_{\nu,3}} = 1$. For $\nu = m$ we have $\frac{4m-2\nu}{n+2\nu} \cdot p_{\nu,1} = p_{\nu,2} = 2$, $2 \leq p_{\nu,3} < \frac{2n}{n-2m}$; i.e. $\nabla^m w_i$ only occurs in the L^2-norm, these terms don't need to be interpolated. For $0 \leq \nu < m$ we have $2 \leq \frac{4m-2\nu}{n+2\nu} \cdot p_{\nu,1} \leq \frac{2n}{n-2(m-\nu)} = p_{\nu,2}$, $2 < p_{\nu,3} < \frac{2n}{n-2m}$; $p_{\nu,2}$ is the limiting exponent for the imbedding $H^{m,2}(\Omega) \hookrightarrow H^{\nu,p}(\Omega)$.

Now we employ the general imbedding and interpolation inequality ([Fr, p. 27]):

$$\|\nabla^\nu u\|_p \leq C(\|u\|_{m,2})^a(\|u\|_{0,2})^{1-a}.$$

Here ν is an integer, $0 \leq \nu < m$, $a \in [\frac{\nu}{m}, 1]$ is a real number, $m - \frac{n}{2} - \nu$ is not a nonnegative integer and $\frac{1}{p} = \frac{n+2\nu}{2n} - a\frac{m}{n}$. The constant C only depends on m, n, N, Ω, a, ν.

We remark that the restrictions on γ_ν ensure all exponents a to be admissible. We obtain:

$$\begin{aligned}
&\sum_{\nu=0}^{m}\int_\Omega \left||\nabla^\nu w_1|^{\frac{n+4m}{n+2\nu}} - |\nabla^\nu w_2|^{\frac{n+4m}{n+2\nu}}\right| \cdot |w_3|\,dx \\
&\leq \sum_{\nu=0}^{m} C(m,n,N,\Omega,\nu,\gamma_\nu)\Bigg(\|w_1\|_{m,2}^{\frac{n+2\nu}{4m-2\nu}\cdot\frac{\gamma_\nu}{2m}}\|w_1\|_{0,2}^{1-\frac{n+2\nu}{4m-2\nu}\cdot\frac{\gamma_\nu}{2m}}
\end{aligned}$$

$$
\begin{aligned}
&+\|w_2\|_{m,2}^{\frac{n+2\nu}{4m-2\nu}\cdot\frac{\gamma_\nu}{2m}} \cdot \|w_2\|_{0,2}^{1-\frac{n+2\nu}{4m-2\nu}\cdot\frac{\gamma_\nu}{2m}}\Big)^{\frac{4m-2\nu}{n+2\nu}} \\
&\cdot \|w_1 - w_2\|_{m,2} \cdot \|w_3\|_{m,2}^{1-\frac{\gamma_\nu}{2m}} \cdot \|w_3\|_{0,2}^{\frac{\gamma_\nu}{2m}} \\
&\leq \sum_{\nu=0}^{m} C(m,n,N,\Omega,L,\nu,\gamma_\nu)(\|w_1\|_{m,2}^{\frac{\gamma_\nu}{2m}} + \|w_2\|_{m,2}^{\frac{\gamma_\nu}{2m}})\|w_1 - w_2\|_{m,2} \\
&\cdot \|w_3\|_{m,2}^{1-\frac{\gamma_\nu}{2m}} \|w_3\|_{0,2}^{\frac{\gamma_\nu}{2m}}.
\end{aligned}
$$

■

Definition. Let $\phi \in L^2(\Omega)$, $u : (0,T) \times \Omega \to \mathbb{R}^N$ be of class $L^2((0,T), H_0^{m,2}(\Omega)) \cap L^\infty((0,T), L^2(\Omega))$. u is called a **weak solution** of the initial boundary value problem

$$
\begin{aligned}
&\frac{\partial}{\partial t}u(t,x) + Au(t,x) + f(t,x,u,...,\nabla^m u) = 0,\ (t,x) \in (0,T)\times\Omega, \\
(4)\qquad &(\frac{\partial u}{\partial \nu})^j|_{\partial\Omega} = 0,\ j = 0,...,m-1,\ t \in [0,T], \\
&u(0,x) = \phi(x),\ x \in \Omega,
\end{aligned}
$$

if the relation

$$
\begin{aligned}
&-\int_0^T (u(s),\chi'(s))\,ds + \sum_{|\alpha|,|\beta|\leq m} \int_0^T (A_{\alpha\beta}D^\beta u(s), D^\alpha\chi(s))\,ds \\
&+\int_0^T (f(s,.,u,...,\nabla^m u),\chi(s))\,ds = (\phi,\chi(0))
\end{aligned}
$$

holds for all $\mathbb{R}^N$-valued functions $\chi \in L^2((0,T), H_0^{m,2}(\Omega))$ with $\chi' \in L^2((0,T), L^2(\Omega)), \chi(T) = 0$.

Remark. By Sobolev's imbedding theorem in one dimension, we have $\chi \in C^0([0,T], L^2(\Omega))$, so $\chi(0)$, $\chi(T)$ make sense. Furthermore $\int_0^T (f(s,.,u,...,\nabla^m u),\chi(s))\,ds$ is well defined, this is verified using (A III) and

Lemma 1. Namely, we put $w_1 := u,\ w_2 := 0,\ w_3 := \chi$ and obtain

$$
\begin{aligned}
&|\int\limits_0^T (f(s,.,u,...,\nabla^m u),\chi(s))\,ds| \\
&\leq C(\operatorname{ess}\sup_{0<s<T} \|u(s)\|_{0,2}) \\
&\quad \cdot \int\limits_0^T \{\|\chi(s)\|_{0,2} + \sum_{\nu=0}^{m} \|u(s)\|_{m,2}^{1+\frac{\gamma_\nu}{2m}} \cdot \|\chi(s)\|_{m,2}^{1-\frac{\gamma_\nu}{2m}} \cdot \|\chi(s)\|_{0,2}^{\frac{\gamma_\nu}{2m}}\}\,ds \\
&\leq C(\operatorname{ess}\sup_{0<s<T} \|u(s)\|_{0,2}) \cdot (1 + \sum_{\nu=0}^{m} (\int\limits_0^T \|u(s)\|_{m,2}^2\,ds)^{\frac{1}{2}+\frac{\gamma_\nu}{4m}}) \\
&\quad \cdot \{(\int\limits_0^T \|\chi(s)\|_{m,2}^2\,ds)^{\frac{1}{2}} + \operatorname{ess}\sup_{0<s<T} \|\chi(s)\|\},
\end{aligned}
$$

the numbers $\gamma_\nu \in (0, 2m),\ \nu = 0, ..., m$ are defined in Lemma 1. This shows further, that $t \mapsto f(t,.,u,...,\nabla^m u)$ is in $[L^\infty((0,T),L^2(\Omega)) \cap L^2((0,T),H_0^{m,2}(\Omega))]'$.

Now we are able to state our main result:

Theorem. Let the assumptions (A I)-(A III) be satisfied, $\phi \in H^{2m,2}(\Omega) \cap H_0^{m,2}(\Omega),\ u \in L^2((0,T),H_0^{m,2}(\Omega)) \cap L^\infty((0,T),L^2(\Omega))$ be a weak solution of the initial boundary value problem (4).

Then for every $1 < r < \infty$ we have

$$
\begin{aligned}
u &\in L^r((0,T),H^{2m,2}(\Omega) \cap H_0^{m,2}(\Omega)), \\
u' &\in L^r((0,T),L^2(\Omega)).
\end{aligned}
$$

Remark. Using the theory of L^p-semigroups, one can obtain higher regularity, namely $u \in L^r((0,T),H^{2m,p}(\Omega))$ and under appropriate smoothness conditions on A, f and ϕ also $u \in C^\mu([0,T],C^{2m,\mu}(\bar{\Omega})),\ u' \in C^\mu([0,T],C^\mu(\bar{\Omega}))$, cf. the corresponding remark in [Wa1].

But in this paper we want to confine ourselves to L^2-theory because the linear results needed here can be proved in a quite elementary manner. The constants in the a-priori estimates as in Lemma 4 below can be calculated explicitly, in particular their dependence on the time interval $[0, T]$. They can't blow up as $T \to 0$, this fact is important for the proof below, see section 3.5.

3. Proof of the theorem

3.1. For $\hat{T} > 0$ let us consider the following modified initial boundary value problem:

$$\begin{aligned} &\frac{\partial}{\partial t} w(t,x) + Aw(t,x) + F(t,x,w,...,\nabla^m w) = 0, \ (t,x) \in (0,\hat{T}) \times \Omega, \\ &(\frac{\partial}{\partial \nu})^j w(t)|_{\partial\Omega} = 0, \ j = 0,...,m-1, \\ &w(0) = \phi, \end{aligned} \tag{5}$$

where $F(t,x,p_0,...,p_m) = K \cdot (1 + \sum_{\nu=0}^{m} |p_\nu|^{\frac{n+4m}{n+2\nu}}) \cdot q(t,x), \ q : [0,\hat{T}] \times \bar{\Omega} \to \mathbb{R}^N$,

$$q(t,x) = \begin{cases} \dfrac{1}{K(1 + \sum_{\nu=0}^{m} |\nabla^\nu u(t,x)|^{\frac{n+4m}{n+2\nu}})} f(t,x,u,...,\nabla^m u), \\ \qquad\qquad\qquad\qquad\qquad \text{if } t \in [0,T] \cap [0,\hat{T}], \\ 0, \qquad\qquad \text{if } t \in (T,\infty) \cap [0,\hat{T}]. \end{cases}$$

We have $|q(t,x)| \leq 1$, so that a weak solution of (5) is defined in exactly the same way as for (4) above.

Interpreting F as inhomogeneity, linear theory (see [Li, pp. 52-55]) yields: $w \in C^0([0,\hat{T}], L^2(\Omega))$, where w is any weak solution of (5). To apply [Li], a standard approximation procedure (truncation of F in each summand) is needed. The truncated nonlinearities with the given weak solution w serve as inhomogeneous terms. Then we take the energy equality for the difference of two approximants. Application of Lemma 1 to the truncated nonlinearities in particular shows the convergence of the approximants in $C^0([0,\hat{T}], L^2(\Omega))$. Note now, that linear systems like $\tilde{w}' + A\tilde{w} + F = 0$ have at most one weak solution, because A is positive. Moreover, the energy equality holds, i.e. if

$w_1, w_2 \in L^2((0,\hat{T}), H_0^{m,2}(\Omega)) \cap L^\infty((0,\hat{T}), L^2(\Omega))$ are weak solutions of (5), then for $t \in [0,\hat{T}]$ we have

$$\frac{1}{2}\|(w_1 - w_2)(t)\|_{0,2}^2 + \int_0^t \sum_{|\alpha|,|\beta| \le m} (A_{\alpha\beta} D^\beta (w_1 - w_2), D^\alpha (w_1 - w_2))(s)\, ds$$

(6)

$$= -\int_0^t (F(s,.,w_1,...,\nabla^m w_1) - F(s,.,w_2,...,\nabla^m w_2), w_1(s) - w_2(s))\, ds.$$

3.2. Lemma 2. For given $\varepsilon > 0$ we have

$$|(F(t,.,w_1,...,\nabla^m w_1) - F(t,.,w_2,...,\nabla^m w_2), w_1(t) - w_2(t))|$$
$$\le C(n,m,N,\Omega,K,\|w_1\|_{0,2},\|w_2\|_{0,2},\varepsilon)(\|w_1\|_{m,2}^2 + \|w_2\|_{m,2}^2)\|w_1 - w_2\|_{0,2}^2$$
$$+\varepsilon\|w_1 - w_2\|_{m,2}^2.$$

Proof. We use Lemma 1 with $w_3 := w_1(t) - w_2(t)$; the numbers $\gamma_\nu \in (0,2m)$ are defined there. Because $|q(t,x)| \le 1$ we have

$$|(F(t,.,w_1,...,\nabla^m w_1) - F(t,.,w_2,...,\nabla^m), w_1(t) - w_2(t))|$$
$$\le K \cdot \sum_{\nu=0}^m \int_\Omega |\, |\nabla^\nu w_1(t)|^{\frac{n+4m}{n+2\nu}} - |\nabla^\nu w_2(t)|^{\frac{n+4m}{n+2\nu}}| \cdot |w_1(t) - w_2(t)|\, dt$$
$$\le C(\|w_1\|_{0,2},\|w_2\|_{0,2})\{\sum_{\nu=0}^m (\|w_1\|_{m,2}^{\frac{\gamma_\nu}{2m}} + \|w_2\|_{m,2}^{\frac{\gamma_\nu}{2m}}) \cdot \|w_1 - w_2\|_{m,2}^{2-\frac{\gamma_\nu}{2m}}$$
$$\cdot\|w_1 - w_2\|_{0,2}^{\frac{\gamma_\nu}{2m}}\}$$
$$\le \varepsilon\|w_1 - w_2\|_{m,2}^2 + C(\varepsilon,\|w_1\|_{0,2},\|w_2\|_{0,2}) \cdot (\|w_1\|_{m,2}^2 + \|w_2\|_{m,2}^2)$$
$$\cdot\|w_1 - w_2\|_{0,2}^2. \qquad \blacksquare$$

An immediate consequence of the energy equality (6), Lemma 2, Gårding's inequality and Gronwall's lemma is the following uniqueness result.

Lemma 3. The initial boundary value problem (5) has at most one weak solution $w \in L^2((0,\hat{T}), H_0^{m,2}(\Omega)) \cap L^\infty((0,\hat{T}), L^2(\Omega))$.

Remark. u is a weak solution of (5) on $[0,T]$ and hence coincides with the unique weak solution of (5) on $[0,T]$. Furthermore we have $u \in C^0([0,T], L^2(\Omega))$.

3.3. In what follows we repeatedly have to interpolate $\|\nabla^\nu v\|_{0,p}$ between $\|v\|_{0,2}$ and $\|v\|_{2m,2}$ according to the general interpolation and imbedding inequality [Fr, p. 27]. We remark that this inequality also holds in the case $2m - \frac{n}{2} - \nu \in \mathbb{N}_0$, if p is not too large, e.g. if $p \le \frac{2n}{n-2m}$.

Our aim is to construct a strong solution to (5) on an interval larger than $[0,T]$. So for the following we fix $\hat{T} > T$, e.g. we may put $\hat{T} := T+1$.

We consider A as closed operator in $L^2(\Omega)$ with domain $D(A) = H^{2m,2}(\Omega) \cap H_0^{m,2}(\Omega)$. It is well known that $-A$ generates an analytic semigroup in $L^2(\Omega)$, which exponentially decays. For the rest of the proof we fix without loss of generality $r > \frac{n+4m}{2m}$. In particular we have $r > 2$. The crucial tool from linear theory is the following result.

Lemma 4. Let $0 < \tilde{T} \le \hat{T},\ f \in L^r((0,\tilde{T}), L^2(\Omega)),\ 1 - \frac{1}{r} < \kappa \le 1$, $\Psi \in D(A^\kappa)$. Then there is exactly one solution $v \in L^r((0,\tilde{T}), H^{2m,2}(\Omega) \cap H_0^{m,2}(\Omega)),\ v' \in L^r((0,\tilde{T}), L^2(\Omega))$ of

$$v' + Av = f,\ v(0) = \Psi.$$

Moreover the following a-priori-estimate holds:

$$\int_0^{\tilde{T}} \|v(s)\|_{2m,2}^r \, ds + \int_0^{\tilde{T}} \|v'(s)\|_{0,2}^r \, ds$$

$$\leq C_1\{\|A^\kappa\Psi\|_{0,2}^r + \int_0^{\tilde{T}} \|f(s)\|_{0,2}^r\, ds\},$$

where C_1 is a constant, which only depends on $r, \Omega, A, \hat{T}$.

This was shown first within an abstract framework in [Si] if $\Psi = 0$. It's easily checked that $\Psi \in D(A^\kappa)$ with $1 - \frac{1}{r} < \kappa \leq 1$ is sufficient for the estimate in question. This condition may be relaxed, cf. [CaV]. Lemma 4 is valid for any $r \in (1, \infty)$, because $L^2(\Omega)$ is a Hilbert space. For the estimate in question with $L^2(\Omega)$ replaced by $L^p(\Omega)$ see [Wa4], [CaV], [Wa3, chapt. I.10 and II.3].

3.4. The following estimate for the nonlinearity F will be used for proving the existence of a local strong solution of (5).

Lemma 5. Let $v, w \in L^r((0,\tilde{T}), H^{2m,2}(\Omega)), v', w' \in L^r((0,\tilde{T}), L^2(\Omega))$. For every $\varepsilon > 0$ we have

$$\begin{aligned}
&\int_0^{\tilde{T}} \|F(s,.,v,...,\nabla^m v) - F(s,.,w,...,\nabla^m w)\|_{0,2}^r\, ds \\
&\leq C(r)\{\varepsilon(\sum_{\nu=0}^{m} \sup_{s\in[0,\tilde{T}]} (\|v(s)\|_{0,2}^{\alpha_\nu} + \|w(s)\|_{0,2}^{\alpha_\nu})) \\
&\qquad\qquad \cdot \int_0^{\tilde{T}} \|v(s) - w(s)\|_{2m,2}^r\, ds\} \\
&+C(\varepsilon, n, m, N, \Omega, K, r)(\sup_{s\in[0,\tilde{T}]} \|v(s) - w(s)\|_{0,2}^r) \\
&\qquad\qquad \cdot \int_0^{\tilde{T}} (\|v(s)\|_{2m,2}^r + \|w(s)\|_{2m,2}^r)\, ds
\end{aligned}$$

with some positive numbers $\alpha_\nu > 0$.

Proof. We use similar arguments as in the proof of Lemma 1. For $\nu = 0, ..., m$ we fix real numbers $p_\nu \in (2, \infty)$ satisfying

$$\frac{2n+4\nu}{n+4\nu-4m} \geq p_\nu \quad \geq \quad \frac{2n^2+4n\nu}{n^2+4n\nu+4\nu^2-4mn+16m^2-16m\nu}, \quad \text{if } \nu > 2m-\frac{n}{2},$$

$$\min\{\frac{2n+4\nu}{n+4\nu-4m}, \frac{2n}{n-2m}\} \geq p_\nu \quad \geq \quad \frac{2n^2+4n\nu}{n^2+4n\nu+8m^2-4mn-4m\nu}, \quad \text{if } \nu > m-\frac{n}{4} \text{ and } \nu \leq 2m-\frac{n}{2},$$

$$\frac{2n}{n-2m} \geq p_\nu \geq \frac{2n^2+4n\nu}{n^2+4n\nu+8m^2-4mn-4m\nu}, \text{ if } \nu \leq m-\frac{n}{4}.$$

We remark that the upper bound for p_ν is greater than the lower bound and than 2. These conditions imply

$$\begin{aligned} &2 \leq p_\nu \leq \frac{2n}{n-4m+2\nu}, \\ &2 \leq \frac{2p_\nu}{p_\nu-2} \cdot \frac{4m-2\nu}{n+2\nu} \leq \frac{2n}{n-4m+2\nu} \text{ if } \nu > 2m-\frac{n}{2}, \end{aligned}$$

$$\begin{aligned} &2 \leq p_\nu \leq \frac{2n}{n-2m}, \\ &2 \leq \frac{2p_\nu}{p_\nu-2} \cdot \frac{4m-2\nu}{n+2\nu} \leq \frac{2n}{n-2m} \text{ if } \nu \leq 2m-\frac{n}{2}, \end{aligned}$$

and the interpolation exponents below are admissable. For $s \in (0, \tilde{T})$ we have

$$\begin{aligned} &\|F(s,.,v,...,\nabla^m v) - F(s,.,w,...,\nabla^m w)\|_{0,2} \\ &\leq K \| \sum_{\nu=0}^{m} (|\nabla^\nu v|^{\frac{n+4m}{n+2\nu}} - |\nabla^\nu w|^{\frac{n+4m}{n+2\nu}})\|_{0,2} \\ &\leq C(m,n,N,K) \sum_{\nu=0}^{m} \|\nabla^\nu (v-w)\|_{0,p_\nu} \\ &\quad \cdot (\| |\nabla^\nu v|^{\frac{4m-2\nu}{n+2\nu}} \|_{0,\frac{2p_\nu}{p_\nu-2}} + \| |\nabla^\nu w|^{\frac{4m-2\nu}{n+2\nu}} \|_{0,\frac{2p_\nu}{p_\nu-2}}) \end{aligned}$$

by Hölder's inequality

$$
\begin{aligned}
&\le C(m,n,N,K)\sum_{\nu=0}^{m}\|\nabla^\nu(v-w)\|_{0,p_\nu}\cdot(\|\nabla^\nu v\|_{0,\frac{2p_\nu}{p_\nu-2}\cdot\frac{4m-2\nu}{n+2\nu}}^{\frac{4m-2\nu}{n+2\nu}} \\
&\quad+\|\nabla^\nu w\|_{0,\frac{2p_\nu}{p_\nu-2}\cdot\frac{4m-2\nu}{n+2\nu}}^{\frac{4m-2\nu}{n+2\nu}}) \\
&\le\sum_{\nu=0}^{m}C(m,n,N,K,\Omega,\nu,p_\nu)(\|v\|_{0,2}^{\frac{4m-2\nu}{n+2\nu}(1-a_\nu)}\|v\|_{2m,2}^{\frac{4m-2\nu}{n+2\nu}a_\nu} \\
&\quad+\|w\|_{0,2}^{\frac{4m-2\nu}{n+2\nu}(1-a_\nu)}\|w\|_{2m,2}^{\frac{4m-2\nu}{n+2\nu}a_\nu})\cdot\|v-w\|_{0,2}^{(1-\frac{\nu}{2m}-\frac{n}{4m}+\frac{n}{2mp_\nu})} \\
&\quad\cdot\|v-w\|_{2m,2}^{(\frac{\nu}{2m}+\frac{n}{4m}-\frac{n}{2mp_\nu})},
\end{aligned}
$$

where $a_\nu=\frac{\nu}{2m}+\frac{n}{4m}-\frac{n}{2m}\cdot\frac{p_\nu-2}{2p_\nu}\cdot\frac{n+2\nu}{4m-2\nu}$ and where we used the general interpolation and imbedding inequality [Fr, p. 27]; we observe that $a_\nu\frac{4m-2\nu}{n+2\nu}=1-\frac{\nu}{2m}-\frac{n}{4m}+\frac{n}{2mp_\nu}$. Hence

$$
\begin{aligned}
&\|F(s,.,v,...,\nabla^m v)-F(s,.,w,...,\nabla^m w)\|_{0,2} \\
&\le C(m,n,N,K,\Omega,p_0,...,p_m)\cdot\sum_{\nu=0}^{m}\{(\|v\|_{0,2}+\|w\|_{0,2})^{(\frac{4m-2\nu}{n+2\nu})(1-a_\nu)} \\
&\cdot\|v-w\|_{2m,2}^{(\frac{v}{2m}+\frac{n}{4m}-\frac{n}{2mp_\nu})}\}\cdot\{(\|v\|_{2m,2}+\|w\|_{2m,2}) \\
&\cdot\|v-w\|_{0,2}\}^{1-\frac{\nu}{2m}-\frac{n}{4m}+\frac{n}{2mp_\nu}} \\
&\le\varepsilon^{\frac{1}{r}}\{\sum_{\nu=0}^{m}(\|v(s)\|_{0,2}^{\beta_\nu}+\|w(s)\|_{0,2}^{\beta_\nu})\}\cdot\|v(s)-w(s)\|_{2m,2} \\
&+C(\varepsilon,m,n,N,K,\Omega,r,p_0,...,p_m)\|v(s)-w(s)\|_{0,2}(\|v(s)\|_{2m,2}+\|w(s)\|_{2m,2})
\end{aligned}
$$

by Young's inequality; β_ν are some positive numbers, determined by n,m,ν,p_ν.

We take this estimate to the power r, integrate with respect to s and immediately obtain Lemma 5. ■

Our local existence result reads as follows.

Lemma 6. Let $1-\frac{1}{r}<\kappa\le 1,\ \Psi\in D(A^\kappa)$. Then there exists a positive time $\tilde{T}\in(0,\hat{T}]$ and a solution

$$w\in L^r((0,\tilde{T}),H^{2m,2}(\Omega)\cap H_0^{m,2}(\Omega)),\ w'\in L^r((0,\tilde{T}),L^2(\Omega))$$

of the initial boundary value problem (5), where $w(0) = \Psi$ instead of $w(0) = \phi$ is requested. $\tilde{T}$ depends in particular on $\|A^\kappa \Psi\|_{0,2}, A, \Omega, K, r$.

Proof. Let $\psi(t) \in L^r((0,\hat{T}), H^{2m,2}(\Omega) \cap H_0^{m,2}(\Omega))$, $\psi' \in L^r((0,\hat{T}), L^2(\Omega))$, $\psi(0) = \Psi$ be an arbitrary auxiliary function, e.g. the solution of $\psi' + A\psi = 0$. We set

$$\begin{aligned}\mathcal{M} := \ & \{w | w \in L^r((0,\tilde{T}), H^{2m,2}(\Omega) \cap H_0^{m,2}(\Omega)), \\ & \quad w' \in L^r((0,\tilde{T}), L^2(\Omega)),\ w(0) = \Psi, \\ & \quad \int_0^{\tilde{T}} (\|w(s)\|_{2m,2}^r + \|w'(s)\|_{0,2}^r)\, ds \leq 1 + C_2\},\end{aligned}$$

where $C_2 := C_1(\|A^\kappa \Psi\|_{0,2}^r + 2^r \int_0^{\hat{T}} \|F(s,.,\psi,...,\nabla^m \psi)\|_{0,2}^r\, ds)$, C_1 is taken from Lemma 4, $\tilde{T}$ has to be determined below, without loss of generality we assume $\tilde{T} \leq 1$.

The map $G : \mathcal{M} \to \mathcal{M}$ is defined as follows: for $w \in \mathcal{M}$, let $v = Gw$ be the solution of

$$\begin{aligned}& v'(t) + Av(t) = -F(t,.,w,...,\nabla^m w), \\ & v(0) = \Psi, \\ & v \in L^r((0,\tilde{T}), H^{2m,2}(\Omega) \cap H_0^{m,2}(\Omega)), \\ & v' \in L^r((0,\tilde{T}), L^2(\Omega)).\end{aligned}$$

First we prove $G(\mathcal{M}) \subset \mathcal{M}$ for sufficiently small $\tilde{T}$. By Lemmata 4 and 5 we have:

$$\begin{aligned}& \int_0^{\tilde{T}} (\|v(s)\|_{2m,2}^r + \|v'(s)\|_{0,2}^r)\, ds \\ & \leq C_1\{\|A^\kappa \Psi\|_{0,2}^r + 2^r \int_0^{\tilde{T}} \|F(s,.,w,...,\nabla^m w) - F(s,.,\psi,...,\nabla^m \psi)\|_{0,2}^r\, ds \\ & \qquad + 2^r \int_0^{\tilde{T}} \|F(s,.,\psi,...,\nabla^m \psi)\|_{0,2}^r\, ds\}\end{aligned}$$

$$
\begin{aligned}
&\leq C_2 + C_1 \cdot 2^r \cdot C(n,m,K,r)\{\varepsilon \sum_{\nu=0}^{m}((\|\Psi\|_{0,2} + (1+C_2)^{\frac{1}{r}})^{\alpha_\nu} \\
&\qquad + \sup_{s\in[0,T]} \|\psi(s)\|_{0,2}^{\alpha_\nu}) \cdot (1 + C_2 + \int_0^{\hat{T}} \|\psi(s)\|_{2m,2}^r\, ds)\} \\
&\qquad + C(\varepsilon)(\sup_{s\in[0,\tilde{T}]} \|\psi(s) - w(s)\|_{0,2}^r) \cdot (1 + C_2 + \int_0^{\hat{T}} \|\psi(s)\|_{2m,2}^r\, ds).
\end{aligned}
$$

For an appropriate choice of ε, the second summand is $\leq \frac{1}{2}$. Furthermore we have

$$
\begin{aligned}
&\|\psi(s) - w(s)\|_{0,2} \leq \|\psi(s) - \Psi\|_{0,2} + \|\Psi - w(s)\|_{0,2} \\
&\leq (\int_0^{s} \|\psi'(s)\|_{0,2}^r\, ds)^{\frac{1}{r}} s^{1-\frac{1}{r}} + (\int_0^{s} \|w'(s)\|_{0,2}^r\, ds)^{\frac{1}{r}} s^{1-\frac{1}{r}} \\
&\leq \{(\int_0^{\hat{T}} \|\psi'(s)\|_{0,2}^r\, ds)^{\frac{1}{r}} + (C_2+1)^{\frac{1}{r}}\} s^{1-\frac{1}{r}}.
\end{aligned}
$$

Consequently $\tilde{T}$ can be chosen in such a way, that the third summand is also $\leq \frac{1}{2}$. Hence $v \in \mathcal{M}$.

Now let $w_1 = \psi,\ w_{n+1} = Gw_n$. In a completely analogous way it is shown, that

$$
\begin{aligned}
&\int_0^{\tilde{T}} (\|w_{n+2}(s) - w_{n+1}(s)\|_{2m,2}^r + \|w_{n+2}'(s) - w_{n+1}'(s)\|_{0,2}^r)\, ds \\
&\leq \varrho \cdot \int_0^{\tilde{T}} (\|w_{n+1}(s) - w_n(s)\|_{2m,2}^r + \|w_{n+1}'(s) - w_n'(s)\|_{0,2}^r)\, ds,
\end{aligned}
$$

with some $\varrho < 1$ for an appropriate choice of $\tilde{T}$. The contraction mapping principle completes the proof. ■

3.5. We define $T_{\max} := \sup\{\tilde{T} |$ on $[0, \tilde{T}]$ there exists a solution w as in Lemma 6$\}$. In particular there is a unique strong solution w on $[0, T_{\max})$. If we can show $T_{\max} > T$, it will follow that $u = w$ on $[0, T]$, thereby proving the theorem.

We assume the contrary: $T_{\max} \leq T$. On $[0, T_{\max})$ we have $u = w$, and u is uniformly continuous on $[0, T]$ in $L^2(\Omega)$. So there is a $\delta > 0$, which only depends on u, such that for $0 < t < t' < T_{\max}$, $|t - t'| < \delta$, $|t - t'| < t$ we have:

$$C(\hat{T}) \cdot \sum_{\nu=0}^{m} \|w(s) - w(2t-s)\|_{0,2}^{r \cdot \frac{4m-2\nu}{n+2\nu}} \leq \frac{1}{2} \text{ for all } s \in [t, t'], \tag{7}$$

where $C(\hat{T})$ is the constant below;

$\hat{w}(s) := w(s) - w(2t - s)$ solves
$\hat{w}'(s) + A\hat{w}(s) = -F(s, ., w(s), ..., \nabla^m w(s)) + w'(2t - s) - Aw(2t - s)$,
$\hat{w}(t) = 0$;

$$\begin{aligned}
&\int_t^{t'} (\|\hat{w}(s)\|_{2m,2}^r + \|\hat{w}'(s)\|_{0,2}^r)\, ds \\
&\leq C(\hat{T})\{ \int_{2t-t'}^{t} (\|w'(s)\|_{0,2}^r + \|w(s)\|_{2m,2}^r)\, ds \\
&\quad + \int_t^{t'} (1 + \sum_{\nu=0}^{m} \| |\nabla^\nu w(s) - \nabla^\nu w(2t-s)|^{\frac{n+4m}{n+2\nu}} \|_{0,2}^r)\, ds \\
&\quad + \int_t^{t'} (\sum_{\nu=0}^{m} \| |\nabla^\nu w(2t-s)|^{\frac{n+4m}{n+2\nu}} \|_{0,2}^r)\, ds \} \\
&\leq C(\hat{T})\{ \int_{2t-t'}^{t} (1 + \|w'(s)\|_{0,2}^r + \|w(s)\|_{2m,2}^r (1 + \sum_{\nu=0}^{m} \|w(s)\|_{0,2}^{r \cdot \frac{4m-2\nu}{n+2\nu}}))\, ds \\
&\quad + \sum_{\nu=0}^{m} \int_t^{t'} (\|w(s) - w(2t-s)\|_{2m,2}^r \cdot \|w(s) - w(2t-s)\|_{0,2}^{r \cdot \frac{4m-2\nu}{n+2\nu}}) ds \}
\end{aligned}$$

by the general interpolation and imbedding inequality [Fr], p. 27

$$\begin{aligned}
&\leq \frac{1}{2} \int_t^{t'} \|\hat{w}(s)\|_{2m,2}^r\, ds + C(\hat{T}) \cdot \int_{2t-t'}^{t} (1 + \|w'(s)\|_{0,2}^r \\
&\qquad + \|w(s)\|_{2m,2}^r (1 + \sum_{\nu=0}^{m} \|w(s)\|_{0,2}^{r \cdot \frac{4m-2\nu}{n+2\nu}}))\, ds \text{ by (7).}
\end{aligned}$$

Hence

$$\int_t^{t'} (\|w'(s)\|_{0,2}^r + \|w(s)\|_{2m,2}^r)\, ds$$

$$\leq C(\hat{T}) \int_{2t-t'}^{t} (1 + \|w'(s)\|_{0,2}^r + \|w(s)\|_{2m,s}^r (1 + \sum_{\nu=0}^{m} \|w(s)\|_{0,2}^{r \cdot \frac{4m-2\nu}{n+2\nu}}))\, ds.$$

In other words: finiteness of the $L^r - H^{2m,2}$-norm of w and the $L^r - L^2$-norm of w' on $[t, t+\delta]$ follows from the finiteness on $[t-\delta, t]$. Starting at $t = \frac{1}{2} T_{\max}$ and proceeding with intervals of uniform length δ, we conclude:

$$\begin{array}{rcl} w & \in & L^r((0, T_{\max}), H^{2m,2}(\Omega) \cap H_0^{m,2}(\Omega)) \\ w' & \in & L^r((0, T_{\max}), L^2(\Omega)). \end{array}$$

By Lemma 5 this yields $F(.,.,w,...,\nabla^m w) \in L^r((0,T_{\max}), L^2(\Omega))$. Using the integral equation for w it follows from $r > \frac{n+4m}{2m}$ that

$$A^\sigma w \in C^0([0, T_{\max}], L^2(\Omega)) \text{ with some } \sigma \in (\frac{n+2m}{n+4m}, 1),$$

and by Sobolev we get $(1 + \sum_{\nu=0}^{m} |\nabla^\nu w|^{\frac{n+4m}{n+2\nu}}) \in C^0([0, T_{\max}], L^2(\Omega))$. Because $q \in L^\infty((0,T) \times \bar{\Omega})$, we only have $F(.,.,w,...,\nabla^m w) \in L^\infty((0, T_{\max}), L^2(\Omega))$.

So in a second step, we obtain $A^\kappa w \in C^0([0, T_{\max}], L^2(\Omega))$ where $\kappa > 1 - \frac{1}{r}$, i.e. $w(T_{\max}) \in D(A^\kappa)$.

Now applying Lemma 6 to the initial value problem (5) on $[T_{\max}, T_{\max} + \tilde{T}]$, we arrive at the desired contradiction.

Remarks.

1) The proof of the theorem shows, that it is possible to obtain an a-priori bound for any solution u in the space $L^r((0,T), H^{2m,2}(\Omega))$, which depends in particular on the modulus of continuity of u with respect to $L^2(\Omega)$:

$$\omega(r) = \sup_{\substack{s,t \in [0,T] \\ |s-t| \leq r}} \|u(t) - u(s)\|_{0,2}.$$

2) The question is also of interest whether it is possible to obtain a-priori bounds in $L^r((0,T), H^{2m,2}(\Omega))$, which only depend on

$$\sup_{s\in[0,T]} \|u(s)\|_{0,2}^2 + \int_0^T \|u(s)\|_{m,2}^2 \, ds.$$

We think that an affirmative answer is only possible under strong additional assumptions on f, like

$$(8) \qquad |\frac{\partial f_i}{\partial u_j}| \le C(1+|u|^{\frac{4m}{n}}),$$

if e.g. f only depends on u. Then bounds could be obtained by differentiating the differential equation with respect to t. Condition (8) however is already violated by simple nonlinearities, e.g. in the case $N=2$, $n>4m$ by $f(u_1,u_2) = (u_2(1+|u_1|)^{\frac{4m}{n}}, u_1(1+|u_2|)^{\frac{4m}{n}})$.

At the first glance, the modified nonlinearity F in system (5) seems to be in appropriate form. But we don't know anything about $\frac{\partial q}{\partial t}$; this prevents us from finding a-priori bounds of the type just mentioned.

Acknowledgement. We are grateful to the referees for some helpful and interesting suggestions.

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Hans-Christoph Grunau
Wolf von Wahl
Lehrstuhl für Angewandte Mathematik
Universität Bayreuth
D-95440 Bayreuth