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Monotonicity and boundedness in the Boussinesq-equations

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ABSTRACT. — The onset of convection from the motionless state of the Boussinesq-approximation to Bénard-convection is studied for both stress-free and rigid boundaries for solutions which are periodic in the horizontal directions with wave-numbers α and β. The critical Rayleigh-numbers $\mathrm{R}(\alpha, \beta)$ for the kinetic energy are displayed graphically as a surface $\{(\alpha, \beta, \mathrm{R}(\alpha, \beta)) \mid \alpha, \beta > 0\}$ in $\mathbb{R}^3$. For stress-free boundaries and small initial-values it is proved that the position of the point $(\alpha, \beta, \mathrm{R})$ relative to the onset governs the behaviour of a generalized energy functional which involves the spatial derivatives of the solution, *i.e.*, below the onset exponential decay takes place. For $(\alpha, \beta, \mathrm{R})$ on the onset it is shown that the motionless state is stable in the sense of Ljapunov with respect to a functional involving even higher order derivatives than the first mentioned functional. Above the onset it becomes unstable. Throughout the paper, the decomposition of the velocity field into a poloidal part, a toroidal part and the mean flow is employed as an essential tool.

1. Introduction, notations. The differential operators in the Boussinesq-equations

We consider the Boussinesq-equations ($\underline{k} = (0, 0, 1)^{\mathrm{T}}$)

$$(1.1) \qquad \begin{cases} \underline{u}' - \Delta \underline{u} + \underline{u} \cdot \nabla \underline{u} - \sqrt{\mathrm{R}}\, \vartheta \underline{k} + \nabla \pi = 0, \quad \nabla \cdot \underline{u} = 0, \\ \mathrm{Pr}\, \vartheta' - \Delta \vartheta + \mathrm{Pr}\, \underline{u} \cdot \nabla \vartheta - \sqrt{\mathrm{R}}\, u_z = 0 \end{cases}$$

for an infinite layer $\mathbb{R}^2 \times (-(1/2), 1/2)$ heated from below. $\mathrm{Pr} > 0$ is the Prandtl-number, $\mathrm{R} > 0$ is the Rayleigh-number, $\underline{u}$, ϑ have the usual meaning, and π is the pressure. The boundary-conditions at $z = \pm(1/2)$ are the usual ones: Stress-free boundaries or rigid boundaries. They are explained below. $'$ refers to the derivative with respect to time, and we also prescribe the initial values $\underline{u}_0$ ϑ_0 at time $t = 0$. $\underline{u}$, ϑ and π are required to be periodic in $(x, y) \in \mathbb{R}^2$ with respect to a rectangle $\mathscr{P} = (-(\pi/\alpha), \pi/\alpha) \times (-(\pi/\beta), \pi/\beta)$ with a wave-number α in x-direction and a wave-number β in y-direction.

The aim of this paper is two-fold. In Section 2 we give a graphical representation of the onset of convection. This is a surface in $(\alpha, \beta, \mathrm{R})$-space referred to as the onset. Its equation has been derived rigourously in [Schmitt & von Wahl, 1992, Proposition 2.3] in the case of stress-free boundaries and in [von Wahl, 1992, Theorem VI.1, (VI.7)] in both cases. The onset of convection is characterized by the following property of the kinetic energy $\mathrm{E}(t) = \| \underline{u}(t) \|^2_{\mathrm{L}^2(\Omega)} + \mathrm{Pr} \| \vartheta(t) \|^2_{\mathrm{L}^2(\Omega)}$, $\Omega = \mathscr{P} \times (-(1/2), 1/2)$, at time t:

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Below or at the onset E is monotonically non increasing for any initial value, above the onset E is strictly monotonically increasing, at least initially, for suitable initial values. The onset is parameterized by the independent variables α, β. A particular part of the onset is a net of lines where the surface is Lipschitz continuous only. On these lines interesting bifurcation phenomena may take place; this has been studied under stress-free boundary conditions and in the two-dimensional case, *i.e.* the case of convection rolls, by [Busse & Or, 1986]. All points on the onset are eigenvalues of even multiplicity of an associated eigenvalue problem (*cf.* Sec. 2 to follow). That the determination of the onset of convection requires a more detailed analysis than simply this was apparently known before (*cf.* [Beck, 1972]). In Section 2 we give the analytical formula for the onset by stating a necessary and sufficient condition for E to behave monotonically non increasing for any initial value. The formula looks quite similar to what has been written in [B, 1972] in a different situation; its rigorous proof becomes straightforward when using abstract tools and the decomposition of $\underline{u}$ into poloidal fields, toroidal fields and the mean flow as is done in [S & W, 1992], [W, 1992].

From the above discussion it is clear that the kinetic energy E can be considered, at least below the onset, as a Ljapunov-functional whose behaviour is governed by the position of (α, β, R) relative to the onset. The norms occuring in E are however too weak to guarantee the global existence of a strong solution even if we start with initial values having small energy-norm below the onset. It might be desirable therefore to construct a Ljapunov-functional L which involves higher order norms of $\underline{u}$, ϑ on one hand and whose behaviour is governed by the same onset we have found for E on the other hand. We construct L in the case of stress-free boundaries and prove that $L(t)$ is steadily decreasing below the onset provided $L(0) \leqq \varepsilon$. In this case the solution exists globally in time. Above the onset L may strictly increase at least initially, no matter how small the initial value is. The formula for L is given in 1.14 below.

$L(t)$ majorizes $c(\|\nabla \underline{u}(t)\|^2 + \|\nabla \vartheta(t)\|^2)$ where c is a positive constant. The idea of introducing $\|\nabla \underline{u}(t)\|^2 + \|\nabla \vartheta(t)\|^2$ into a so-called generalized energy-functional is not new. We refer to the comprehensive work [Galdi & Padula, 1990] and the references therein, in particular to section 11 for the Boussinesq-equations under stress-free boundary conditions. Whereas these authors refer to $R_c = \min \{ R(\alpha^2, \beta^2) | (\alpha, \beta, R(\alpha^2, \beta^2))$ as defining the onset$\}$ (*cf.* our results. In fact, in [G & P, 1990] the system (1.1) is studied with additional terms which are due to rotation for example) for the behaviour of a sufficiently regular solution, here we study the existence and behaviour of a solution with a dependence on (α, β, R) relative to the onset. It may be added that $E(t)$ is known to be bounded, uniformly in t, above or below the onset, whether the initial values are small or not (*cf.* [Temam, 1988, p. 132]). $L(t)$ however cannot in general be excluded from blowing up at a finite time unless $Pr = +\infty$ (*cf.* [von Wahl. 1991]; in this case $L(t)$ stays bounded, uniformly in t) or the problem is two-dimensional, *i.e.* we have convection rolls.

On the onset itself $L(t)$ stays bounded if the initial values are small enough. In Section 4 we even construct an absorbing set in a stronger norm (than that represented by L) for a basin of small initial values. This set also turns out to be small, its bound depending on R.

Since the onset precisely consists of the smallest positive eigenvalues of a selfadjoint eigenvalue-problem associated with (1.1) (*cf.* Sec. 2 to follow) it turns out that the energy-stability (*i.e.* with reference to $E(t)$) of the motionless state $\underline{u}=0$, $\vartheta=0$ with respect to the particular disturbance $\underline{u}(t), \vartheta(t)$ under consideration is governed by the position of (α, β, R) with respect to the onset. This is in principle known. As for the rigid case we refer to [Kirchgässner & Kielhöfer, 1973]; since we treat (1.1) within a different formulation we will give a proof elsewhere, both in the terms we use here and in a more general connection. As for the case of the stress-free condition we indicate this very briefly in the end of Section 4 for $\mathrm{Pr}=1$; we take L instead of E but otherwise also follow the ideas in [K & K, 1973, p. 307].

When constructing the onset and L the decomposition

$$\begin{aligned} \underline{u} &= \operatorname{curl}\operatorname{curl}\varphi\,\underline{k}+\operatorname{curl}\psi\,\underline{k}+\underline{f} \\ &= P+T+\underline{f} \end{aligned} \tag{1.2}$$

into a poloidal field P, a toroidal one T and the mean flow $\underline{f}$ can be used to advantage, as already mentioned. (1.2) holds for solenoidal (*i.e.* $\operatorname{div}\underline{u}=0$) vector fields periodic in x, y with respect to $\mathscr{P}$. φ, ψ are uniquely determined if we require them to be periodic in x, y and to have vanishing mean value over $\mathscr{P}$. The mean flow $\underline{f}$ depends only on z and has constant third component f_3. When applying (1.2) to (1.1) the boundary conditions on $\underline{u}$ become equivalent ones on φ, ψ and $\underline{f}$. Moreover $f_3\equiv 0$. The boundary conditions on $\underline{u}$ at $z=\pm(1/2)$ are

$$\partial_z u_x=\partial_z u_y=u_z=0 \text{ in the case of stress-free boundaries,} \tag{1.3}$$

$$\underline{u}=0 \text{ in the case of rigid boundaries,} \tag{1.4}$$

and the new ones read correspondingly

$$\varphi=\partial_z^2\varphi=\partial_z\psi=\partial_z f_1=\partial_z f_2=0 \tag{1.3'}$$

$$\varphi=\partial_z\varphi=\psi=f_1=f_2=0 \tag{1.4'}$$

at $z=\pm(1/2)$. As for ϑ we have $\vartheta=0$ at $z=\pm(1/2)$ in either case. The system (1.1) itself is transformed into an equivalent one for $\Phi=(\varphi, \psi, \vartheta, f_1, f_2)^T$. It has the form

$$\mathscr{B}\Phi'+\mathscr{A}\Phi-\sqrt{R}\,\mathscr{C}\Phi+\mathscr{M}(\Phi)=0 \tag{1.5}$$

with matrix operators $\mathscr{B}$, $\mathscr{A}$, $\mathscr{C}$ and a nonlinear term $\mathscr{M}$. $\mathscr{A}$, $\mathscr{B}$ turn out to be diagonal and strictly positive definite selfadjoint operators in an appropriate Hilbert space H. H is simply the product $L^2_M(\Omega)\times L^2_M(\Omega)\times L^2(\Omega)\times(L^2_M((-(1/2), 1/2)))^2$ or $L^2_M(\Omega)\times L^2_M(\Omega)\times L^2(\Omega)\times(L^2((-(1/2), 1/2)))^2$ for stress-free boundaries or rigid boundaries with $\Omega=\mathscr{P}\times(-(1/2), 1/2)$. The subscript $._M$ indicates that φ, ψ have vanishing mean value over $\mathscr{P}$, whereas $\underline{f}$ is required to have vanishing mean value over $(-(1/2), 1/2)$ in the case of stress-free boundaries. The pressure is eliminated. The highest order derivatives of $u_z=-(\partial_x^2+\partial_y^2)\varphi$ are isolated in a single equation, the nonlinearity $\mathscr{M}(\Phi)$ is almost local. This approach was discussed in detail in [S & W, 1992].

We now introduce some **notation** and then discuss the differential operators occurring in (1.5). A vector field $\underline{u}$ or $\underline{f}$ is usually written as a column, *i.e.* $\underline{u}=(u_1, u_2, u_3)^{\mathrm{T}}=(u_x, u_y, u_z)^{\mathrm{T}}$, $\underline{f}=(f_1, f_2, f_3)^{\mathrm{T}}=(f_x, f_y, f_z)^{\mathrm{T}}$ with the symbol $.^{\mathrm{T}}$ for transposition. Correspondingly we sometimes write (x, y, z) as (x_1, x_2, x_3). As for differentiation we use

$$\partial_x^{\tilde\alpha},\ \partial_y^{\tilde\beta},\ \partial_z^{\gamma} \text{ for the } \tilde\alpha \text{ times applied operator } \partial_x, \ldots$$

$$\partial_z,\ \partial_{zz}^2,\ \partial_x,\ \partial_{xy}^2,\ \partial_{xz}^2,\ \ldots,$$

$\Delta_2=\partial_x^2+\partial_y^2$, $\underline{\delta}.=\operatorname{curl}\operatorname{curl}.k=(\partial_{zx}.,\ \partial_{yz}.,\ -\Delta_2.)^{\mathrm{T}}$, $\underline{\varepsilon}.=\operatorname{curl}.\underline{k}=(\partial_y.,\ -\partial_x.,\ 0)^{\mathrm{T}}$ for functions. and

$$\partial,\ \partial^2,\ \ldots \text{ for any first, second, } \ldots \text{ order derivative.}$$

If no confusion can arise we also use ∂_{xy}^q for any q-th order derivative with respect to x, y. When $\mathscr{P}=(-(\pi/\alpha), \pi/\alpha)\times(-(\pi/\beta), \pi/\beta)$ is the periodicity cell then we consider Sobolev-spaces $\mathrm{H}^{k,p}(\Omega)$ of x, y-periodic functions over the layer $\Omega=\mathscr{P}\times(-(1/2), 1/2)$, $k\in\mathbb{N}$, $p>1$. These have the usual meaning. In most cases we deal with $p=2$. Then, if $\varphi\in\mathrm{H}^{k,2}(\Omega)$ and

$$\varphi(x, y, z)=\frac{1}{\sqrt{|\mathscr{P}|}}\sum_{\kappa\in\mathbb{Z}^2} a_\kappa(z)\, e^{i(\alpha\kappa_1 x+\beta\kappa_2 y)} \quad \text{in} \quad \mathrm{L}^2(\Omega),$$

we obtain equivalently

$$\partial_x^{\tilde\alpha}\partial_y^{\tilde\beta}\partial_z^{\gamma}\varphi(x, y, z)=\frac{1}{\sqrt{|\mathscr{P}|}}\sum_{\kappa\in\mathbb{Z}^2}\partial_z^{\gamma}a_\kappa(z)\,(i\alpha\kappa_1)^{\tilde\alpha}(i\beta\kappa_2)^{\tilde\beta}\, e^{i(\alpha\kappa_1 x+\beta\kappa_2 y)} \quad \text{in} \quad \mathrm{L}^2(\Omega)$$

for $\tilde\alpha+\tilde\beta+\gamma\leqq k$. The norm is given by

$$\|\partial_x^{\tilde\alpha}\partial_y^{\tilde\beta}\partial_z^{\gamma}\varphi\|^2=\|\partial_x^{\tilde\alpha}\partial_y^{\tilde\beta}\partial_z^{\gamma}\varphi\|_{\mathrm{L}^2(\Omega)}^2=\sum_{\kappa\in\mathbb{Z}^2}\int_{-1/2}^{1/2}|\partial_z^{\gamma}a_\kappa(z)|^2\,dz\,(\alpha\kappa_1)^{2\tilde\alpha}(\beta\kappa_2)^{2\tilde\beta}$$

by Levi's Theorem. It is clear now how Parseval's equation reads. Equivalently we can introduce the Sobolev spaces $\mathrm{H}_{\mathscr{P}}^{l}=\mathrm{H}_{\mathscr{P}}^{l,2}$ of $\mathscr{P}$-periodic functions in the plane with exponent of integration 2 (*cf.* [S & W, 1992, Sect. 1]) and consider the spaces

$$\mathrm{W}^k((a, b), \mathrm{H}_{\mathscr{P}}^{l})$$

where (a, b) is an open interval on the z-axis. They consist of the mappings $f:(a, b)\to\mathrm{H}_{\mathscr{P}}^{l}$ with $\partial_z^p\partial_{xy}^q f\in\mathrm{L}^2((a, b), \mathrm{H}_{\mathscr{P}}^0)=\mathrm{L}^2((a, b), \mathrm{L}^2(\mathscr{P}))$ for any integers p, $q\geqq 0$ with $p\leqq k$, $q\leqq l$ and $p+q\leqq\max\{k, l\}$. They become a Hilbert space in the usual way. A selfadjoint operator A in a Hilbert space $\mathscr{H}$ is called strictly positive definite iff $(\mathrm{A}u, u)\geqq\gamma\|u\|^2$, $u\in\mathscr{D}(\mathrm{A})$, for some $\gamma>0$. $\mathrm{C}^k([a, b], \mathscr{H})$ is the usual space of k-times continuously differentiable functions on $[a, b]$ with values in the space $\mathscr{H}$. The matrices in (1.5) have the following form:

$$\mathscr{B}=\begin{pmatrix} (-\Delta)(-\Delta_2) & 0 & 0 & 0 & 0 \\ 0 & (-\Delta_2) & 0 & 0 & 0 \\ 0 & 0 & \Pr \mathrm{I} & 0 & 0 \\ 0 & 0 & 0 & \mathrm{I} & 0 \\ 0 & 0 & 0 & 0 & \mathrm{I} \end{pmatrix},$$

$$\mathscr{A}=\begin{pmatrix} \Delta^2(-\Delta_2) & 0 & 0 & 0 & 0 \\ 0 & (-\Delta)(-\Delta_2) & 0 & 0 & 0 \\ 0 & 0 & (-\Delta) & 0 & 0 \\ 0 & 0 & 0 & (-\partial_z^2) & 0 \\ 0 & 0 & 0 & 0 & (-\partial_z^2) \end{pmatrix},$$

$$\mathscr{C}=\begin{pmatrix} 0 & 0 & (-\Delta_2) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ (-\Delta_2) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix};$$

the nonlinearity $\mathscr{M}(\Phi)$ is given by

$$\mathscr{M}(\Phi)=\begin{pmatrix} \underline{\delta}.((\underline{\delta}\varphi+\underline{\varepsilon}\psi+\underline{f})\cdot\nabla(\underline{\delta}\varphi+\underline{\varepsilon}\psi+\underline{f})) \\ -\underline{\varepsilon}.((\underline{\delta}\varphi+\underline{\varepsilon}\psi+\underline{f})\cdot\nabla(\underline{\delta}\varphi+\underline{\varepsilon}\psi+\underline{f})) \\ \Pr(\underline{\delta}\varphi+\underline{\varepsilon}\psi+\underline{f})\cdot\nabla\vartheta \\ \dfrac{1}{|\mathscr{P}|}\displaystyle\int_{\mathscr{P}}(\underline{\delta}\varphi+\underline{\varepsilon}\psi)\cdot\nabla(\underline{\delta}\varphi+\underline{\varepsilon}\psi)_x\,dx\,dy \\ \dfrac{1}{|\mathscr{P}|}\displaystyle\int_{\mathscr{P}}(\underline{\delta}\varphi+\underline{\varepsilon}\psi)\cdot\nabla(\underline{\delta}\varphi+\underline{\varepsilon}\psi)_y\,dx\,dy \end{pmatrix}.$$

Beside $\underline{u}=\underline{\delta}\varphi+\underline{\varepsilon}\psi+\underline{f}$ we will also use the notation

$$\tilde{\underline{u}}=\underline{\delta}\varphi+\underline{\varepsilon}\psi.$$

The system (1.5) is most easily treated within an appropriate Hilbert space H, where $\mathscr{A}$, $\mathscr{B}$ become strictly positive definite selfadjoint operators and $\mathscr{C}$ is hermitian. As the Hilbert space H with norm $\|.\|$ we take

$$\mathrm{H}=\mathscr{H}_{\mathrm{M}}\times\mathscr{H}_{\mathrm{M}}\times\mathscr{H}\times\mathscr{H}^1\times\mathscr{H}^1$$

with $\varphi\in\mathscr{H}_{\mathrm{M}}$, $\psi\in\mathscr{H}_{\mathrm{M}}$, $\vartheta\in\mathscr{H}$, $f_1\in\mathscr{H}^1$, $f_2\in\mathscr{H}^1$ in the case of rigid boundaries. Here

$$\mathscr{H}_{\mathrm{M}}=\left\{\tilde{\varphi}\,|\,\tilde{\varphi}\in \mathrm{W}^0\left(\left(-\frac{1}{2},\frac{1}{2}\right),\mathrm{L}^2_{\mathscr{P}}\right),\int_{\mathscr{P}}\tilde{\varphi}\,dx\,dy=0\right\},$$

$$\mathscr{H} = \mathrm{W}^0\left(\left(-\frac{1}{2}, \frac{1}{2}\right), \mathrm{L}^2_{\mathscr{P}}\right),$$

$$\mathscr{H}^1 = \left\{ f \mid f \in \mathrm{L}^2\left(\left(-\frac{1}{2}, \frac{1}{2}\right)\right)\right\}.$$

$\mathscr{H}_{\mathrm{M}}$, $\mathscr{H}$ are made Hilbert spaces in the usual way. For $\mathscr{H}^1$ we choose the inner product

$$(f, g) = |\mathscr{P}| \int_{-1/2}^{1/2} f.\bar{g}\, dz.$$

In the case of stress-free boundaries we take

$$\mathrm{H} = \mathscr{H}_{\mathrm{M}} \times \mathscr{H}_{\mathrm{M}} \times \mathscr{H} \times \mathscr{H}^1_{\mathrm{M}} \times \mathscr{H}^1_{\mathrm{M}}$$

with $\mathscr{H}^1_{\mathrm{M}}$ being the closed subspace of $\mathscr{H}^1$ which consists of the f with vanishing mean value over $(-(1/2), 1/2)$. Now we can define $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$ by defining $\mathrm{A} = \Delta^2(-\Delta_2)$, $\breve{\mathrm{B}} = (-\Delta)(-\Delta_2)$ for φ, $\mathrm{B} = (-\Delta)(-\Delta_2)$ for ψ and $-\Delta$ for ϑ, $-\partial_z^2$ for f_1, f_2, $-\Delta_2$ for φ, ψ and also ϑ. Observe that we have two different kinds of operators $(-\Delta)(-\Delta_2)$ in the case of stress-free boundaries. To emphasize this we will write

$$(-\Delta_{\mathrm{N}}) \text{ instead of } (-\Delta)$$

when dealing with ψ in the stress-free case.

Definition 1.1. – *We expand φ, ψ, ϑ into series*

$$\varphi(x, y, z) = \frac{1}{\sqrt{|\mathscr{P}|}} \sum_{\kappa \in \mathbb{Z}^2 \setminus \{0\}} a_\kappa(z)\, e^{i\kappa \cdot (\alpha x, \beta y)}, \tag{1.6}$$

$$\psi(x, y, z) = \frac{1}{\sqrt{|\mathscr{P}|}} \sum_{\kappa \in \mathbb{Z}^2 \setminus \{0\}} b_\kappa(z)\, e^{i\kappa \cdot (\alpha x, \beta y)}, \tag{1.7}$$

$$\vartheta(x, y, z) = \frac{1}{\sqrt{|\mathscr{P}|}} \sum_{\kappa \in \mathbb{Z}^2} c_\kappa(z)\, e^{i\kappa \cdot (\alpha x, \beta y)}, \tag{1.8}$$

which are convergent in $\mathrm{W}^0((-(1/2), 1/2), \mathrm{L}^2_{\mathscr{P}})$. *Set*

$$\begin{aligned}\mathrm{A}_\kappa &= (\alpha^2 \kappa_1^2 + \beta^2 \kappa_2^2 - \partial_z^2)^2 \\ &= (\alpha^2 \kappa_1^2 + \beta^2 \kappa_2^2)^2 - 2(\alpha^2 \kappa_1^2 + \beta^2 \kappa_2^2)\partial_z^2 + \partial_z^4, \qquad \kappa \in \mathbb{Z}^2 \setminus \{0\},\end{aligned}$$

$$\mathscr{D}(\mathrm{A}_\kappa) = \left\{ f \mid f \in \mathrm{H}^4\left(\left(-\frac{1}{2}, \frac{1}{2}\right)\right) \textit{ with either } f = \partial_z f = 0 \textit{ at } z = \pm\frac{1}{2} \textit{ or } f = \partial_z^2 f = 0 \textit{ at } z = \pm\frac{1}{2}\right\}.$$

Then A_κ *is a strictly positive definite selfadjoint operator in* $L^2((-1/2, 1/2))$. *We define* $A = \Delta^2(-\Delta_2)$ *on*

$$\mathcal{D}(A) = \Big\{ \varphi \mid \varphi \in \mathcal{H}_M,\ \varphi \text{ is expanded as in } (1.6),\ a_\kappa \in \mathcal{D}(A_\kappa),\ \kappa \in \mathbb{Z}^2 \setminus \{0\},$$
$$\sum_{\kappa \in \mathbb{Z}^2 \setminus \{0\}} \int_{-1/2}^{1/2} (\alpha^2 \kappa_1^2 + \beta^2 \kappa_2^2)^2 \, |A_\kappa a_\kappa|^2 \, dz < +\infty \Big\}$$

by

$$A\varphi = \frac{1}{\sqrt{|\mathcal{P}|}} \sum_{\kappa \in \mathbb{Z}^2 \setminus \{0\}} (\alpha^2 \kappa_1^2 + \beta^2 \kappa_2^2) A_\kappa a_\kappa e^{i\kappa \cdot (\alpha \cdot, \beta \cdot)}.$$

Set

$$\check{B}_\kappa = \alpha^2 \kappa_1^2 + \beta^2 \kappa_2^2 - \partial_z^2, \qquad \kappa \in \mathbb{Z}^2 \setminus \{0\},$$
$$\mathcal{D}(\check{B}_\kappa) = \Big\{ f \mid f \in H^2\Big(\Big(-\frac{1}{2}, \frac{1}{2}\Big)\Big) \text{ with } f = 0 \text{ at } z = \pm\frac{1}{2} \Big\}.$$

Then $\check{B}_\kappa$ *is a strictly positive definite selfadjoint operator in* $L^2((-(1/2), 1/2))$. *We define* $\check{B} = (-\Delta)(-\Delta_2)$ *on*

$$\mathcal{D}(\check{B}) = \Big\{ \varphi \mid \varphi \in \mathcal{H}_M,\ \varphi \text{ is expanded as in } (1.6),\ a_\kappa \in \mathcal{D}(\check{B}_\kappa),\ \kappa \in \mathbb{Z}^2 \setminus \{0\},$$
$$\sum_{\kappa \in \mathbb{Z}^2 \setminus \{0\}} \int_{-1/2}^{1/2} (\alpha^2 \kappa_1^2 + \beta^2 \kappa_2^2)^2 \, |\check{B}_\kappa a_\kappa|^2 \, dz < +\infty \Big\}$$

by

$$\check{B}\varphi = \frac{1}{\sqrt{|\mathcal{P}|}} \sum_{\kappa \in \mathbb{Z}^2 \setminus \{0\}} (\alpha^2 \kappa_1^2 + \beta^2 \kappa_2^2) \check{B}_\kappa a_\kappa e^{i\kappa \cdot (\alpha \cdot, \beta \cdot)}.$$

Let

$$B_\kappa = \alpha^2 \kappa_1^2 + \beta^2 \kappa_2^2 - \partial_z^2, \quad \kappa \in \mathbb{Z}^2 \setminus \{0\},$$
$$\mathcal{D}(B_\kappa) = \Big\{ f \mid f \in H^2\Big(\Big(-\frac{1}{2}, \frac{1}{2}\Big)\Big) \text{ with either } f = 0 \text{ at } z = \pm\frac{1}{2} \text{ or } \partial_z f = 0 \text{ at } z = \pm\frac{1}{2} \Big\}.$$

Then B_κ *is a strictly positive definite selfadjoint operator in* $L^2((-(1/2), 1/2))$ $(|\kappa| \geqq 1\,!)$. *We define* $B = (-\Delta)(-\Delta_2)$ *on*

$$\mathcal{D}(B) = \Big\{ \psi \mid \psi \in \mathcal{H}_M,\ \psi \text{ is expanded as in } (1.7),\ b_\kappa \in \mathcal{D}(B_\kappa),\ \kappa \in \mathbb{Z}^2 \setminus \{0\},$$
$$\sum_{\kappa \in \mathbb{Z}^2 \setminus \{0\}} \int_{-1/2}^{1/2} (\alpha^2 \kappa_1^2 + \beta^2 \kappa_2^2)^2 \, |\check{B}_\kappa b_\kappa|^2 \, dz < +\infty \Big\}$$

by

$$B\psi = \frac{1}{\sqrt{|\mathcal{P}|}} \sum_{\kappa \in \mathbb{Z}^2 \setminus \{0\}} (\alpha^2 \kappa_1^2 + \beta^2 \kappa_2^2) B_\kappa b_\kappa e^{i\kappa \cdot (\alpha \cdot, \beta \cdot)}.$$

It is now obvious how $-\Delta$ *is defined for* ϑ, i.e. *in* $\mathcal{H}$, $-\Delta_2$ *in* $\mathcal{H}_M$ *or* $\mathcal{H}$, $-\partial_z^2$ *in* $\mathcal{H}^1$ *or* $\mathcal{H}_M^1$.

Now it's easy to prove that A, B, $\check{B}$, $-\Delta$, $-\Delta_N$, $-\Delta_2$, $-\partial_z^2$ are strictly positive definite selfadjoint operators in the corresponding Hilbert spaces The same applies to Δ^2 under various boundary conditions. Observe that in $\mathscr{H}$ the operator $(-\Delta_2)$ is only nonnegative selfadjoint. The fractional powers $(-\Delta_2)^\rho$, $\rho>0$, are nevertheless well defined. One can use the series (1.10) to follow.

As a consequence we find that $\mathscr{A}$, $\mathscr{B}$ are strictly positive definite selfadjoint operators in H. As for the fractional powers we have, for example,

$$(1.9)\qquad (-\Delta)^\rho \psi \text{ or } (-\Delta_N)^\rho \psi = \frac{1}{\sqrt{|\mathscr{P}|}} \sum_{\kappa\in\mathbb{Z}^2\setminus\{0\}} B_\kappa^\rho b_\kappa e^{i\kappa\cdot(\alpha.,\beta.)},$$

$$(1.10)\qquad (-\Delta_2)^\rho \psi = \frac{1}{\sqrt{|\mathscr{P}|}} \sum_{\kappa\in\mathbb{Z}^2\setminus\{0\}} (a^2\kappa_1^2+\beta^2\kappa_2^2)^\rho b_\kappa e^{i\kappa\cdot(\alpha.,\beta.)},$$

where ψ is expanded as in (1.7). ρ is any real number. It is obvious that ∂_{xy}^q commutes with A, B, ... on suitable subspaces of $L^2((-(1/2), 1/2), L^2(\mathscr{P}))$ (which is identified with $L^2(\Omega)$). This material was dealt with in [S & W, 1992], [W, 1992].

The choice of the various Hilbert spaces of functions with vanishing mean values corresponds to the invariance properties of the nonlinear terms. For these and other invariance properties see [W, 1992, ch. IV]. The norm $\|A.\|$ is equivalent with the norm of $W^4((-(1/2), 1/2), H_{\mathscr{P}}^6)$. Corresponding equivalences hold for the other operators. See [W, 1992, ch. III]. The spaces within which we solve (1.5) are now at hand. We are looking for solutions Φ with

$$(1.11)\qquad \Phi\in L^2((0, T), \mathscr{D}(\mathscr{A})),$$

$$(1.12)\qquad \Phi'\in L^2((0, T), \mathscr{D}(\mathscr{B})),$$

and, as a consequence of (1.11), (1.12),

$$(1.13)\qquad \mathscr{D}\Phi\in C^0([0, T], H)$$

with

$$\mathscr{D}=\begin{pmatrix} \nabla\check{B} & 0 & 0 & 0 & 0 \\ 0 & (-\Delta_2)^{1/2}B^{1/2} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{Pr}\,\nabla & 0 & 0 \\ 0 & 0 & 0 & \partial_z & 0 \\ 0 & 0 & 0 & 0 & \partial_z \end{pmatrix},$$

($\mathscr{D}\Phi\in C^0([0, T], H)$ means that each component of $\nabla\check{B}\varphi$ is continuous from [0, T] into $\mathscr{H}_M^1$ and so on for $\nabla\vartheta$, $\nabla\mathscr{B}\Phi$.)

$$\mathscr{D}(\mathscr{D})=\{\varphi\mid\varphi\in\mathscr{D}\check{B}),\ \nabla\varphi\in\mathscr{D}(\check{B}) \text{ for rigid boundaries and } \check{B}\varphi\in\mathscr{D}((-\Delta)^{1/2}) \text{ for stress-free boundaries}\}$$
$$\times\mathscr{D}((-\Delta_2)^{1/2}B^{1/2})\times\mathscr{D}((-\Delta)^{1/2})\times\mathscr{D}((-\partial_z^2)^{1/2})\times\mathscr{D}((-\partial_z^2)^{1/2}).$$

The initial value is also taken from $\mathcal{D}(\mathcal{D})$. $\|\mathcal{D}.\|$ is simply $\|\nabla\mathcal{B}.\|$. Thus (1.13) implies $\nabla\mathcal{B}\Phi\in C^0([0, T], H)$. In what follows we will call a solution with properties (1.11), (1.12), (1.13) a strong solution. Sometimes we simply speak of a solution.

The Ljapunov-functional $L(t)$ can now be expressed in terms of the operators we have introduced. It reads $(\|.\|=\|.\|_{L^2(\Omega)})$

$$(1.14)\quad L(t)=\|(-\Delta)(-\Delta_2)^{1/2}\varphi(t)\|^2+\|(-\Delta_N)^{1/2}(-\Delta_2)^{1/2}\psi(t)\|^2 + \Pr\|(-\Delta)^{1/2}\vartheta(t)\|^2+\|\partial_z f_1(t)\|^2+\|\partial_z f_2(t)\|^2$$

(stress-free boundaries). For its construction we need some auxiliary operators which are introduced below. We set for stress-free boundaries

$$\tilde{\mathcal{A}}\Phi=\begin{pmatrix}(-\Delta) & 0 & 0 & 0 & 0\\ 0 & (-\Delta_N) & 0 & 0 & 0\\ 0 & 0 & (-\Delta) & 0 & 0\\ 0 & 0 & 0 & (-\partial_z^2) & 0\\ 0 & 0 & 0 & 0 & (-\partial_z^2)\end{pmatrix}\cdot\begin{pmatrix}\varphi\\ \psi\\ \vartheta\\ f_1\\ f_2\end{pmatrix},$$

$$\tilde{\mathcal{D}}\Phi=\begin{pmatrix}(-\Delta)(-\Delta_2)^{1/2} & 0 & 0 & 0 & 0\\ 0 & (-\Delta_N)^{1/2}(-\Delta_2)^{1/2} & 0 & 0 & 0\\ 0 & 0 & \sqrt{\Pr}(-\Delta)^{1/2} & 0 & 0\\ 0 & 0 & 0 & \partial_z & 0\\ 0 & 0 & 0 & 0 & \partial_z\end{pmatrix}\cdot\begin{pmatrix}\varphi\\ \psi\\ \vartheta\\ f_1\\ f_2\end{pmatrix},$$

$$\tilde{\tilde{\mathcal{A}}}\Phi=\begin{pmatrix}(-\Delta)^{3/2}(-\Delta_2)^{1/2} & 0 & 0 & 0 & 0\\ 0 & (-\Delta_N)(-\Delta_2)^{1/2} & 0 & 0 & 0\\ 0 & 0 & (-\Delta) & 0 & 0\\ 0 & 0 & 0 & -\partial_z^2 & 0\\ 0 & 0 & 0 & 0 & -\partial_z^2\end{pmatrix}\cdot\begin{pmatrix}\varphi\\ \psi\\ \vartheta\\ f_1\\ f_2\end{pmatrix},$$

$\tilde{A}=(-\Delta)$, $\tilde{B}=(-\Delta_N)$, $\tilde{\tilde{A}}=(-\Delta)^{3/2}(-\Delta_2)^{1/2}$, $\tilde{\tilde{B}}=(-\Delta_N)(-\Delta_2)^{1/2}$. The definitions corresponds to those we have already given for $\mathcal{A}$, $\mathcal{B}$, A, B, ...

Acknowledgement

The valuable advice of Prof. M. Wiegner (University of Bayreuth) during the preparation of Section 2 is gratefully achnowledged.

2. The onset of convection

We refer to the definition of the onset of convection we have given in Section 1. Here we provide the mathematical background. The eigenvalue-problem ($\gamma > 0$)

$$H_\gamma \begin{pmatrix} f \\ g \end{pmatrix} := \begin{pmatrix} \gamma^2 (\gamma^2 - \partial_z^2)^2 f \\ (\gamma^2 - \partial_z^2) g \end{pmatrix} = \begin{pmatrix} \lambda \gamma^2 g \\ \lambda \gamma^2 f \end{pmatrix}$$

in $(L^2((-(1/2), 1/2)))^2$ with $f = g = \partial_z f = 0$ at $z = \pm(1/2)$ in the case of rigid boundaries and $f = g = \partial_z^2 f = 0$ at $z = \pm(1/2)$ in the case of stress-free boundaries has a minimal positive eigenvalue $\mu(\gamma^2)$. This can be proved by Courant's method (*cf.* [W, 1992, ch. VI] and Sec. 3 to follow). We can express the onset of convection in terms of $\mu(\gamma^2)$. The following assertion was proved in [W, 1992, ch. VI].

THEOREM 2.1. – *Let* $T > 0$, *let* Φ *be a solution of*

$$(2.1) \qquad \begin{cases} \mathscr{B}\Phi' + \mathscr{A}\Phi - \sqrt{R}\,\mathscr{C}\Phi + \mathscr{M}(\Phi) = 0 \\ \Phi(0) = \Phi_0 \end{cases}$$

over $(0, T)$ *as introduced in Section* 1. *Set*

$$\sqrt{R_{\min}(\alpha^2, \beta^2)} = \left(\sup_{\Phi \in \mathscr{D}(A) \setminus \{0\}} \frac{|(\mathscr{C}\Phi, \Phi)|}{(\mathscr{A}\Phi, \Phi)} \right)^{-1},$$

where φ, ψ, ϑ *in* $\Phi = (\varphi, \psi, \vartheta, f_1, f_2)^T$ *are expanded as in* (1.6), (1.7), (1.8). *Then*

$$(2.2) \qquad R_{\min}(\alpha^2, \beta^2) = \min_{\kappa = (\kappa_1, \kappa_2) \in \mathbb{Z}^2 \setminus \{0\}} \mu^2 (\alpha^2 \kappa_1^2 + \beta^2 \kappa_2^2).$$

Φ *has monotonically non-increasing kinetic energy* $\|\mathscr{B}^{1/2}\Phi(t)\|^2 = \|\underline{u}(t)\|^2 + \Pr\|\vartheta(t)\|^2$ *for any initial value* $\Phi_0 \in \mathscr{D}(\mathscr{D})$ *if and only if*

$$(2.3) \qquad R \leqq R_{\min}(\alpha^2, \beta^2).$$

If (2.3) holds then we obtain

$$(2.4) \qquad \|\mathscr{B}^{1/2}\Phi(t)\|^2 \leqq \|\mathscr{B}^{1/2}\Phi_0\|^2, \quad t \in [0, T].$$

The onset of convection is thus the surface $(\mathbb{R}^+ \times \mathbb{R}^+, R_{\min}(\mathbb{R}^+ \times \mathbb{R}^+))$ in $\mathbb{R}^3$ with $\mathbb{R}^+$ being the positive reals and with $R_{\min}$ being the mapping given by (2.2). As can be seen from the plots to follow an interesting part of the onset is a net of lines where the surface is Lipschitz-continuous only. According to [B, 1972] those lines can be understood as the set of points where the preferred cellular pattern changes.

As it is evident from Theorem 2.1 the onset of convection consists of the smallest positive eigenvalues $\sqrt{R_{\min}(\alpha^2, \beta^2)}$ of $\mathscr{A}\Phi = \lambda\mathscr{C}\Phi$ (again use Courant's method on $|(\mathscr{A}^{-1/2}\mathscr{C}\mathscr{A}^{-1/2}\Psi, \Psi)| / \|\Psi\|^2$ and observe that $\mathscr{A}^{-1/2}\mathscr{C}\mathscr{A}^{-1/2}$ is compact and hermitian).

The method of calculating the eigenvalue $\mu(\gamma^2)$ is well known. In the stress-free case we have $\mu(\gamma^2)=(\gamma^2+\pi^2)^{3/2}/\gamma$ (*see* e.g. [Chandrasekhar, 1961, p. 35]), thus μ^2 attains its global minimum at

(2.5) $$\gamma_c=\pi/\sqrt{2} \text{ with minimal value } R_c:=\mu^2(\gamma_c^2)=27\pi^4/4.$$

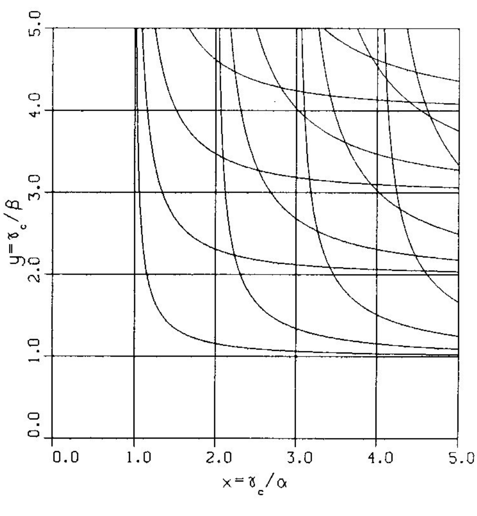

Fig. 1. – The set of points where $r_{\min}$ attains its minimal value R_c is given by a system of straight lines and hyperbolas. For γ_c and $r_{\min}$ see (2.5), (2.5'), (2.6).

In the case of rigid boundaries the characteristic equation for $\mu(\gamma^2)$ turns out to be transcendental, hence $\mu(\gamma^2)$ is known only from numerical computations, *cf.* [C, 1961, p. 36-43]. As for the minimum we have

(2.5') $$\gamma_c=3.116\ldots \text{ with minimal value } R_c=1\,707.762\ldots$$

Following [B, 1972] we wish to describe the onset in terms of the box geometry parameters for $\mathscr{P}$. Therefore we use the transformation

(2.6) $$x=\frac{\gamma_c}{\alpha}, \qquad y=\frac{\gamma_c}{\beta}, \qquad r_{\min}(x, y)=R_{\min}(\alpha^2, \beta^2),$$

allowing us to treat stress-free and rigid boundary conditions in a more unified way. In either case we get $r_{\min}(x, y)=R_c$ if and only if

$$\frac{\kappa_1^2}{x^2}+\frac{\kappa_2^2}{y^2}=1 \text{ for some } \kappa\in\mathbb{N}_0^2\setminus\{0\},$$

so the set of points where $r_{\min}$ attains its minimal value R_c consists of a system of straight lines and hyperbolas as is shown in Figure 1.

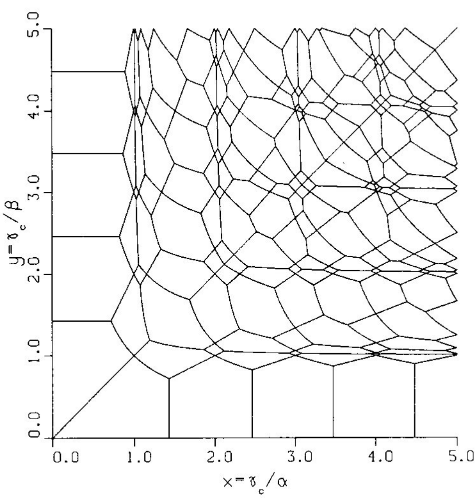

Fig. 2. – The net of lines where r_{min} is Lipschitz continuous only. The change between the modes $(k, 0)$ and $(k+1, 0)$ occurs at $x=[k^{2/3}(k+1)^{2/3}((k+1)^{2/3}+k^{2/3})/2]^{1/2}$ (stress-free boundaries). See (2.6) for r_{min}.

The net of lines in Figure 2 is produced by marking each $(x, y) \in \mathbb{R}^+ \times \mathbb{R}^+$ that satisfies $r_{min}(x, y)=\mu^2(((\kappa_1^2/x^2)+(\kappa_2^2/y^2))\gamma_c^2)$ for at least two different modes $\kappa \in \mathbb{N}_0^2 \setminus \{0\}$, whereas the connected domains marked out by the net are related to single modes κ (the preferred cellular patterns) so that r_{min} restricted to these domains is given uniquely by $\mu^2(((\kappa_1^2/x^2)+(\kappa_2^2/y^2))\gamma_c^2)$. The structure of the net is not affected by the differing choice of μ for stress-free and rigid boundaries, thus it is common to both cases as well as to the situation studied in [B, 1972].

As a result the three-dimensional plots in Figure 3 and Figure 4 on the next pages possess similar shapes. The plots are intended to show the qualitative behaviour of r_{min}. Because $r_{min}(x, y) \to R_c$ rather quickly as $x^2+y^2 \to \infty$ we have rescaled the z-axis by drawing

$$(x, y) \mapsto [r_{min}(x, y)-\rho R_c]^\sigma-[(1-\rho) R_c]^\sigma$$

with appropriate scaling parameters $\sigma \in (0, 1]$, $\rho \in [0, 1)$. Figures 3, 4 show the results for $\sigma=0.5$, $\rho=0.97$. The singularity occuring for $x=y=0$ is cut off at the level $z=14.0$.

3. A Ljapunov-functional in the case of stress-free boundaries

In this section we are going to prove that $\|\tilde{\mathscr{D}}\Phi(t)\|$ is a Ljapunov-functional for the nonlinear Boussinesq-equations, in a sense which is made precise in Theorem 3.1 below.

THEOREM 3.1. – *Let Φ be a (strong) solution of*

$$(3.1)\qquad \left\{\begin{array}{l} \mathscr{B}\Phi'+\mathscr{A}\Phi-\sqrt{R}\,\mathscr{C}\Phi+\mathscr{M}(\Phi)=0, \\ \Phi(0)=\Phi_0 \in \mathscr{D}(\mathscr{D}) \end{array}\right.$$

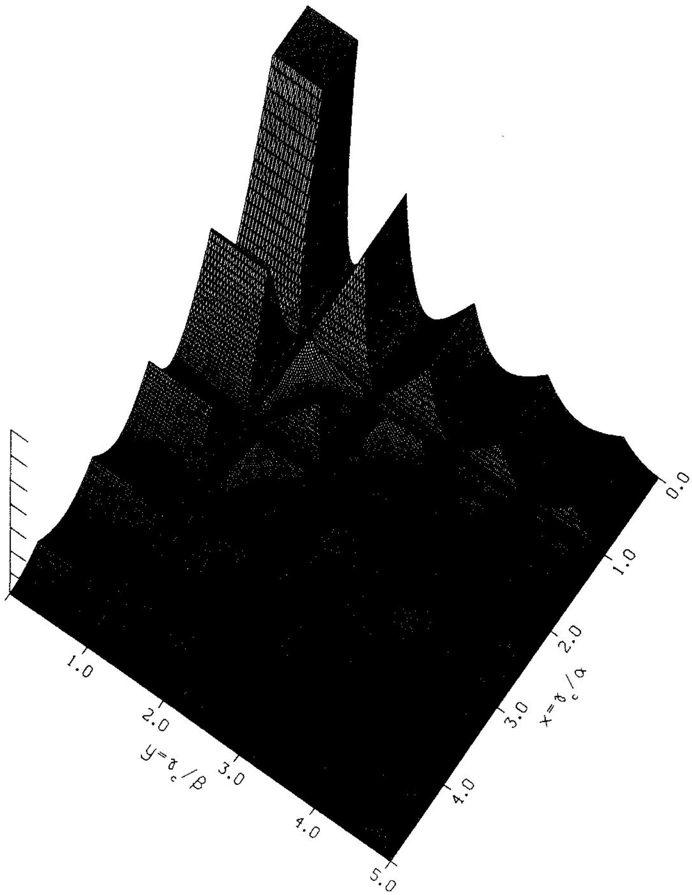

Fig. 3. – The onset of convection as a function $r_{\min}$ of the scaled box geometry parameters x and y (stress-free boundaries).

over (0, T) *as introduced in sect.* 1. *The boundaries are assumed to be stress free. Let* $R < R_{\min}(\alpha^2, \beta^2)$. *Then there is an* $\varepsilon = \varepsilon(1 - \sqrt{R/R_{\min}(\alpha^2, \beta^2)}, \alpha, \beta, \Pr) > 0$ *such that*

$\| \tilde{\mathscr{D}} \Phi(t) \|$ *is steadily decreasing if* $\| \tilde{\mathscr{D}} \Phi_0 \| \leqq \varepsilon$, *and,*

in this case, has an exponentially decreasing bound.

If conversely $R > R_{\min}(\alpha^2, \beta^2)$ *then there are initial values* $\Phi_0 \in \mathscr{D}(\mathscr{A})$ *with* $\| \tilde{\mathscr{D}} \Phi_0 \|$ *arbitrarily small such that*

$\| \tilde{\mathscr{D}} \Phi(t) \|$ *strictly increases at least initially.*

provided $\Phi \in C^0([0, T], \mathscr{D}(\tilde{\tilde{\mathscr{A}}}))$.

It is not hard to show that in the first case the (strong) solution exists globally in time. This will be done afterwards, as well as showing the property $\Phi \in C^0([0, T], \mathscr{D}(\tilde{\tilde{\mathscr{A}}}))$ in the second case.

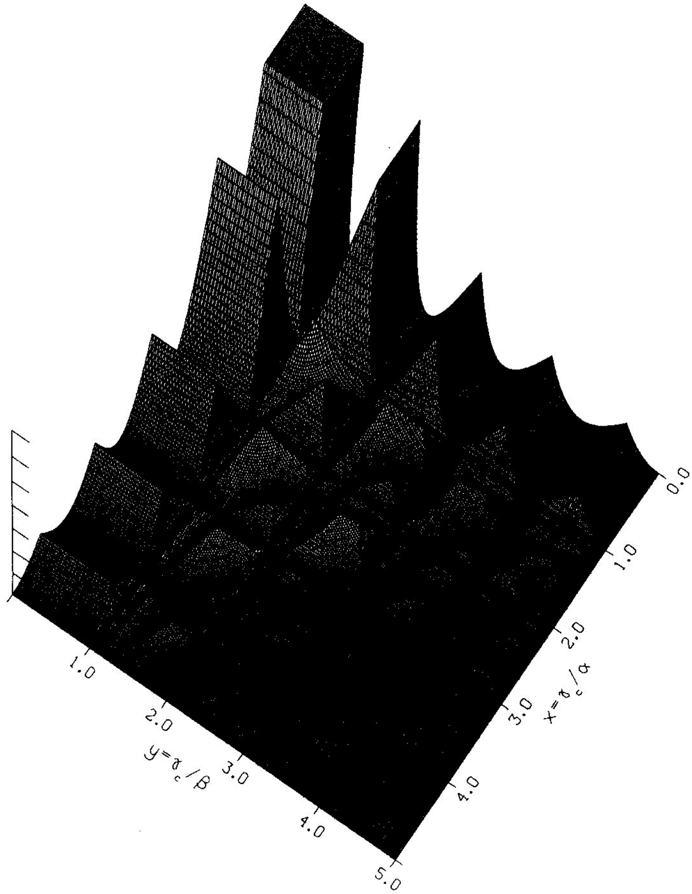

Fig. 4. – The onset of convection as a function $r_{\min}$ of the scaled box geometry parameters x and y (rigid boundaries).

Proof of Theorem 3.1. – (3.1) is multiplied scalarly by $\tilde{\mathscr{A}}\Phi$. This gives

$$\frac{d}{dt}\|\tilde{\mathscr{D}}\Phi\|^2(t)+2\|\tilde{\tilde{\mathscr{A}}}\Phi(t)\|^2\left(1-\sqrt{R}\frac{(\mathscr{C}\Phi(t),\,\tilde{\mathscr{A}}\Phi(t))}{\|\tilde{\tilde{\mathscr{A}}}\Phi(t)\|^2}+\frac{(\mathscr{M}(\Phi(t)),\,\tilde{\mathscr{A}}\Phi(t))}{\|\tilde{\tilde{\mathscr{A}}}\Phi(t)\|^2}\right)=0.$$

For $\Phi\in\mathscr{D}(\mathscr{A})$, $\Phi=(\varphi,\,\psi,\,\vartheta,f_1,f_2)^{\mathrm{T}}$ and with φ, ϑ expanded as in Section 1, we obtain

$$(3.2)\quad \|\tilde{\tilde{\mathscr{A}}}\Phi\|^2\geqq\sum_{\kappa\in\mathbb{Z}^2\setminus\{0\}}\int_{-1/2}^{+1/2}[(\alpha^2\kappa_1^2+\beta^2\kappa_2^2)\cdot((\alpha^2\kappa_1^2+\beta^2\kappa_2^2)-\partial_z^2)^3 a_\kappa\cdot\bar{a}_\kappa + ((\alpha^2\kappa_1^2+\beta^2\kappa_2^2)-\partial_z^2)^2 c_\kappa\cdot\bar{c}_\kappa]\,dz,$$

$$(3.3)\quad (\mathscr{C}\Phi,\,\tilde{\mathscr{A}}\Phi)=\sum_{\kappa\in\mathbb{Z}^2\setminus\{0\}}\int_{-1/2}^{+1/2}[(\alpha^2\kappa_1^2+\beta^2\kappa_2^2)\cdot((\alpha^2\kappa_1^2+\beta^2\kappa_2^2)-\partial_z^2)\,c_\kappa\cdot\bar{a}_\kappa + (\alpha^2\kappa_1^2+\beta^2\kappa_2^2)((\alpha^2\kappa_1^2+\beta^2\kappa_2^2)-\partial_z^2)\,a_\kappa\,\bar{c}_\kappa]\,dz.$$

The eigenvalue-problem $(\gamma>0)$

$$H_\gamma\begin{pmatrix} f \\ g \end{pmatrix}=\begin{cases}\gamma^2(\gamma^2-\partial^2)^3 f=\lambda\gamma^2(\gamma^2-\partial_z^2)g, \\ (\gamma^2-\partial_z^2)^2 g=\lambda\gamma^2(\gamma^2-\partial_z^2)f\end{cases} \tag{3.4}$$

in $(L^2((-(1/2), 1/2)))^2$ with $f=g=0$ at $z=\pm(1/2)$, $f\in\mathscr{D}((\gamma^2-\partial_z^2)^3)$, $g\in\mathscr{D}((\gamma^2-\partial_z^2)^2)$, has a minimal positive eigenvalue $\mu(\gamma^2)$. Due to the symmetry of the operator

$$S_\gamma=\begin{pmatrix} 0 & \gamma^2(\gamma^2-\partial_z^2) \\ \gamma^2(\gamma^2-\partial_z^2) & 0 \end{pmatrix}$$

in $(L^2((-(1/2), 1/2)))^2$ this is most easily checked by applying Courant's method for finding the eigenvalue of a compact selfadjoint operator which has largest modulus. At the same time we get $(F=(f, g)^T)$

$$\mu(\gamma^2)=\left(\max_{F\in\mathscr{D}(H_\gamma),\, F\neq 0}\frac{|(S_\gamma F, F)|}{(H_\gamma F, F)}\right)^{-1}. \tag{3.5}$$

$\mu(\gamma^2)$ is well-known, its value being $(\gamma^2+\pi^2)^{3/2}/\gamma$ (*cf.* [W, 1992, ch. VI] for this method, [C, 1961, p. 35] for another and the value of $\mu(\gamma^2)$). (3.5) immediately yields

$$\begin{aligned}\|\tilde{\tilde{\mathscr{A}}}\Phi\|^2 \geqq & \sum_{\kappa\in\mathbb{Z}^2\setminus\{0\}} \mu(\alpha^2\kappa_1^2+\beta^2\kappa_2^2). \\ & \times\left|\int_{-1/2}^{+1/2}[(\alpha^2[(\alpha^2\kappa_1^2+\beta^2\kappa_2^2).((\alpha^2\kappa_1^2+\beta^2\kappa_2^2)-\partial_z^2)c_\kappa\cdot\bar{a}_\kappa \right. \\ & \left. +(\alpha^2\kappa_1^2+\beta^2\kappa_2^2)((\alpha^2\kappa_1^2+\beta^2\kappa_2^2)-\partial_2^2)\alpha_\kappa\cdot\bar{c}_\kappa]\,dz\right| \\ \geqq & \min_{\kappa\in\mathbb{Z}^2\setminus\{0\}} \mu(\alpha^2\kappa_1^2+\beta^2\kappa_2^2).|(\mathscr{C}\Phi, \tilde{\mathscr{A}}\Phi)| \\ = & \sqrt{R_{\min}(\alpha^2, \beta^2)}\,|(\mathscr{C}\Phi, \tilde{\mathscr{A}}\Phi)|.\end{aligned} \tag{3.6}$$

Now we have to estimate $|(\mathscr{M}(\Phi), \tilde{\mathscr{A}}\Phi)|$. With $\underline{u}=\underline{\delta}\varphi+\underline{\varepsilon}\psi+\underline{f}$ we obtain, for some $\rho\in(0, 1)$, the estimates

$$\begin{aligned}\|\underline{u}\cdot\nabla\underline{u}\| &\leqq c\,\|\underline{\delta}\varphi+\underline{\varepsilon}\psi+\underline{f}\|_{C^0(\bar{\Omega})}\cdot\|\nabla(\underline{\delta}\varphi+\underline{\varepsilon}\psi+\underline{f})\| \\ &\leqq c\,\|\tilde{\tilde{\mathscr{A}}}^{1-\rho}\Phi\|\,\|\tilde{\mathscr{D}}\Phi\|,\end{aligned} \tag{3.7}$$

$$\|\underline{u}\cdot\nabla\vartheta\|\leqq c\,\|\tilde{\tilde{\mathscr{A}}}^{1-\rho}\Phi\|\,\|\tilde{\mathscr{D}}\Phi\|. \tag{3.8}$$

For the proof of $\|\underline{u}\|_{C^0(\bar{\Omega})}\leqq c\,\|\tilde{\tilde{\mathscr{A}}}^{1-\rho}\Phi\|$ *cf.* the Gagliardo-Nirenberg inequality and [Friedman, 1969, p. 177]. Moreover we have, as a consequence of (3.7), (3.8),

$$|(\underline{\delta}\cdot(\underline{u}\cdot\nabla\underline{u}), \tilde{A}\varphi)|\leqq c\,\|\underline{u}\cdot\nabla\underline{u}\|\,\|\tilde{\tilde{A}}\varphi\| \tag{3.9}$$

$$|(\underline{\varepsilon}\cdot(\underline{u}\cdot\nabla\underline{u}), B\psi)|\leqq c\,\|\underline{u}\cdot\nabla\underline{u}\|\,\|\tilde{\tilde{B}}\psi\| \tag{3.10}$$

$$|(\underline{u}\cdot\nabla\vartheta, \Delta\vartheta)|\leqq\|\underline{u}\cdot\nabla\vartheta\|\,\|\Delta\vartheta\| \tag{3.11}$$

$$(3.12)\quad \left|\int_{-1/2}^{1/2}\int_{\mathscr{P}} (\underline{\delta}\varphi+\underline{\varepsilon}\psi)\cdot\nabla(\underline{\delta}\varphi+\underline{\varepsilon}\psi)_x\,dx\,dy\,(-\partial_z^2 f_1)\,dz\right|$$
$$\leq c\,\|(\underline{\delta}\varphi+\underline{\varepsilon}\psi)\cdot\nabla(\underline{\delta}\varphi+\underline{\varepsilon}\psi)_x\|\,\|\partial_z^2 f_1\|\leq c\,\|\tilde{\tilde{\mathscr{A}}}^{1-\rho}\Phi\|\,\|\tilde{\mathscr{D}}\Phi\|\,\|\tilde{\tilde{\mathscr{A}}}\Phi\|,$$

$$(3.13)\quad \left|\int_{-1/2}^{1/2}\int_{\mathscr{P}} (\underline{\delta}\varphi+\underline{\varepsilon}\psi)\cdot\nabla(\underline{\delta}\varphi+\underline{\varepsilon}\psi)_y\,dx\,dy\,(-\partial_z^2 f_2)\,dz\right|\leq c\,\|\tilde{\tilde{\mathscr{A}}}^{1-\rho}\Phi\|\,\|\tilde{\mathscr{D}}\Phi\|\,\|\tilde{\tilde{\mathscr{A}}}\Phi\|.$$

Combining (3.7), (3.8) with (3.9)-(3.13) we find

$$|(\mathscr{M}(\Phi),\ \tilde{\mathscr{A}}\Phi)|\leq \mathrm{M}\,\|\tilde{\mathrm{D}}\Phi\|\,\|\tilde{\tilde{\mathscr{A}}}^{1-\rho}\Phi\|\,\|\tilde{\tilde{\mathscr{A}}}\Phi\|,$$

where $\mathrm{M}=\mathrm{M}(\alpha,\beta,\mathrm{Pr})$ depends on α, β and Pr. This gives

$$(3.14)\quad \frac{d}{dt}\|\tilde{\mathscr{D}}\Phi\|^2(t)+2\,\|\tilde{\tilde{\mathscr{A}}}\Phi(t)\|^2\left(1-\frac{\sqrt{\mathrm{R}}}{\sqrt{\mathrm{R}_{\min}(\alpha^2,\beta^2)}}-\mathrm{M}\,\|\tilde{\mathscr{D}}\Phi(t)\|\right)\leq 0.$$

This proves the first assertion. If $\mathrm{R}>\mathrm{R}_{\min}(\alpha_2,\beta^2)$ we choose an eigenvector $\tilde{\Phi}_0$ with $\tilde{\Phi}_0\in\mathscr{D}(\mathscr{A})$, $\mathscr{A}\tilde{\Phi}_0=\sqrt{\mathrm{R}_{\min}(\alpha^2,\beta^2)}\,\mathscr{C}\tilde{\Phi}_0$. Then we set

$$\Phi(0)=\tilde{\rho}\,\tilde{\Phi}_0=\Phi_0$$

where $\tilde{\rho}$ is any positive number. We have

$$\|\tilde{\tilde{\mathscr{A}}}\Phi_0\|^2=(\mathscr{A}\Phi_{0,},\ \tilde{\mathscr{A}}\Phi_0)=\sqrt{\mathrm{R}_{\min}(\alpha^2,\beta^2)}\,(\mathscr{C}\Phi_0,\ \tilde{\mathscr{A}}\Phi_0).$$

Consider the (strong) solution of (3.1) with initial value Φ_0. Then

$$(3.15)\quad \frac{d}{dt}\|\tilde{\mathscr{D}}\Phi\|^2(0)+2\,\|\tilde{\tilde{\mathscr{A}}}\Phi_0\|^2\left(1-\frac{\sqrt{\mathrm{R}}}{\sqrt{\mathrm{R}_{\min}(\alpha^2,\beta^2)}}+O(\tilde{\rho})\right)=0.$$

For sufficiently small $\tilde{\rho}$ we find $d/dt\,\|\tilde{\mathscr{D}}\Phi\|^2(0)>0$. Let us make a final remark concerning the preceding reasoning. For a strong solution Φ we have by definition

$$\Phi\in\mathrm{C}^0([0,\mathrm{T}],\ \mathscr{D}(\mathscr{D})),$$

and in particular

$$\Phi\in\mathrm{C}^0([0,\mathrm{T}],\ \mathscr{D}(\tilde{\mathscr{D}})).$$

$d/dt\,\|\tilde{\mathscr{D}}\Phi\|^2$ in (3.14) is the distributional derivative of $\|\tilde{\mathscr{D}}\Phi\|^2$. Thus the conclusion after (3.14) is justified. To get (3.15) we observe that from

$$\Phi\in\mathrm{C}^0([0,\mathrm{T}),\ \mathscr{D}(\tilde{\tilde{\mathscr{A}}}))$$

it follows that $(\mathscr{M}(\Phi(.)),\ \tilde{\mathscr{A}}\Phi(.))$ is continuous on $[0,\mathrm{T})$; we infer this from (3.7)-(3.13). Thus $\|\tilde{\mathscr{D}}\Phi\|^2(.)\in\mathrm{C}^1([0,\mathrm{T}))$. □

THEOREM 3.2. — *Let* Φ *be the strong solution of*

$$\mathscr{B}\Phi' + \mathscr{A}\Phi - \sqrt{\mathrm{R}}\,\mathscr{C}\Phi + \mathscr{M}(\Phi) = 0,$$
$$\Phi(0) = \Phi_0 \in \mathscr{D}(\mathscr{D})$$

with maximal internal of existence $[0, \mathrm{T}(\Phi_0))$. *Assume that*

$$\mathrm{R} < \mathrm{R}_{\min}(\alpha^2, \beta^2), \qquad \|\tilde{\mathscr{D}}\Phi_0\| \leqq \varepsilon,$$

where ε *is the bound of Theorem* 3.1. *Then* $\mathrm{T}(\Phi_0) = +\infty$.

Proof. — Assume that $\mathrm{T}(\Phi_0) < +\infty$. Then we have to bound $\|\mathscr{D}\Phi(t)\|$, *cf.* [W, 1992, Theorem IV.5]. The proof of Theorem 3.1 gives

$$\frac{d}{dt}\|\tilde{\mathscr{D}}\Phi\|^2(t) + \gamma\|\tilde{\tilde{\mathscr{A}}}\Phi(t)\|^2 \leqq 0$$

a.e. in $(0, \mathrm{T}(\Phi_0))$. γ is a positive constant with $\gamma = \gamma(1 - \sqrt{\mathrm{R}/\mathrm{R}_{\min}(\alpha^2, \beta^2}), \varepsilon)$. In particular

$$(3.16) \qquad \int_0^{T(\Phi_0)} \|\tilde{\tilde{\mathscr{A}}}\Phi(\tau)\|^2 d\tau \leqq \frac{\varepsilon^2}{\gamma}, \quad i.e.\ h = \|\tilde{\tilde{\mathscr{A}}}\Phi(.)\|^2 \in \mathrm{L}^2((0, \mathrm{T}(\Phi_0)))$$

Theorem IV.2 in [W, 1992] shows that

$$(3.17) \quad \|\mathscr{D}\Phi(t)\|^2 + \int_s^t \|\mathscr{A}\Phi(\tau)\|^2 d\tau$$
$$\leqq \|\mathscr{D}\Phi(s)\|^2 + c\int_s^t (\|\mathscr{M}(\Phi(\tau))\|^2 + \|\mathscr{D}\Phi(\tau)\|^2)\,d\tau, \quad 0 \leqq s < t < \mathrm{T}(\Phi_0).$$

In order to bound $\|\mathscr{D}\Phi(t)\|$ we need to estimate $\|\mathscr{M}(\Phi(\tau))\|$ in terms of $\|\tilde{\tilde{\mathscr{A}}}\Phi(\tau)\|$ and $\|\mathscr{D}\Phi(\tau)\|$. We have ($\underline{u} = \underline{\delta}\varphi + \underline{\varepsilon}\psi + \underline{f}$, $\underline{f}\cdot\nabla\underline{f} = 0$)

$$(3.18) \qquad \underline{\delta}\cdot(\underline{u}\cdot\nabla\underline{u}) = \underline{\delta}\cdot((\underline{\delta}\varphi + \underline{\varepsilon}\psi)\cdot\nabla(\underline{\delta}\varphi + \underline{\varepsilon}\psi)$$
$$+ \underline{f}\cdot\nabla(\underline{\delta}\varphi + \underline{\varepsilon}\psi) + (\underline{\delta}\varphi + \underline{\varepsilon}\psi)\cdot\nabla\underline{f}).$$

With an obvious notation we get

$$(3.19) \qquad \underline{\delta}\cdot(\underline{\delta}\varphi\cdot\nabla\underline{\delta}\varphi) = \partial^2\underline{\delta}\varphi\cdot\nabla\underline{\delta}\varphi + \partial\underline{\delta}\varphi\cdot\nabla\partial\underline{\delta}\varphi + \underline{\delta}\varphi\cdot\nabla\partial^2\underline{\delta}\varphi,$$
$$(3.20) \qquad \underline{\delta}\cdot(\underline{\delta}\varphi\cdot\nabla\underline{\varepsilon}\psi) = \partial^2\underline{\delta}\varphi\cdot\nabla\underline{\varepsilon}\psi + \partial\underline{\delta}\varphi\cdot\nabla\partial\underline{\varepsilon}\psi,$$
$$(3.21) \qquad \underline{\delta}\cdot(\underline{\varepsilon}\psi\cdot\nabla\underline{\delta}\varphi) = \partial^2\underline{\varepsilon}\psi\cdot\nabla\underline{\delta}\varphi + \partial\underline{\varepsilon}\psi\cdot\nabla\partial\underline{\delta} + \underline{\varepsilon}\psi\cdot\nabla\partial^2\underline{\delta}\varphi,$$
$$(3.22) \qquad \underline{\delta}\cdot(\underline{\varepsilon}\psi\cdot\nabla\underline{\varepsilon}\psi) = \partial^2\underline{\varepsilon}\psi\cdot\nabla\underline{\varepsilon}\psi + \partial\underline{\varepsilon}\psi\cdot\nabla\partial\underline{\varepsilon}\psi.$$

Observe that in (3.20), (3.22) terms $\underline{\delta}\varphi\cdot\nabla\partial^2\underline{\varepsilon}\psi$, $\underline{\varepsilon}\psi\cdot\nabla\partial^2\underline{\varepsilon}\psi$ do not occur (*cf*, [W, 1992, ch. II]). By Sobolev we have

$$(3.23) \qquad \|\partial^2\underline{\delta}\varphi\cdot\nabla\underline{\delta}\varphi\| \leqq c\|\nabla\underline{\delta}\varphi\|_{C^0(\bar{\Omega})}\|\partial^2\underline{\delta}\varphi\| \leqq c\|\mathscr{D}\Phi\|\,\|\tilde{\tilde{\mathscr{A}}}\Phi\|,$$
$$(3.24) \qquad \|\partial\underline{\delta}\varphi\cdot\nabla\partial\underline{\delta}\varphi\| \leqq c\|\partial\underline{\delta}\varphi\|_{C^0(\bar{\Omega})}\|\nabla\partial\underline{\delta}\varphi\| \leqq c\|\mathscr{D}\Phi\|\,\|\tilde{\tilde{\mathscr{A}}}\Phi\|$$

if $\partial\underline{\delta}\varphi$ contains at most one z-derivative, and also

(3.25) $\|\partial\underline{\delta}\varphi\cdot\nabla\underline{\partial}\varphi\|\leqq c\|\partial\underline{\delta}\varphi\|_{C^0(\bar{\Omega})}\|\nabla\partial\underline{\delta}\varphi\|\leqq c\|\mathscr{D}\Phi\|\|\tilde{\tilde{\mathscr{A}}}\Phi\|$

if $\partial\underline{\delta}\varphi$ contains two z-derivatives (then $\nabla\partial\underline{\delta}\varphi$ contains at most two z-derivatives).

The latter estimate is due to an improved version of Sobolev's inequality for x, y-periodic functions, namely

$$\|\underline{\delta}\varphi\|_{C^0(\bar{\Omega})}\leqq c\|(-\Delta)(-\Delta_2)\varphi\|, \tag{3.26}$$

i.e. we need only the derivatives ∂_{xz}, ∂_{yz}, ∂^2_{xx}, ∂^2_{yy} to majorize the $C^0(\bar{\Omega})$-norm. Correspondingly

$$\|\underline{\varepsilon}\psi\|_{C^0(\bar{\Omega})}\leqq c\|(-\Delta_N)^{1/2}(-\Delta_2)\psi\|. \tag{3.27}$$

Cf. [W, 1992, (III.23)] for (3.26), (3.27). Evidently

$$\|\underline{\delta}\varphi\cdot\nabla\partial^2\underline{\delta}\varphi\|\leqq c\|\mathscr{D}\Phi\|\|\tilde{\tilde{\mathscr{A}}}\Phi\|. \tag{3.28}$$

As for (3.20) we have

$$\|\partial^2\underline{\delta}\varphi\cdot\nabla\underline{\varepsilon}\psi\|\leqq c\|\partial^2\underline{\delta}\varphi\|_{L^6(\Omega)}\|\nabla\underline{\varepsilon}\psi\|_{L^3(\Omega)} \tag{3.29}$$

$$\leqq c\|\mathscr{D}\Phi\|\|\tilde{\tilde{\mathscr{A}}}\Phi\| \text{ by Sobolev,} \tag{3.30}$$

$$\|\partial\underline{\delta}\varphi\cdot\nabla\partial\underline{\varepsilon}\psi\|\leqq c\|\mathscr{D}\Phi\|\|\tilde{\tilde{\mathscr{A}}}\Phi\| \text{ by (3.26).} \tag{3.31}$$

For (3.21) we obtain, by Sobolev,

$$\|\partial^2\underline{\varepsilon}\psi\cdot\nabla\underline{\delta}\varphi\|\leqq c\|\tilde{\tilde{\mathscr{A}}}\Phi\|\|\nabla\underline{\delta}\varphi\|_{C^0(\bar{\Omega})} \tag{3.32}$$

$$\leqq c\|\tilde{\tilde{\mathscr{A}}}\Phi\|\|\mathscr{D}\Phi\| \text{ since } (\underline{\varepsilon}\psi)_z=0,$$

$$\|\partial\underline{\varepsilon}\psi\cdot\nabla\partial\underline{\delta}\varphi\|\leqq c\|\tilde{\tilde{\mathscr{A}}}\Phi\|\|\mathscr{D}\Phi\| \text{ as in (3.29), (3.30),} \tag{3.33}$$

$$\|\underline{\varepsilon}\psi\cdot\nabla\partial^2\underline{\delta}\varphi\|\leqq c\,|\tilde{\tilde{\mathscr{A}}}\Phi\|\|\mathscr{D}\Phi\|. \tag{3.34}$$

The second order derivatives of ψ in $\underline{\partial}\cdot(\underline{\varepsilon}\psi\cdot\nabla\underline{\varepsilon}\psi)$ do not contain a z-derivative, *cf.* [W, 1992, (II.21) and the reasoning below]. If ∂^2_{xy} is any second order derivative with respect to x or y we have, if $b_\kappa(z)$, $\kappa\in\mathbb{Z}^2\setminus\{0\}$, are the coefficients in the expansion for ψ,

$$\begin{aligned}|\partial^2_{wy}\psi(x,y,z)|&\leqq c\sum_{\kappa\in\mathbb{Z}^2\setminus\{0\}}|b_\kappa(z)|\kappa|^2\\ &\leqq c\sum_{\kappa\in\mathbb{Z}^2\setminus\{0\}}\|b_\kappa\|^{1/4}_{H^{2,2}((-(1/2),\,1/2))}|\kappa|^{1/2}\\ &\qquad\cdot\|b_\kappa\|^{3/4}_{L^2((-(1/2),\,1/2))}\cdot|\kappa|^{(9/4)+(3/8)}\frac{1}{|\kappa|^{(3/4)+(3/8)}}\end{aligned} \tag{3.35}$$

by Gagliardo-Nirenberg in one dimension,

$$\leqq c(\|\partial^2_z(-\Delta_2)\psi\|+\|(-\Delta_2)\psi\|)^{1/4}.\|(-\Delta_2)^{(3/2)+(1/4)}\psi\|^{3/4}$$

by Hölder's inequality with $\frac{1}{p_1}=\frac{1}{8}$, $\frac{1}{p_2}=\frac{3}{8}$, $\frac{1}{p_3}=\frac{1}{2}$,

$$\leqq c\|(-\Delta_N)(-\Delta_2)^{1/2}\psi\|^{1/4}\|(-\Delta_2)^{(3/2)+(1/4)}\psi\|^{3/4},$$

$$\leqq c\|B\psi\|^{13/16}\|\tilde{\mathscr{D}}\Phi\|^{3/16}.$$

Concerning (3.22) we thus obtain

$$(3.36)\qquad \|\underline{\delta}\cdot(\underline{\varepsilon}\psi\cdot\nabla\underline{\varepsilon}\psi)\|\leqq c\|\mathrm{B}\psi\|^{13/16}\|(-\Delta_N)^{1/2}(-\Delta_2)\psi\|\|\tilde{\mathscr{D}}\Phi\|^{3/16}.$$

It is now easy to see that

$$(3.37)\qquad \|\underline{\delta}\cdot(\underline{f}\cdot\nabla(\underline{\delta}\varphi+\underline{\varepsilon}\psi))\|\leqq c\|\tilde{\mathscr{A}}\Phi\|\|\mathscr{D}\Phi\|,$$

$$\|\underline{\delta}\cdot((\underline{\delta}\varphi+\underline{\varepsilon}\psi)\cdot\nabla\underline{f})\|\leqq\|\underline{\delta}\cdot(\underline{\delta}\varphi\cdot\nabla\underline{f})\|+\|\underline{\delta}\cdot(\underline{\varepsilon}\psi\cdot\nabla\underline{f})\|$$

$$(3.38)\qquad \leqq c\|\mathscr{D}\Phi\|\|\tilde{\tilde{\mathscr{A}}}\Phi\| \text{ since } \underline{\varepsilon}\psi\cdot\nabla\underline{f} \text{ vanishes.}$$

Thus we find

$$(3.39)\qquad \|\underline{\delta}\cdot(\underline{u}\cdot\nabla\underline{u})\|\leqq c(\|\tilde{\tilde{\mathscr{A}}}\Phi\|\|\mathscr{D}\Phi\|+\|\mathscr{A}\Phi\|^{13/16}.\|(-\Delta_N)^{1/2}(-\Delta_2)\psi\|\|\tilde{\mathscr{D}}\Phi\|^{3/16}).$$

It is easier to prove in a similar fashion

$$\|\underline{\varepsilon}\cdot(\underline{u}\cdot\nabla\underline{u})\|\leqq c\|\tilde{\tilde{\mathscr{A}}}\Phi\|\|\mathscr{D}\Phi\|,$$

and find corresponding estimates for $(\underline{\delta}\varphi+\underline{\varepsilon}\psi+\underline{f})\cdot\nabla\vartheta$, $\int_{\mathscr{P}}(\underline{\delta}\varphi+\underline{\varepsilon}\psi)\cdot\nabla(\underline{\delta}\varphi+\underline{\varepsilon}\psi)_x\,dx\,dy$, $\int_{\mathscr{P}}(\underline{\delta}\varphi+\underline{\varepsilon}\psi)\cdot\nabla(\underline{\delta}\varphi+\underline{\varepsilon}\psi)_y\,dx\,dy$. We arrive at

$$(3.40)\qquad \|\mathscr{M}(\Phi(\tau))\|^2\leqq c(\|\tilde{\tilde{\mathscr{A}}}\Phi(\tau)\|^2\|\mathscr{D}\Phi(\tau)\|^2 + \|\mathscr{A}\Phi(\tau)\|^{13/8}\|(-\Delta_N)^{1/2}(-\Delta_2)\psi(\tau)\|^2.\|\tilde{\mathscr{D}}\Phi(\tau)\|^{3/8}).$$

Substituting this into (3.17) it follows that

$$(3.41)\qquad \left\{\begin{aligned}&\|\mathscr{D}\Phi(t)\|^2+\int_s^t\|\mathscr{A}\Phi(\tau)\|^2\,d\tau\\ &\qquad\leqq\|\mathscr{D}\Phi(s)\|^2+c\int_s^t[\|\tilde{\tilde{\mathscr{A}}}\Phi(\tau)\|^2\|\mathscr{D}\Phi(\tau)\|^2\\ &\qquad+\varepsilon^{3/8}\|\mathscr{A}\Phi(\tau)\|^{13/8}\|(-\Delta_N)^{1/2}(-\Delta_2)\psi(\tau)\|^2+\|\mathscr{D}\Phi(\tau)\|^2]\,d\tau,\\ &\qquad 0\leqq s<t<\mathrm{T}(\Phi_0).\end{aligned}\right.$$

Applying Hölder's inequality for real numbers to the third integrand in (3.41) we obtain

$$(3.42)\qquad \begin{aligned}&\|\mathscr{D}\Phi(t)\|^2+\int_s^t\|\mathscr{A}\Phi(\tau)\|^2\,d\tau\\ &\qquad\leqq\|\mathscr{D}\Phi(s)\|^2+c\int_s^t[\|\tilde{\tilde{\mathscr{A}}}\Phi(\tau)\|^2\|\mathscr{D}\Phi(\tau)\|^2\\ &\qquad+\|(-\Delta_N)^{1/2}(-\Delta_2)\psi(\tau)\|^{26/3}\|\mathscr{D}\Phi(\tau)\|^2+\|\mathscr{D}\Phi(\tau)\|^2]\,d\tau.\end{aligned}$$

If we want to show the boundedness of $\|\mathscr{D}\Phi(t)\|$ when approaching $\mathrm{T}(\Phi_0)$ from below we therefore have to prove the integrability of $\|(-\Delta_N)^{1/2}(-\Delta_2)\psi(.)\|^{26/3}$ at $\mathrm{T}(\Phi_0)$. This is done as follows: We apply $(-\Delta)^{-1/2}(-\Delta_2)^{-1/2}$ to the row for φ and $(-\Delta_2)^{-1/2}$

to the row for ψ. This gives

$$\mathscr{B}^* \Phi' + \mathscr{A}^* \Phi - \sqrt{R}\, \mathscr{C}^* \Phi + \mathscr{M}^* (\Phi, \Phi) = 0 \tag{3.43}$$

with

$$\mathscr{B}^* = \begin{pmatrix} (-\Delta)^{1/2}(-\Delta_2)^{1/2} & 0 & 0 & 0 & 0 \\ 0 & (-\Delta_2) & 0 & 0 & 0 \\ 0 & 0 & \Pr I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{pmatrix},$$

$$\tilde{\tilde{\mathscr{A}}} = \mathscr{A}^* = \begin{pmatrix} (-\Delta)^{3/2}(-\Delta_2)^{1/2} & 0 & 0 & 0 & 0 \\ 0 & (-\Delta_N)(-\Delta_2)^{1/2} & 0 & 0 & 0 \\ 0 & 0 & (-\Delta) & 0 & 0 \\ 0 & 0 & 0 & -\partial_z^2 & 0 \\ 0 & 0 & 0 & 0 & -\partial_z^2 \end{pmatrix},$$

$$\mathscr{C}^* = \begin{pmatrix} 0 & 0 & (-\Delta)^{-1/2}(-\Delta_2)^{1/2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ (-\Delta_2) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathscr{M}^* (\Phi, \Phi) = \begin{pmatrix} (-\Delta)^{-(1/2)}(-\Delta_2)^{-(1/2)}\, \underline{\delta} \cdot (\underline{u} \cdot \nabla \underline{u}) \\ -(-\Delta_2)^{-(1/2)}\, \underline{\varepsilon} \cdot (\underline{u} \cdot \nabla \underline{u}) \\ \Pr \underline{u} \cdot \nabla \vartheta \\ |\mathscr{P}|^{-1} \int_{\mathscr{P}} \tilde{\underline{u}} \cdot \nabla \tilde{u}_x \, dx\, dy \\ |\mathscr{P}|^{-1} \int_{\mathscr{P}} \tilde{\underline{u}} \cdot \nabla \tilde{u}_y \, dx\, dy \end{pmatrix}, \tag{3.44}$$

with $\underline{u} = \underline{\delta}\varphi + \underline{\varepsilon}\psi + f$, $\tilde{\underline{u}} = \underline{\delta}\varphi + \underline{\varepsilon}\psi$. We have written $\mathscr{M}^* (\Phi, \Phi)$ instead of $\mathscr{M}^* (\Phi)$ to emphasize the bilinear character of $\mathscr{M}^*$. The first Φ refers to the factors in front of ∇, the second to the factors ∇. Due to Theorem 3.1 we have

$$\mathscr{B}^* \Phi \in L^\infty ((0, T(\Phi_0)), H), \tag{3.45}$$

$$\mathscr{C}^* \Phi \in L^\infty ((0, T(\Phi_0)), H). \tag{3.46}$$

Now we take a cut-off function with respect to time: Let $0 < 2\delta < T(\Phi_0)$ and let ζ be continuously differentiable on $[0, +\infty)$,

$$\zeta(t) = 0, \quad 0 \leqq t \leqq \delta,$$

$$\zeta(t) = 1, \quad t \geqq 2\delta.$$

Then we obtain

$$(3.47)\qquad \mathscr{B}^*(\zeta\Phi)'+\mathscr{A}^*\zeta\Phi=\sqrt{R}\,\mathscr{C}^*\zeta\Phi-\zeta'\mathscr{B}^*\Phi-\mathscr{M}'(\zeta\Phi,\Phi)=F,$$

On the left hand side of (3.47) we introduce $((-\Delta)^{1/2}(-\Delta_2)^{1/2}\varphi, (-\Delta_2)^{1/2}\psi, \vartheta, f_1, f_2)^T$ as a new unknown vector-function. Now we use an abstract result in [de Simon, 1964]. It assures that for any $r>1$ we have

$$\int_0^T \|\mathscr{B}^*(\zeta\Phi)'(\tau)\|^r d\tau+\int_0^T \|\mathscr{A}^*\zeta\Phi(\tau)\|^r d\tau$$

$$\leqq c(r)\int_0^T \|F(\tau)\|^r d\tau,$$

$$\leqq c(r,\varepsilon).T+\int_0^T \|\mathscr{M}^*(\zeta\Phi(\tau),\Phi(\tau))\|^r d\tau \text{ by (3.45), (3.46) and Theorem 3.1,}$$

$$\leqq c(r,\varepsilon).T+c\int_0^T \|\mathscr{A}^{*1-\rho}\zeta\Phi(\tau)\|^r.\|\tilde{\mathscr{D}}\Phi(\tau)\|^r d\tau$$

by using (3.7), (3.8), (3.12), (3.13), $0\leqq T<T(\Phi_0)$, provided $F\in L^r((0,T),H)$. Since

$$(3.48)\qquad \|\mathscr{A}^{*1-\rho}\zeta\Phi\|\leqq\|\mathscr{A}^*\zeta\Phi\|^{1-\rho}\|\zeta\Phi\|^{\rho}$$

we immediately obtain an *a priori* estimate for

$$\int_0^T \|\mathscr{A}^*\zeta\Phi(\tau)\|^r d\tau$$

where the bound stays bounded if T approaches $T(\Phi_0)$. Since we already know that

$$\int_0^{T(\Phi_0)} \|\mathscr{A}^*\zeta\Phi(\tau)\|^2 d\tau<+\infty$$

we can improve on the exponent 2 in the first step to get $r_1=2/(1-\rho)$. In the second we obtain $r_2=2/(1-\rho)^2$ and so on. Finally we arrive at

$$\mathscr{A}^*\Phi=\tilde{\tilde{\mathscr{A}}}\Phi\in\bigcap_{\substack{\delta,\,0<2\delta<T(\Phi_0),\\ r,\,1\leqq r<+\infty}} L^r((\delta,T(\Phi_0)),H).$$

In view of (3.42) and $\|(-\Delta_N)^{1/2}(-\Delta_2)\psi(\tau)\|\leqq c\|(-\Delta_N)(-\Delta_2)^{1/2}\psi(\tau)\|\leqq c\|\tilde{\tilde{\mathscr{A}}}\Phi(\tau)\|$ this proves the assertion. □

Now we deal with the assumption $\Phi\in C^0([0,T),\mathscr{D}(\tilde{\tilde{\mathscr{A}}}))$ in Theorem 3.1. We show

PROPOSITION 3.1. — *Let Φ be a solution of* (3.1) *over* (0, T) *as introduced in Section* 1 *with initial value $\Phi_0\in\mathscr{D}(\mathscr{A})$. Then*

$$\Phi\in C^0([0,T],\mathscr{D}(\tilde{\tilde{\mathscr{A}}})).$$

Proof. – Applying $(-\Delta)^{-1/2}(-\Delta_2)^{-1/2}$ to the first row and $(-\Delta_2)^{1/2}$ to the second row of the Boussinesq-equations, we obtain

$$(3.49)\qquad \Phi^*(t)=e^{-t\tilde{\mathscr{A}}}\Phi^*(0)+\int_0^t e^{-(t-\sigma)\tilde{\mathscr{A}}}(\sqrt{\mathrm{R}}\,\hat{\mathscr{C}}\,\Phi(\sigma)-\hat{\mathscr{M}}(\Phi(\sigma),\Phi(\sigma)))\,d\sigma$$

with $\Phi^*=((-\Delta)^{1/2}(-\Delta_2)^{1/2}\varphi, (-\Delta_2)^{1/2}\psi, \vartheta, f_1, f_2)^{\mathrm{T}}$,

$$\hat{\mathscr{A}}=\begin{pmatrix}(-\Delta) & 0 & 0 & 0 & 0\\ 0 & (-\Delta_{\mathrm{N}}) & 0 & 0 & 0\\ 0 & 0 & \mathrm{Pr}^{-1}(-\Delta) & 0 & 0\\ 0 & 0 & 0 & -\partial_z^2 & 0\\ 0 & 0 & 0 & 0 & -\partial_z^2\end{pmatrix},$$

$$\hat{\mathscr{C}}=\begin{pmatrix}0 & 0 & (-\Delta)^{-(1/2)}(-\Delta_2)^{1/2} & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ \mathrm{Pr}^{-1}(-\Delta_2) & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{pmatrix},$$

$$\hat{\mathscr{M}}(\Phi,\Phi)=\begin{pmatrix}(-\Delta)^{-(1/2)}(-\Delta_2)^{-(1/2)}\underline{\delta}\cdot(\underline{u}\cdot\nabla\underline{u})\\ -(-\Delta_2)^{-(1/2)}\underline{\varepsilon}\cdot(\underline{u}\cdot\nabla\underline{u})\\ \underline{u}\cdot\nabla\vartheta\\ |\mathscr{P}|^{-1}\int_{\mathscr{P}}\underline{\tilde{u}}\cdot\nabla\tilde{u}_x\,dx\,dy\\ |\mathscr{P}|^{-1}\int_{\mathscr{P}}\underline{\tilde{u}}\cdot\nabla\tilde{u}_y\,dx\,dy\end{pmatrix}.$$

Now we apply $\hat{\mathscr{A}}$ to both sides of (3.49). We obtain with ($\underline{u}=\underline{\delta}\varphi+\underline{\varepsilon}\psi+\underline{f}$, $\tilde{u}=\underline{\delta}\varphi+\underline{\varepsilon}\psi$)

$$(3.50)\qquad (-\Delta)\int_0^t e^{-(t-\sigma)(-\Delta)}(\sqrt{\mathrm{R}}(-\Delta)^{-(1/2)}(-\Delta_2)^{1/2}\vartheta(\sigma)$$
$$-(-\Delta_2)^{-(1/2)}(-\Delta)^{-(1/2)}\underline{\delta}\cdot(\underline{u}\cdot\nabla\underline{u})(\sigma))\,d\sigma$$
$$=\int_0^t(-\Delta)^{1/2}e^{-(t-\sigma)(-\Delta)}(\sqrt{\mathrm{R}}(-\Delta_2)^{1/2}\vartheta(\sigma)-(-\Delta_2)^{-(1/2)}\underline{\delta}\cdot(\underline{u}\cdot\nabla\underline{u})(\sigma))\,d\sigma,$$

$$(3.51)\qquad (-\Delta_{\mathrm{N}}\int_0^t e^{-(t-\sigma)(-\Delta_{\mathrm{N}})}((-\Delta_2)^{-(1/2)}\underline{\varepsilon}\cdot(\underline{u}\cdot\nabla\underline{u})(\sigma))\,d\sigma$$
$$=\int_0^t(-\Delta_{\mathrm{N}})^{1/2}e^{-(t-\sigma)(-\Delta_{\mathrm{N}})}((-\Delta_{\mathrm{N}})^{1/2}(-\Delta_2)^{-(1/2)}\underline{\varepsilon}\cdot(\underline{u}\cdot\nabla\underline{u})(\sigma))\,d\sigma,$$

$$(3.52)\qquad (-\Delta)\int_0^t e^{-(t-\sigma)(-\Delta)}(\sqrt{\mathrm{R}}(-\Delta_2)\varphi(\sigma)-\underline{u}\cdot\nabla\vartheta(\sigma))\,d\sigma$$
$$=\int_0^t(-\Delta)^{1-\rho_1}e^{-(t-\sigma)(-\Delta)}(-\Delta)^{\rho_1}(\sqrt{\mathrm{R}}(-\Delta_2)\varphi(\sigma)-\underline{u}\cdot\nabla\vartheta(\sigma))\,d\sigma$$

for some $\rho_1 \in (0, 1/2)$ which is to be specified later,

$$(3.53)\quad (-\partial_z^2)\int_0^t e^{-(t-\sigma)(-\partial_x^2)}\left(-|\mathscr{P}|^{-1}\int_{\mathscr{P}} \underline{\tilde{u}}\cdot\nabla\underline{\tilde{u}}_i\,dx\,dy\right)(\sigma)\,d\sigma$$
$$= \int_0^t (-\partial_z^2)^{1/2} e^{-(t-\sigma)(-\partial_z^2)}\left(-|\mathscr{P}|^{-1}(-\partial_z^2)^{1/2}\int_{\mathscr{P}} \underline{\tilde{u}}\cdot\nabla\underline{\tilde{u}}_i\,dx\,dy\right)(\sigma)\,d\sigma,\quad i=1, 2.$$

Due to the properties of the solution under consideration we have

$$(3.54)\quad \begin{cases} (-\Delta_2)^{1/2}\vartheta\in C^0([0, T], H), \\ (-\Delta)^{\rho_1}(-\Delta_2\varphi)\in C^0([0, T], H), \end{cases}$$

$$(3.55)\quad \begin{aligned} \|(-\Delta_2)^{-1/2}\underline{\delta}\cdot(\underline{u}\cdot\nabla\underline{u})\| &\leqq \|\partial_z(\underline{u}\cdot\nabla\underline{u})\|, \\ &\leqq \|\partial_z(\underline{\delta}\varphi\cdot\nabla\underline{u})\| + \|\partial_z(\underline{\varepsilon}\psi\cdot\nabla\underline{u})\| + \|\partial_z(\underline{f}\cdot\nabla\underline{u})\|, \\ &\leqq c\|\mathscr{D}\Phi\|\|\tilde{\tilde{\mathscr{A}}}\Phi\| \end{aligned}$$

by using (3.26) for $\partial_x\partial_z^2\varphi$, $\partial_y\partial_z^2\varphi$, the estimate $\|\underline{u}\|_{C^0(\bar{\Omega})}\leqq c\|\tilde{\tilde{\mathscr{A}}}\Phi\|$ and the fact $(\underline{\varepsilon}\psi)_z=0$. Since $(-\Delta_N)^{1/2}$, $(-\partial_z^2)^{1/2}$ do not involve boundary values we obtain in the same way

$$(3.56)\quad \|(-\Delta_N)^{1/2}(-\Delta_2)^{-(1/2)}\underline{\varepsilon}\cdot(\underline{u}\cdot\nabla\underline{u})\|\leqq c\|\mathscr{D}\Phi\|\|\tilde{\tilde{\mathscr{A}}}\Phi\|,$$

$$(3.57)\quad \|(-\partial_z^2)^{1/2}\int_{\mathscr{P}}\underline{\tilde{u}}\cdot\nabla\underline{\tilde{u}}_i\,dx\,dy\|\leqq c\|\mathscr{D}\Phi\|\|\tilde{\tilde{\mathscr{A}}}\Phi\|,\qquad i=1, 2.$$

As for $(-\Delta)^{\rho_1}$ we observe that for $\rho_1<1/4$ the power $(-\Delta)^{\rho_1}$ does not involve boundary values. In this case

$$(3.58)\quad \|(-\Delta)^{\rho_1}\underline{u}\cdot\nabla\vartheta\|\leqq\|\nabla(\underline{u}\cdot\nabla\vartheta)\|\leqq c\|\mathscr{D}\Phi\|\|\tilde{\tilde{\mathscr{A}}}\Phi\|.$$

The kernels in (3.50)-(3.53) have at most the singularity $c/(t-\sigma)^{1-\rho_1}$. Together with $\tilde{\mathscr{A}}e^{-\cdot\tilde{\mathscr{A}}}\Phi^*(0)\in C^0([0, T], H)$ $(\Phi_0\in\mathscr{D}(\mathscr{A})!)$, $\mathscr{D}\Phi\in C^0([0, T], H)$ we infer that

$$\tilde{\tilde{\mathscr{A}}}\Phi\in L^\infty((0, T), H).$$

Then however the terms in (3.50)-(3.53) are in $C^{\rho_2}([0, T], H)$ for some $\rho_2\in(0, 1)$. The assertion is proved. □

4. The global existence on the onset of convection for small initial values in the case of stress-free boundaries. The construction of an absorbing set

We assume here that (α, β, R) is on the onset of convection introduced in Section 2. The theorem to follow is however also true in the case that (α, β, R) is below the onset. This is, in view of the proof below, a triviality and does not need further consideration. We are going to prove

THEOREM 4.1. — *Let* (α, β, R) *be on the onset of convection. Let* $T>0$, *let* Φ *be a solution over* $(0, T)$ *of*

$$\left\{\begin{aligned} &\mathscr{B}\Phi'+\mathscr{A}\Phi-\sqrt{R}\,\mathscr{C}\Phi+\mathscr{M}(\Phi)=0,\\ &\Phi(0)=\Phi_0\end{aligned}\right. \tag{4.1}$$

under stress-free boundaries. Set $\hat{\Phi}=(\varphi, \psi, \sqrt{R}\,\vartheta, f_1, f_2)^T$. *There is an* $\varepsilon=\varepsilon(R, Pr)>0$ *and a* $c=c(R)>0$ *such that if*

$$\|\mathscr{D}\hat{\Phi}_0\|\leqq\varepsilon$$

then $\|\mathscr{D}\hat{\Phi}(t)\|<(1+c)\,\varepsilon$, $t\in[0, T]$. *In particular* (4.1) *has a unique global solution if* $\|\mathscr{D}\hat{\Phi}_0\|\leqq\varepsilon$.

Proof. — Let us introduce $\hat{\vartheta}=\sqrt{R}\,\vartheta$ instead of ϑ. Then (4.1) gives

$$\frac{d}{dt}\|\mathscr{D}\hat{\Phi}\|^2(t)+2\|\mathscr{A}\hat{\Phi}(t)\|^2=-2(\mathscr{M}(\hat{\Phi}(t)), \mathscr{A}\hat{\Phi}(t))+2(\hat{\mathscr{C}}\hat{\Phi}(t), \mathscr{A}\hat{\Phi}(t))=0$$

with $\hat{\Phi}=(\varphi, \psi, \hat{\vartheta}, f_1, f_2)^T$ and

$$\hat{\mathscr{C}}=\begin{pmatrix} 0 & 0 & (-\Delta_2) & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ R(-\Delta_2) & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{pmatrix}.$$

According to [W, 1992, Prop. IV.1-IV.3, Proofs of Prop. IV.5, 6] we have

$$2\|\mathscr{M}(\hat{\Phi})\|\leqq 2M\|\mathscr{A}\hat{\Phi}\|\,\|\mathscr{D}\hat{\Phi}\|$$

with $M=M(\alpha, \beta, Pr)$. Evidently

$$\begin{aligned}2|(\hat{\mathscr{C}}\hat{\Phi}, \mathscr{A}\hat{\Phi})|&\leqq 2(\|(-\Delta_2)\hat{\vartheta}\|\,\|A\varphi\|+R\|(-\Delta_2)\varphi\|\,\|(-\Delta)\hat{\vartheta}\|)\\ &\leqq\lambda_1^2\|(-\Delta)\hat{\vartheta}\|^2+\frac{1}{\lambda_1^2}\|A\varphi\|^2+\lambda_2^2\|(-\Delta)\hat{\vartheta}\|^2+\frac{R^2}{\lambda_2^2}\|(-\Delta_2)\varphi\|^2,\end{aligned}$$

$$\lambda_1, \lambda_2>0.$$

Choosing λ_1, λ_2 such that $\lambda_1^2+\lambda_2^2<2$, $1/\lambda_1^2<2$ we arrive at

$$\frac{d}{dt}\|\mathscr{D}\hat{\Phi}\|^2(t)+\gamma\|\mathscr{A}\hat{\Phi}(t)\|^2\leqq 2M\|\mathscr{A}\hat{\Phi}(t)\|^2.\|\mathscr{D}\hat{\Phi}(t)\|+\frac{R^2}{\lambda_2^2}\|(-\Delta_2)\varphi(t)\|^2$$

for some $\gamma=\gamma(\lambda_1, \lambda_2)>0$. Moreover we have $\|\mathscr{A}\hat{\Phi}\|\geqq\tilde{\gamma}\|\mathscr{D}\hat{\Phi}\|$ for some $\tilde{\gamma}>0$. Now choose an $\varepsilon>0$ such that $2(1+(2R^2/\lambda_2^2\gamma\tilde{\gamma}))^{1/2}M\varepsilon<\gamma/2$. If $\|\mathscr{D}\hat{\Phi}_0\|\leqq\varepsilon$ and if for some $t_1\in(0, T]$ we have $\|\mathscr{D}\hat{\Phi}(t_1)\|^2\geqq\varepsilon^2+\varepsilon^2(2R^2/\lambda_2^2\gamma\tilde{\gamma})$, then there is a $T_1\in(0, T]$ with

$$\|\mathscr{D}\hat{\Phi}(T_1)\|^2=\varepsilon^2+\varepsilon^2\frac{2R^2}{\lambda_2^2\gamma\tilde{\gamma}},$$

$$\|\mathscr{D}\hat{\Phi}(t)\|^2<\varepsilon^2+\varepsilon^2\frac{2R^2}{\lambda_2^2\gamma\tilde{\gamma}},\quad t\in[0, T_1)$$

since $\mathscr{D}\hat{\Phi}\in C^0([0, T], H)$. On $(0, T_1]$ we obtain

$$\frac{d}{dt}\|\mathscr{D}\hat{\Phi}\|^2(t)+\frac{\gamma}{2}\|\mathscr{A}\hat{\Phi}(t)\|^2\leqq\frac{R^2}{\lambda_2^2}\|(-\Delta_2)\varphi(t)\|^2,$$

$$\frac{d}{dt}\|\mathscr{D}\hat{\Phi}\|^2(t)+\frac{\gamma\tilde{\gamma}}{2}\|\mathscr{D}\hat{\Phi}(t)\|^2\leqq\frac{R^2}{\lambda_2^2}\|(-\Delta_2)\varphi(t)\|^2,$$

$$(4.2)\qquad \|\mathscr{D}\hat{\Phi}(t)\|^2\leqq e^{-(\gamma\tilde{\gamma}/2)t}\|\mathscr{D}\hat{\Phi}_0\|^2+\frac{R^2}{\lambda_2^2}e^{-(\gamma\tilde{\gamma}/2)t}\cdot\frac{e^{(\gamma\tilde{\gamma}/2)t}-1}{\gamma\tilde{\gamma}/2}\sup_{0\leqq t\leqq T_1}\|(-\Delta_2)\varphi(t)\|^2$$

by integration,

$$<\varepsilon^2+\frac{2R^2}{\lambda_2^2\gamma\tilde{\gamma}}\|\mathscr{D}\hat{\Phi}_0\|^2 \text{ since we are on the onset (\textit{cf.} (2.4))},$$

$$\leqq\varepsilon^2+\varepsilon^2\frac{2R^2}{\lambda_2^2\gamma\tilde{\gamma}}.$$

This is a contradiction to our assumption. The assertion is proved. □

As it is seen from the proof of the preceding theorem the assumption "(α, β, R) is on or below the onset" was only needed to show (4.2). In other words, this assumption is needed to guarantee the smallness of $(-\Delta_2)\varphi(t)=u_z(t)$. If this could be done in a different way, then our assumption can be dropped. What we have done in Theorem 4.1 is to construct an absorbing set for the basin $\|\mathscr{D}\hat{\Phi}_0\|\leqq\varepsilon$ of initial values $\hat{\Phi}_0$. The constant $c(R)$, which determines the size of the absorbing set, grows linearly with R. Due to the imbeddings we have used when deriving the estimate $\|\mathscr{M}(\hat{\Phi})\|\leqq M\|\mathscr{A}\hat{\Phi}\|\|\mathscr{D}\hat{\Phi}\|$, the quantities $\varepsilon(R)$, $c(R)$ depends on a β too.

Let (α, β, R) be above the onset of convection. This means that

$$R>R_{\min}(\alpha^2, \beta^2).$$

Then, if $\tilde{\Phi}$ is in the eigenspace of the problem $\mathscr{A}\Phi=\pm\sqrt{R_{\min}(\alpha^2, \beta^2)}\mathscr{C}\Phi$, we have

$$\mathscr{A}\tilde{\Phi}-\sqrt{\mathscr{R}}\mathscr{C}\tilde{\Phi}=-\frac{\sqrt{R}\mp\sqrt{R_{\min}(\alpha^2, \beta^2)}}{\pm\sqrt{R_{\min}(\alpha^2, \beta^2)}}\mathscr{A}\tilde{\Phi}=-(\sqrt{R}\mp\sqrt{R_{\min}(\alpha^2, \beta^2)})\mathscr{C}\tilde{\Phi}.$$

Let E_+, E_- be the eigenspaces just mentioned. Let $E=E_++E_-$ $(E_+\cap E_-=\{0\})$ and $Pr=1$. The orthogonal projection P onto the space E commutes with $\mathscr{A}-\sqrt{R}\mathscr{C}$ on E. It is the same case with $\mathscr{B}.\mathscr{A}-\sqrt{R}\mathscr{C}$ and $\mathscr{B}$ are also reduced by E. Thus we obtain for a solution Φ of (1.5) the equation

$$(4.3)\quad \frac{1}{2}\frac{d}{dt}\|\mathscr{B}^{1/2}\Phi_+\|^2(t)-\frac{\sqrt{R}-\sqrt{R_{\min}(\alpha^2, \beta^2)}}{\sqrt{R_{\min}(\alpha^2, \beta^2)}}\|\mathscr{A}^{1/2}\Phi_+(t)\|^2+(\mathscr{M}(\Phi), \Phi_+)=0$$

with Φ_+ being that part of Φ which is in E_+. If Φ does not exist for all times then $L(t)$ blows up at a finite time and the pure conducting state $\Phi\equiv 0$ (*i.e.* the motionless state $\underline{u}=0$, $\vartheta=0$) becomes unstable with respect to the particular perturbation Φ. If Φ exists for all times we refer to L and E and can follow the method in [K & K, 1973, p. 307] by using (4.3); thereby one shows that $\Phi \equiv 0$ cannot be stable with respect to all globally existing perturbations with wave numbers α, β and Rayleigh-number R.

REFERENCES

BECK J. L., 1972, Convection in a Box of Porous Material Saturated with Fluid, *Phys. Fluids*, **15**, 1377-1383.

BUSSE F. H., OR A. C., 1986, Subharmonic and asymmetric convection rolls, *Z. Angew. Math. Phys.*, **37**, 608-623.

CHANDRASEKHAR S., 1961, *Hydrodynamic and Hydromagnetic Stability*. Clarendon Press, Oxford.

FRIEDMAN A. E., 1969, *Partial Differential Equations*, Holt, Rinehart & Winston, New York.

GALDI G. P., PADULA, M., 1990, A New Approach to Energy Theory in the Stability of Fluid Motion, *Arch. Rational Mech. Anal.*, **110**, 187-286.

KIRCHGÄSSNER K., KIELHÖFER H., 1973, Stability and Bifurcation in Fluid Dynamics, *Rocky Mountain J. Math.*, 3, 275-318.

SCHMITT B. J., von WAHL W., 1992, Decomposition of Solenoidal Fields into Poloidal Fields, Toroidal Fields and the Mean Flow. Applications to the Boussinesq-Equations. The Navier-Stokes Equations II - Theory and Numerical Methods. Proceedings, Oberwolfach 1991. I. G. Heywood, K. Masuda, R. Rautmann, S. A. Solonnikov (eds.), *Lecture Notes in Math.*, **1530**, 291-305, Springer, Berlin, Heidelberg, New York.

DE SIMON L., 1964, Un'applicazione della teoria degli integrali singolari alla studio delle equazioni diffenziali lineari astratte del primo ordine, *Rend. Sem. Mat. Univ. Padova*, **34**, 205-223.

TEMAM R., 1988, *Infinite-Dimensional Dynamical Systems in Mechanics and Physics*. Applied Mathematical Sciences, **68**, Springer, Berlin, Heidelberg, New York.

VON WAHL W., 1991, On a Problem from Thermal Convection, Preprint, to appear in *Proceedings of the 3rd International Workshop-Conference on Evolution Equations, Control Theory and Biomathematics*, Han-sur-Lesse, Belgium.

VON WAHL W., 1992, The Boussinesq-Equations in Terms of Poloidal and Toroidal Fields and the Mean Flow. Lecture Notes, *Bayreuth. Math. Schr.*, **40**, 203-290.

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Monotonicity and boundedness in the Boussinesq-equations

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ABSTRACT. – The onset of convection from the motionless state of the Boussinesq-approximation to Bénard-convection is studied for both stress-free and rigid boundaries for solutions which are periodic in the horizontal directions with wave-numbers α and β. The critical Rayleigh-numbers $R(\alpha, \beta)$ for the kinetic energy are displayed graphically as a surface $\{(\alpha, \beta, R(\alpha, \beta)) \mid \alpha, \beta > 0\}$ in $\mathbb{R}^3$. For stress-free boundaries and small initial-values it is proved that the position of the point (α, β, R) relative to the onset governs the behaviour of a generalized energy functional which involves the spatial derivatives of the solution, *i.e.*, below the onset exponential decay takes place. For (α, β, R) on the onset it is shown that the motionless state is stable in the sense of Ljapunov with respect to a functional involving even higher order derivatives than the first mentioned functional. Above the onset it becomes unstable. Throughout the paper, the decomposition of the velocity field into a poloidal part, a toroidal part and the mean flow is employed as an essential tool.

1. Introduction, notations. The differential operators in the Boussinesq-equations

We consider the Boussinesq-equations ($\underline{k}=(0, 0, 1)^T$)

$$(1.1)\qquad \left\{\begin{array}{l} \underline{u}' - \Delta \underline{u} + \underline{u}\cdot\nabla \underline{u} - \sqrt{R}\,\vartheta \underline{k} + \nabla \pi = 0, \quad \nabla\cdot\underline{u} = 0, \\ \Pr \vartheta' - \Delta\vartheta + \Pr \underline{u}\cdot\nabla\vartheta - \sqrt{R}\, u_z = 0 \end{array}\right.$$

for an infinite layer $\mathbb{R}^2 \times (-(1/2), 1/2)$ heated from below. $\Pr > 0$ is the Prandtl-number, $R > 0$ is the Rayleigh-number, $\underline{u}$, ϑ have the usual meaning, and π is the pressure. The boundary-conditions at $z = \pm(1/2)$ are the usual ones: Stress-free boundaries or rigid boundaries. They are explained below. ′ refers to the derivative with respect to time, and we also prescribe the initial values $\underline{u}_0$ ϑ_0 at time $t=0$. $\underline{u}$, ϑ and π are required to be periodic in $(x, y) \in \mathbb{R}^2$ with respect to a rectangle $\mathscr{P} = (-(\pi/\alpha), \pi/\alpha) \times (-(\pi/\beta), \pi/\beta)$ with a wave-number α in x-direction and a wave-number β in y-direction.

The aim of this paper is two-fold. In Section 2 we give a graphical representation of the onset of convection. This is a surface in (α, β, R)-space referred to as the onset. Its equation has been derived rigourously in [Schmitt & von Wahl, 1992, Proposition 2.3] in the case of stress-free boundaries and in [von Wahl, 1992, Theorem VI.1, (VI.7)] in both cases. The onset of convection is characterized by the following property of the kinetic energy $E(t) = \| \underline{u}(t) \|^2_{L^2(\Omega)} + \Pr \| \vartheta(t) \|^2_{L^2(\Omega)}$, $\Omega = \mathscr{P} \times (-(1/2), 1/2)$, at time t:

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