Estimating ∇u by div u and curl u

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In this paper we obtain necessary and sufficient conditions for the validity of the estimate

$$\|\nabla u\|_{L^p} \leqslant c(\|\operatorname{div} u\|_{L^p} + \|\operatorname{curl} u\|_{L^p}).$$

The constant c does not depend on u. u is a vector field with values in $\mathbb{R}^3$. It is defined on a bounded set G of $\mathbb{R}^3$ or an unbounded one, denoted by $\hat{G}$. The boundary conditions are as follows: either the normal component of u vanishes or the tangential one does. Our conditions are expressed in terms of the Betti numbers of G or $\hat{G}$.

0. Introduction, notation

In this paper the question is studied of the conditions under which it is possible to prove an estimate (c = constant that is independent of u)

$$\|\nabla u\|_{L^p} \leqslant c(\|\operatorname{div} u\|_{L^p} + \|\operatorname{curl} u\|_{L^p}). \tag{0.1}$$

for vector fields u in $\mathbb{R}^3$; these vector fields are defined either on a bounded subset G of $\mathbb{R}^3$ or on an unbounded one, say $\hat{G} = \mathbb{R}^3 - \bar{G}$. Moreover, we assume that either the normal component (n, u) vanishes on $\partial G = \partial \hat{G}$ or the tangential one $[n, u]$ does so and that $u(x) \to 0$ as $|x| \to \infty$ on unbounded sets $\hat{G}$. Here n is the exterior normal on ∂G, G is assumed to be smoothly bounded. $[.,.]$ is the cross product in $\mathbb{R}^3$. As we shall prove, the validity of (0.1) in the case $[n, u] = 0$ is equivalent to the vanishing of the second Betti number of G (respectively, $\hat{G}$), whereas in the case $(n, u) = 0$ the estimate (0.1) holds if and only if the first Betti number of G (the same as the first Betti number of $\hat{G}$) vanishes.

Since the second Betti number of the unbounded set $\hat{G}$ always satisfies $\hat{G} \geqslant 1$ this means that the estimate (0.1) on $\hat{G}$ becomes incorrect under the boundary condition $[n, u] = 0$. Counterexamples to (0.1) in the cases that first or second Betti numbers do not vanish are provided by the harmonic vector fields. In the case of a bounded set the integration exponent p refers to any number on $(1, +\infty)$, whereas for an unbounded set p essentially has to be confined to the interval $(1, 3)$. This represents a characteristic difficulty when dealing with infinite domains.

For applications of the problems being treated here we refer the reader to [1]. There the stationary Stokes equations and the stationary Navier–Stokes equations are dealt with under boundary conditions of three different types:

(i) the velocity is given on a portion Γ_1 of the boundary,
(ii) the pressure and the tangential component of the velocity are given on a second portion Γ_2 of the boundary,
(iii) The normal component of the velocity and the tangential component of the vorticity are given on the remainder Γ_3 of the boundary.

If, for example, (ii) holds throughout the boundary our result eventually guarantees the existence and uniqueness of the velocity field of a weak solution of the stationary Stokes system in $W^{1,p}(G)$ provided G has second Betti number 0. At least this holds for $W^{1,2}(G)$.

As for the method of proof we employ classical representation formulae for vector fields in $\mathbb{R}^3$ by means of convolution integrals involving $\operatorname{div} u$, $\operatorname{curl} u$ and the representation of the solution of the stationary Maxwell equations as given by Kress [4].

On these representation formulae we apply estimates with maximal regularity, both on volume integrals and on surface integrals over ∂G. We employ C^α spaces, Sobolev spaces of integer and of fractional order, and when speaking of maximal regularity we think of the Hölder–Korn–Lichtenstein–Giraud or the Calderón–Zygmund inequalities and their variants and on the trace theorem. This method has been presented in [7, 8] to which we mostly refer in our calculations.

First we deal with (0.1) for vector fields u being of class $C^{1+\alpha}$ with $[n, u] = 0$ or $(n, u) = 0$ on ∂G, secondly with vector fields having their gradients in $L^p(G)$ or $L^p(\hat{G})$ and fulfilling the same boundary conditions. This we do by approximating such vector fields by more regular ones having the property that $[n, u]$ or (n, u) vanishes on ∂G. This step seems to be of independent interest.

It may be worth noting that (0.1) is evidently true for vector fields in $\overset{\circ}{W}{}^{1,p}(G)$ or $\overset{\circ}{W}{}^{1,p}(\hat{G})$, $+\infty > p > 1$. This is easily seen from the representation

$$
\begin{aligned}
u(x) = &-\frac{1}{4\pi} \operatorname{grad}_x \int_{\substack{G \\ (\hat{G})}} \frac{1}{|x - x'|} \operatorname{div}_{x'} u(x')\,dx' \\
&+ \frac{1}{4\pi} \operatorname{curl}_x \int_{\substack{G \\ (\hat{G})}} \frac{1}{|x - x'|} \operatorname{curl}_{x'} u(x')\,dx'
\end{aligned}
$$

by applying the Calderón–Zygmund inequality. Here u is in $C_0^\infty(G)$ or in $C_0^\infty(\hat{G})$.

We introduce some notation and explain how it is used. $G \subset \mathbb{R}^3$ is a bounded open set of $\mathbb{R}^3$, its boundary ∂G being of class C^∞. G has the structure

$$
G = \bigcup_{i=1}^{\hat{m}} G_i
$$

with the components of connectedness G_i, $\bar{G}_i \cap \bar{G}_j = \varnothing$, $i \neq j$. Each boundary ∂G_i consists of finitely many closed connected surfaces. $\hat{m}$ is the second Betti number of the complement $\hat{G} = \mathbb{R}^3 - \bar{G}$. l is the number of handles of G, that is the first Betti number of G. The complement $\hat{G}$ then has the following structure:

$$
\hat{G} = \bigcup_{i=1}^{m} \hat{G}_i \bigcup \hat{G}_{m+1},
$$

with the bounded components of connectedness $\hat{G}_1, \ldots, \hat{G}_m$ and the one unbounded component of connectedness $\hat{G}_{m+1}$. Thus m is the second Betti number of G. The sets $\hat{G}_i$ have properties corresponding to those of the G_i. The first Betti number $\hat{l}$ of $\hat{G}$ equals l. This can be shown using Alexander's duality theorem. On the basis of Alexander's handle model it is possible to construct a basis of the space of the Neumann fields, that is the harmonic vector fields in G or $\hat{G}$ with vanishing normal component (cf. [6] p. 229 and [8] p. 94); in the case of an unbounded set $\hat{G}$ these vector fields are also required to tend to zero if $|x| \to \infty$. The dimension of this space is $l = \hat{l}$, as well in the case of a bounded set G as of an unbounded set $\hat{G}$. This is in principle well known (cf. [3]). However, we need more detailed information, which we extract from the so-called fundamental theorem of vector analysis: if n is the exterior normal with respect to G, $\hat{n}$ the exterior normal with respect to $\hat{G}$, then we have ([6] p. 97)

$$u = - \operatorname{grad} \frac{1}{4\pi} \left(\int_G \frac{1}{r} \operatorname{div}' u \, dx' + \int_{\partial G} - \frac{1}{r} (n, u)' \, d\Omega' \right)$$
$$+ \operatorname{curl} \frac{1}{4\pi} \left(\int_G \frac{1}{r} \operatorname{curl}' u \, dx' + \int_{\partial G} - \frac{1}{r} [n, u]' \, d\Omega' \right), \tag{0.2}$$

in the case of a bounded set G, and

$$u = - \operatorname{grad} \frac{1}{4\pi} \left(\int_{\hat{G}} \frac{1}{r} \operatorname{div}' u \, dx' + \int_{\partial \hat{G}} - \frac{1}{r} (\hat{n}, u)' \, d\Omega' \right)$$
$$+ \operatorname{curl} \frac{1}{4\pi} \left(\int_{\hat{G}} \frac{1}{r} \operatorname{curl}' u \, dx' + \int_{\partial \hat{G}} - \frac{1}{r} [\hat{n}, u]' \, d\Omega' \right), \tag{0.3}$$

in the case of the unbounded set $\hat{G}$ if $u(x) \to 0$ as $|x| \to \infty$. Here we adopt the following notation:

$$\int_G \frac{1}{r} \operatorname{div}' u \, dx' \quad \text{stands for} \quad \int_G \frac{1}{|x - x'|} \operatorname{div}_{x'} u(x') \, dx',$$

$$\int_{\partial G} \frac{1}{r} (n, u)' \, d\Omega' \quad \text{stands for} \quad \int_G \frac{1}{|x - \xi'|} (n(\xi'), u(\xi')) \, d\Omega,$$

and so on, and more generally,

$$\int_G \frac{1}{r} f' \, dx' \quad \text{means} \quad \int_G \frac{1}{|x - x'|} f(x') \, dx',$$

$$\int_{\partial G} \frac{1}{r} f' \, d\Omega' \quad \text{means} \quad \int_{\partial G} \frac{1}{|x - \xi'|} f(\xi') \, d\Omega,$$

and so on. The formulae above hold for all sufficiently regular vector fields u and are a consequence of Green's formula. In the case of the unbounded set $\hat{G}$ we need, however, a certain asymptotic behaviour of $\hat{u}$ as $|x| \to \infty$ (cf. [8] p. 103). These formulae are easily generalized to vector fields having their gradients in $L^p(G)$ for some p, $1 < p < +\infty$, or in $L^p(\hat{G})$ for some p, $1 < p < 3$. This is discussed later in this chapter in the case of the unbounded set $\hat{G}$. The space of Dirichlet fields, that is the space of harmonic vector fields with vanishing tangential component, has dimension m in the

case of a bounded set G and dimension $\hat{m}$ in the case of the unbounded set $\hat{G}$. For these vector fields more or less what was stated in connection with the Neumann fields holds ([6] p. 212, [8] pp. 124, 191).

We set

$$\frac{\partial r^{-1}(x,\xi')}{\partial n} = \left(n(\xi'), \operatorname{grad}_x \frac{1}{|x-\xi'|} \right) = \left(n(\xi'), -\frac{x-\xi'}{|x-\xi'|^3} \right),$$

$x \in \mathbb{R}^3 - \partial G$, $\xi' \in \partial G$. Consequently,

$$\int_{\partial G} \frac{\partial r^{-1}}{\partial n} f' \, d\Omega' \text{ stands for } \int_{\partial G} \frac{\partial r^{-1}(x,\xi')}{\partial n} f(\xi') \, d\Omega.$$

For

$$\frac{\partial r^{-1}(x,\xi')}{\partial n'} = \left(n(\xi'), \operatorname{grad}_{\xi'} \frac{1}{|x-\xi'|} \right)$$

we use the analogous notation.

$W^{1,p}(G)$, $W^{1,p}(\hat{G})$, $W^{1-1/p,p}(\partial G)$, $W^{2-1/p,p}(\partial G)$ are the usual Sobolev spaces, $C^{k+\alpha}(G)$, $C^{k+\alpha}(\bar{G})$, $C^{k+\alpha}(\partial G)$ are the usual spaces of Hölder-continuous functions, $k \in \mathbb{N} \cup \{0\}$, $0 < \alpha < 1$. When referring to the *trace theorem* we mean that a function in $W^{1,p}(G)$ or in $W^{1,p}(\hat{G})$ has a trace $u|\partial G$ on $\partial G = \partial \hat{G}$ with $(1 < p < +\infty)$

$$u|\partial G \in W^{1-1/p,p}(\partial G),$$

$$\|u|\partial G\|_{W^{1-1/p,p}(\partial G)} \leqslant c \|u\|_{W^{1,p}(G)} \quad \text{or} \quad \leqslant c \|u\|_{W^{1,p}(\hat{G})};$$

on the other hand, each element $u \in W^{1-1/p,p}(\partial G)$ has a continuation, also denoted by u, to G or to $\hat{G}$ or to the boundary strip G_ρ or $\hat{G}_\rho$, being adjacent to $\partial G = \partial \hat{G}$, lying in G or $\hat{G}$ and having the width $\rho > 0$, such that $(1 < p < +\infty)$

$$\|u\|_{W^{1,p}(G)} \leqslant c \|u\|_{W^{1-1/p,p}(\partial G)},$$

$$\|u\|_{W^{1,p}(\hat{G})} \leqslant c \|u\|_{W^{1-1/p,p}(\partial G)},$$

$$\|u\|_{W^{1,p}(G_\rho)} \leqslant c(\rho) \|u\|_{W^{1-1/p,p}(\partial G)},$$

$$\|u\|_{W^{1,p}(\hat{G}_\rho)} \leqslant c(\rho) \|u\|_{W^{1-1/p,p}(\partial G)}.$$

When referring to the *Calderón–Zygmund inequality* we mean that $(1 < p < +\infty)$

$$\frac{\partial}{\partial x_i} \int_G \frac{1}{r} f' \, dx' \in W^{1,p}(G), \qquad f \in L^p(G),$$

$$\left\| \frac{\partial^2}{\partial x_i \partial x_j} \int_G \frac{1}{r} f' \, dx' \right\|_{L^p(G)} \leqslant c \|f\|_{L^p(G)},$$

$\dfrac{\partial}{\partial x_i} \displaystyle\int_{\hat{G}} \frac{1}{r} f' \, dx'$ has its gradient in $L^p(\hat{G})$ for any $f \in L^p(\hat{G})$ with compact support, and

$$\left\| \frac{\partial^2}{\partial x_i \partial x_j} \int_{\hat{G}} \frac{1}{r} f' \, dx' \right\|_{L^p(\hat{G})} \leqslant c \|f\|_{L^p(\hat{G})},$$

where c does not depend on the size of the support of f.

When referring to the *Hardy–Littlewood inequality* we mean that

$$\|r^{-\lambda} * f\|_{L^{\tilde{p}}(\mathbb{R}^3)} \leqslant c\|f\|_{L^{\tilde{q}}(\mathbb{R}^3)},$$

$$0 < \frac{1}{\tilde{p}} = \frac{1}{\tilde{q}} + \frac{\lambda}{3} - 1, \quad 0 < \lambda < 3, \quad f \in L^{\tilde{q}}(\mathbb{R}^3).$$

If we want to emphasize that we are considering a vector field u whose components lie in a Banach space $\mathscr{B}$ we write $u \in \mathscr{B}^3$. If no confusion is to be feared we use simply $\mathscr{B}$. $K_R(0)$ is the open ball of radius $R > 0$ around zero in $\mathbb{R}^3$.

The reader may observe that the integrals in (0.3) are not always well defined. Therefore some remarks on the validity of (0.3) are in order. Let u be a vector field with

$$u \in \bigcap_{R, R \geqslant R_0 > 0} W^{1,p}(K_R(0) \cap \hat{G}),$$

and

$$\nabla u \in L^p(\hat{G})$$

for some p, $1 < p < 3$. Then (cf. [7] p. 21, [8] p. 215)

$$u + \tilde{c} \in L^q(\hat{G}), \quad \frac{1}{q} = \frac{1}{p} - \frac{1}{3},$$

where $\tilde{c}$ is a vector field that is constant on the components of connectedness $\hat{G}_i$ of $\hat{G}$, $1 \leqslant i \leqslant m + 1$. $\tilde{c}$ may be chosen as to be zero on $\hat{G}_1, \ldots, \hat{G}_m$. Moreover, we can find a sequence $(\hat{u}_\nu)$ of vector fields in $W^{1,p}(\hat{G})$, being twice continuously differentiable in $\bar{\hat{G}}$, vanishing outside a compact set, such that

$$\nabla \hat{u}_\nu \to \nabla u \quad \text{in } L^p(\hat{G}),$$

$$\hat{u}_\nu \to u + \tilde{c} \quad \text{in } L^q(\hat{G}) \quad \text{as } \nu \to \infty$$

(see the proof of Theorem 3.2 in section 3 to follow). For $\hat{u}_\nu$ the formula (0.3) holds as is easily seen from [8], Theorem I.6.1, p. 184. The Hardy–Littlewood and Calderón–Zygmund inequalities imply

$$\text{grad} \int_{\hat{G}} \frac{1}{r} \text{div}' u_\nu \, dx' \to \int_{\hat{G}} \text{grad} \frac{1}{r} \text{div}' u \, dx' \quad \text{in } L^q(\hat{G}),$$

$$\frac{\partial}{\partial x_i} \text{grad} \int_{\hat{G}} \frac{1}{r} \text{div } u_\nu \, dx' \to \frac{\partial}{\partial x_i} \int_{\hat{G}} \text{grad} \frac{1}{r} \text{div}' u \, dx' \quad \text{in } L^p(\hat{G}),$$

$$\text{curl} \int_{\hat{G}} \frac{1}{r} \text{curl}' u_\nu \, dx' \to \int_{\hat{G}} \text{curl} \frac{1}{r} \text{curl}' u \, dx' \quad \text{in } L^q(\hat{G}),$$

$$\frac{\partial}{\partial x_i} \text{curl} \int_{\hat{G}} \frac{1}{r} \text{curl}' u_\nu \, dx' \to \frac{\partial}{\partial x_i} \int_{\hat{G}} \text{curl} \frac{1}{r} \text{curl}' u \, dx' \quad \text{in } L^p(\hat{G})$$

as $\nu \to \infty$. Evidently

$$(n, u_\nu) \to (n, u + \tilde{c}) \quad \text{in } W^{1-1/p, p}(\partial \hat{G}),$$

$$[n, u_\nu] \to [n, u + \tilde{c}] \quad \text{in } W^{1-1/p, p}(\partial \hat{G}).$$

Thus

$$\left\| \int_{\partial \hat{G}} \frac{1}{r} (n, u_\nu)' \, d\Omega' - \int_{\partial \hat{G}} \frac{1}{r} (n, u + \tilde{c})' \, d\Omega' \right\|_{W^{2-1/p,p}(\partial \hat{G})}$$

$$\leqslant c \, \| (n, u_\nu) - (n, u + \tilde{c}) \|_{W^{1-1/p,p}(\partial G)} \to 0 \quad \text{as} \quad \nu \to \infty,$$

and similarly

$$\left\| \int_{\partial \hat{G}} \frac{1}{r} [n, u_\nu]' \, d\Omega' - \int_{\partial \hat{G}} \frac{1}{r} [n, u + \tilde{c}]' \, d\Omega' \right\|_{W^{2-1/p,p}(\partial \hat{G})} \to 0$$

as $\nu \to \infty$ (cf. [7] pp. 9–13). By well known estimates for the Dirichlet's problem in $L^p(\hat{G})$ we obtain

$$\frac{\partial^2}{\partial x_i \partial x_j} \int_{\partial \hat{G}} \frac{1}{r} (n, u_\nu)' \, d\Omega' \to \frac{\partial^2}{\partial x_i \partial x_j} \int_{\partial \hat{G}} \frac{1}{r} (n, u + \tilde{c})' \, d\Omega' \quad \text{in } L^p(\hat{G}),$$

$$\frac{\partial}{\partial x_i} \int_{\partial \hat{G}} \frac{1}{r} (n, u_\nu)' \, d\Omega' \to \frac{\partial}{\partial x_i} \int_{\partial \hat{G}} \frac{1}{r} (n, u + \tilde{c})' \, d\Omega' \quad \text{in } L^q(\hat{G}) \quad \text{as } \nu \to \infty;$$

corresponding statements hold if (n, u_ν) is replaced by $[n, u_\nu]$. Thus we end up with

$$u + \tilde{c} = -\frac{1}{4\pi} \int_{\hat{G}} \operatorname{grad} \frac{1}{r} \operatorname{div}' u \, dx' + \operatorname{grad} \frac{1}{4\pi} \int_{\partial \hat{G}} -\frac{1}{r} (n, u + \tilde{c})' \, d\Omega'$$

$$+ \frac{1}{4\pi} \int_{\hat{G}} \operatorname{curl} \frac{1}{r} \operatorname{curl}' u \, dx' + \operatorname{curl} \frac{1}{4\pi} \int_{\partial \hat{G}} -\frac{1}{r} [n, u + \tilde{c}]' \, d\Omega'. \tag{0.4}$$

1. The case of vanishing tangential component

We are going to prove the following theorem.

Theorem 1.1. *Let $\hat{m} \in \mathbb{N}$ be arbitrary, but let $\hat{G}$ have no bounded component of connectedness. Let $u \in (C^{1+\alpha}(\bar{G}))^3$ for some $\alpha \in (0, 1)$. Let $[n, u] = 0$ on ∂G, that is u has vanishing tangential component on ∂G. Then the following estimate holds*

$$\| \nabla u \|_{L^p(G)} \leqslant c(\| \operatorname{div} u \|_{L^p(G)} + \| \operatorname{curl} u \|_{L^p(G)}), \quad 1 < p < +\infty$$

where the constant c depends on p and G but not on u.

Proof. If $\hat{G}$ has no bounded components of connectedness, then $m = 0$. According to Theorem I.3.5 in [8] p. 124 the problem

$$\operatorname{curl} v = \gamma,$$

$$\operatorname{div} v = \varepsilon,$$

$$-[n, v] = 0,$$

has one and only one classical solution v; here we have set $\gamma = \operatorname{curl} u$ and $\varepsilon = \operatorname{div} u$. Of course $v = u$. By Theorem I.3.3 in [8] p. 113 the quantity $\varepsilon^* = -(n, u)$ satisfies on ∂G the integral equation

$$\varepsilon^* - \frac{1}{2\pi} \int_{\partial G} \frac{\partial r^{-1}}{\partial n} \varepsilon^{*\prime} \, d\Omega' = \frac{1}{2\pi} \left(n, \operatorname{grad} \int_G \frac{\varepsilon'}{r} \, dx' - \operatorname{curl} \int_G \frac{\gamma'}{r} \, dx' \right). \tag{1.1}$$

The operator $I + K$ with

$$K\varepsilon^* = -\frac{1}{2\pi} \int_{\partial G} \frac{\partial r^{-1}}{\partial n} \varepsilon^{*\prime} \, d\Omega'$$

is the integral operator that governs the exterior Neumann problem, that is the Neumann problem on $\hat{G}$. Since $m = 0$, the null space of $I + K$, whose dimension is m (see [8] pp. 62, 69), consists only of the element zero. Therefore

$$\|\varepsilon^*\|_{L^p(\partial G)} \leqslant c\left(\left\|\operatorname{grad} \int_G \frac{\varepsilon'}{r} dx'\right\|_{L^p(\partial G)} + \left\|\operatorname{curl} \int_G \frac{\gamma'}{r} dx'\right\|_{L^p(\partial G)}\right)$$

$$\leqslant c\left(\left\|\operatorname{grad} \int_G \frac{\varepsilon'}{r} dx'\right\|_{W^{1,p}(G)} + \left\|\operatorname{curl} \int_G \frac{\gamma'}{r} dx'\right\|_{W^{1,p}(G)}\right)$$

by the trace theorem, which is

$$\leqslant c(\|\varepsilon\|_{L^p(G)} + \|\gamma\|_{L^p(G)})$$

by the Calderón–Zygmund and the Hardy–Littlewood inequalities.

Since

$$\left|\frac{\partial r^{-1}}{\partial n}(\xi, \xi')\right| \leqslant \frac{c}{|\xi - \xi'|}, \quad \xi, \xi' \in \partial G, \quad \xi' \neq \xi,$$

$$\left|\frac{\partial r^{-1}}{\partial n}(\eta, \xi') - \frac{\partial r^{-1}}{\partial n}(\tilde{\eta}, \xi')\right| \leqslant c \frac{|\eta - \tilde{\eta}|}{\min(|\eta - \xi'|^2, |\tilde{\eta} - \xi'|^2)}, \quad \eta \neq \xi', \quad \tilde{\eta} \neq \xi',$$

$|\eta - \xi'|, |\tilde{\eta} - \xi'| \leqslant 1$, it is well known (see [2] p. 202, in particular [5] p. 74), that

$$\left\|\int_{\partial G} \frac{\partial r^{-1}}{\partial n} \varepsilon^{*\prime} \, d\Omega'\right\|_{W^{1-1/p,p}(\partial G)}$$

$$\leqslant c\|\varepsilon^*\|_{L^p(\partial G)},$$

$$\leqslant c(\|\varepsilon\|_{L^p(G)} + \|\gamma\|_{L^p(G)}).$$

According to the integral equation for ε^* we obtain

$$\|\varepsilon^*\|_{W^{1-1/p,p}(\partial G)}$$

$$\leqslant c\left(\|\varepsilon\|_{L^p(G)} + \|\gamma\|_{L^p(G)} + \left\|\operatorname{grad} \int_G \frac{1}{r} \varepsilon' dx'\right\|_{W^{1-1/p,p}(\partial G)}\right.$$

$$\left. + \left\|\operatorname{curl} \int_G \frac{1}{r} \gamma' dx'\right\|_{W^{1-1/p,p}(\partial G)}\right),$$

$$\leqslant c\left(\|\varepsilon\|_{L^p(G)} + \|\gamma\|_{L^p(G)} + \left\|\operatorname{grad} \int_G \frac{1}{r} \varepsilon' dx'\right\|_{W^{1,p}(G)}\right.$$

$$\left. + \left\|\operatorname{curl} \int_G \frac{1}{r} \gamma' dx'\right\|_{W^{1,p}(G)}\right)$$

by the trace theorem, which is

$$\leqslant c(\|\varepsilon\|_{L^p(G)} + \|\gamma\|_{L^p(G)})$$

by the Calderón–Zygmund and the Hardy–Littlewood inequalities.

Since $[n, u] = 0$ on ∂G, the vectors $n(\xi)$, $u(\xi)$ are linearly dependent for each $\xi \in \partial G$. Let

$$\lambda_1(\xi)n(\xi) + \lambda_2(\xi)u(\xi) = 0$$

for some $\xi \in \partial G$ with $\lambda_1(\xi) \neq 0$, $\lambda_2(\xi) = 0$. It follows that $\lambda_1(\xi) = 0$, which is a contradiction. Thus

$$u(\xi) = -(n(\xi), u(\xi))n(\xi),$$
$$= \varepsilon^*(\xi)n(\xi), \quad \xi \in \partial G,$$
$$u \in W^{1-1/p, p}(\partial G).$$

According to the trace theorem there is a $w \in W^{1, p}(G)$ with

$$w = u \text{ on } \partial G,$$
$$\|w\|_{W^{1, p}(G)} \leqslant c\|w\|_{W^{1-1/p, p}(\partial G)},$$
$$= c\|u\|_{W^{1-1/p, p}(\partial G)},$$
$$\leqslant c\|\varepsilon^*\|_{W^{1-1/p, p}(\partial G)},$$
$$\leqslant c(\|\varepsilon\|_{L^p(G)} + \|\gamma\|_{L^p(G)}).$$

In particular, we have $u - w \in \mathring{W}^{1, p}(G)$. For any element $h \in \mathring{W}^{1, p}(G)$ we have the decomposition

$$h = -\operatorname{grad} \frac{1}{4\pi} \int_G \frac{\operatorname{div}'h}{r} \, dx' + \operatorname{curl} \frac{1}{4\pi} \int_G \frac{\operatorname{curl}'h}{r} \, dx';$$

from this it follows by the Calderón–Zygmund inequality that

$$\|\nabla h\|_{L^p(G)} \leqslant c(\|\operatorname{div} h\|_{L^p(G)} + \|\operatorname{curl} h\|_{L^p(G)}),$$

thus

$$\|\nabla(u - w)\|_{L^p(G)} \leqslant c(\|\operatorname{div} w\|_{L^p(G)} + \|\varepsilon\|_{L^p(G)} + \|\operatorname{curl} w\|_{L^p(G)} + \|\gamma\|_{L^p(G)}),$$
$$\leqslant c(\|\varepsilon\|_{L^p(G)} + \|\gamma\|_{L^p(G)}),$$
$$\|\nabla u\|_{L^p(G)} \leqslant c(\|\varepsilon\|_{L^p(G)} + \|\gamma\|_{L^p(G)} + \|\nabla w\|_{L^p(G)})$$
$$\leqslant c(\|\varepsilon\|_{L^p(G)} + \|\gamma\|_{L^p(G)}),$$

which is the assertion of the Theorem 1.1. $\square$

If $m \geqslant 1$ Theorem 1.1 becomes false. A counterexample is provided by the fact that the vector space

$$Y(G) = \{u \mid u \in (C^{1+\alpha}(\bar{G}))^3, \operatorname{curl} u = 0 \text{ in } \bar{G}, \operatorname{div} u = 0 \text{ in } \bar{G}, [n, u] = 0 \text{ on } \partial G\}$$

has dimension m. For this see [8] p. 124. It also follows in the same way by using Theorem I.7.1 in [8] p. 192, which is easily seen to be valid also in the case that the productivities $E^1, \ldots, E^{\hat{m}}$ introduced there do not vanish, that

$$Y(\hat{G}) = \{u \mid u \in \bigcap_{R, R \geqslant R_0 > 0} (C^{1+\alpha}(\bar{\hat{G}} \cap \overline{K_R(0)}))^3, \operatorname{curl} u = 0 \text{ in } \bar{\hat{G}}, \operatorname{div} u = 0 \text{ in } \bar{\hat{G}},$$
$$[n, u] = 0 \text{ on } \partial \hat{G} = \partial G, |u(x)| \to 0 \text{ as } |x| \to \infty\}$$

has dimension $\hat{m}$. Since $\hat{m} \geqslant 1$ is always true it follows that the assertion of Theorem 1.1 becomes false if we replace G by $\hat{G}$. To express it in a more precise way: let $1 < p < +\infty$. Then there is no estimate

$$\| \nabla u \|_{L^p(\hat{G})} \leqslant c(\| \operatorname{curl} u \|_{L^p(\hat{G})} + \| \operatorname{div} u \|_{L^p(\hat{G})})$$

for all

$$u \in \bigcap_{R, R \geqslant R_0 > 0} (C^{1+\alpha}(\bar{\hat{G}} \cap \overline{K_R(0)}))^3$$

with

$$\nabla u \in L^p(\hat{G}).$$

As for the preceding argument we have to make two remarks. The first one concerns the regularity of the vector fields of $Y(G)$ or $Y(\hat{G})$. In [8] it was not shown that they are in

$$(C^{1+\alpha}(\bar{G}))^3, \quad \bigcap_{R, R \geqslant R_0 > 0} (C^{1+\alpha}(\bar{\hat{G}} \cap \overline{K_R(0)}))^3,$$

respectively. However, if w is any vector field from $C^1(G) \cap C^\alpha(\bar{G})$ with curl $w = 0$ in G, div $w = 0$ in G, $[n, w] = 0$ on ∂G, then we obtain with $\varepsilon^* = -(n, w)$ that $(I + K)\varepsilon^* = 0$ by (1.1). From this we get $\varepsilon^* \in C^{1+\alpha}(\partial G)$ (see [7] pp. 9–13 and p. 2). For w we have the representation formula (in $\bar{G}$)

$$w = -\frac{1}{4\pi} \operatorname{grad} \int_{\partial G} \frac{1}{r} \varepsilon^{*\prime} \, d\Omega'.$$

According to [7] pp. 9–13 and p. 2 we obtain

$$\int_{\partial G} \frac{1}{r} \varepsilon^{*\prime} \, d\Omega' \bigg|_{\partial G} \in C^{2+\alpha}(\partial G).$$

Using

$$\Delta \int_G \frac{1}{r} \varepsilon^* \, d\Omega' = 0$$

we arrive at $w \in C^{1+\alpha}(\bar{G})$. If we replace G by $\hat{G}$ the argument is similar since for any w from $C^1(\hat{G}) \cap C^\alpha(\bar{\hat{G}} \cap \overline{K_R(0)})$, $R \geqslant R_0 > 0$ with curl $w = 0$, div $w = 0$ in $\hat{G}$, $[n, w] = 0$, $|w(x)| \to 0$ as $|x| \to \infty$ on ∂G we have (in $\bar{\hat{G}}$)

$$w = \frac{1}{4\pi} \operatorname{grad} \int_{\partial G} \frac{1}{r} \varepsilon^{*\prime} \, d\Omega', \tag{1.2}$$

where again $\varepsilon^* = -(n, w)$ on ∂G, cf. [8] p. 184. The second remark concerns the integrability properties at infinity of the elements of $Y(\hat{G})$. From (1.2) it follows that

$$\nabla w \in \bigcap_{p, +\infty \geqslant p > 1} L^p(\hat{G}), \quad w \in Y(\hat{G}).$$

The reader may observe that in the case that $\hat{G}$ is the exterior of the unit ball the space $Y(\hat{G})$ is spanned by the vector field $x_i/|x|$, $i = 1, 2, 3$.

2. The case of vanishing normal component

Whereas the case of vanishing tangential component is governed by the second Betti number, the case of vanishing normal component is completely described in terms of the first Betti number. We are going to prove the following theorem.

Theorem 2.1. *We assume that the first Betti number l of G vanishes. Let $u \in (C^{1+\alpha}(\bar{G}))^3$ for some $\alpha \in (0, 1)$. Let $(n, u) = 0$ on ∂G, that is u has a vanishing normal component on ∂G. Then there holds an estimate.*

$$\| \nabla u \|_{L^p(G)} \leqslant c(\| \operatorname{curl} u \|_{L^p(G)} + \| \operatorname{div} u \|_{L^p(G)}), \quad 1 < p < +\infty,$$

where the constant c depends on G and p but not on u.

Proof. According to Theorem I.3.8 in [8] p. 143 the problem

$$\operatorname{curl} v = \gamma,$$

$$\operatorname{div} v = \varepsilon,$$

$$-(n, v) = 0$$

has one and only one classical solution v; again we have set $\gamma = \operatorname{curl} u$, $\varepsilon = \operatorname{div} u$. Of course $v = u$. By Theorem I.3.6 in [8] p. 126 the quantity $\gamma^* = -[n, u]$ satisfies the integral equation

$$\gamma^* + \mathcal{R}\gamma^* = \frac{1}{2}\left[n, \operatorname{grad} \int_G \frac{1}{r}\varepsilon' \, dx' - \operatorname{curl} \int_G \frac{1}{r}\gamma' \, dx' \right] \tag{2.1}$$

on ∂G. $\mathcal{R}$ is the integral operator defined by

$$\mathcal{R}\gamma^* = \frac{1}{2\pi} \int_{\partial G} \left[n, \operatorname{curl} \frac{\gamma^{*\prime}}{r} \right] d\Omega',$$

$$= \frac{1}{2\pi} \int_{\partial G} \left[n, \left[\operatorname{grad} \frac{1}{r}, \gamma^{*\prime} \right] \right] d\Omega',$$

$$= \frac{1}{2\pi} \int_{\partial G} (n, \gamma^{*\prime}) \operatorname{grad} \frac{1}{r} d\Omega' - \frac{1}{2\pi} \int_{\partial G} \left(n, \operatorname{grad} \frac{1}{r} \right) \gamma^{*\prime} \, d\Omega'.$$

Since

$$(n(\xi), \gamma^*(\xi')) \operatorname{grad} r^{-1}(\xi, \xi') = (n(\xi) - n(\xi'), \gamma^*(\xi')) \operatorname{grad} r^{-1}(\xi, \xi'),$$

the integral operator $\mathcal{R}$ has the property that it can be decomposed into integral operators whose kernels $\tilde{\mathcal{R}}$ satisfy the estimates

$$|\tilde{\mathcal{R}}(\xi, \xi')| \leqslant c/|\xi - \xi'|' \, \xi \neq \xi' \tag{2.2}$$

$$|\mathcal{R}(\eta, \xi') - \tilde{\mathcal{R}}(\tilde{\eta}, \xi') \leqslant c \frac{|\eta - \tilde{\eta}|}{\min(|\eta - \xi'|^2, |\tilde{\eta} - \xi'|^2)},$$

$$\eta \neq \xi', \tilde{\eta} \neq \xi', |\eta - \xi'|, |\tilde{\eta} - \xi'| \leqslant 1. \tag{2.3}$$

This provides the estimate

$$\| \mathcal{R}\gamma^* \|_{W^{1-1/p, p}(\partial G)} \leqslant c \| \gamma^* \|_{L^p(\partial G)} \tag{2.4}$$

(cf. [5], p. 74). By Hilfssatz I.3.6 in [8], p. 150 the null space of $I + \mathcal{R}$ has dimension $l = 0$. Thus

$$\| \gamma^* \|_{L^p(\partial G)} \leqslant c(\| \varepsilon \|_{L^p(\partial G)} + \| \gamma \|_{L^p(\partial G)}),$$

$$\| \gamma^* \|_{W^{1 - 1/p, p}(\partial G)} \leqslant c(\| \varepsilon \|_{L^p(\partial G)} + \| \gamma \|_{L^p(\partial G)})$$

as in the proof of Theorem 1.1. Since

$$
\begin{aligned}
- u(\xi) = & - u(\xi) - (n(\xi), u(\xi)) \, n(\xi), \\
= & - [n(\xi), [n(\xi), u(\xi)]] \\
= & [n(\xi), \gamma^*(\xi)], \quad \xi \in \partial G
\end{aligned}
$$

we can proceed as in the proof of Theorem 1.1 and thus arrive at the assertion.   □

By Alexander's duality theorem the first Betti number l of G equals the first Betti number $\hat{l}$ of $\hat{G}$ (in three space dimensions). Thus it is natural to ask whether if Theorem 2.1 remains valid if G is replaced by $\hat{G}$. The answer is given by the following theorem.

Theorem 2.2. *We assume that the first Betti number $\hat{l}$ of $\hat{G}$ vanishes. Let*

$$u \in \bigcap_{R, \, R \geqslant R_0 > 0} [C^{1 + \alpha}(\overline{\hat{G} \cap \overline{K_R(0)}})]^3$$

for some $\alpha \in (0, 1)$. Let $(n, u) = 0$ on $\partial \hat{G} = \partial G$. Let $3 > p > 1$, and

$$\nabla u \in L^p(\hat{G}), u \in L^q(\hat{G}), \frac{1}{q} = \frac{1}{p} - \frac{1}{3}.$$

Then there holds an estimate

$$\| \nabla u \|_{L^p(\hat{G})} \leqslant c(\| \operatorname{curl} u \|_{L^p(\hat{G})} + \| \operatorname{div} u \|_{L^p(\hat{G})})$$

with c depending on p and $\hat{G}$ but not on u. In the case $+ \infty > p \geqslant 3$ we assume not only $\nabla u \in L^p(\hat{G})$ but also

$$\operatorname{curl} u \in L^{p_0}(\hat{G}),$$

$$\operatorname{div} u \in L^{p_0}(\hat{G}), u \in L^{q_0}(\hat{G}), \frac{1}{q_0} = \frac{1}{p_0} - \frac{1}{3},$$

for some p_0 with $3 > p_0 > 1$. Then there holds an estimate

$$
\begin{aligned}
& \| \nabla u \|_{L^{p_0}(\hat{G})} + \| \nabla u \|_{L^p(\hat{G})} \\
& \leqslant c(\| \operatorname{curl} u \|_{L^p(\hat{G})} + \| \operatorname{curl} u \|_{L^{p_0}(\hat{G})} + \| \operatorname{div} u \|_{L^p(\hat{G})} + \| \operatorname{div} u \|_{L^{p_0}(\hat{G})})
\end{aligned}
$$

with c as above. In particular $\nabla u \in L^{p_0}(\hat{G})$.

Proof. We proceed as in the proof of Theorem 2.1. The integral equation for $\gamma^* = - [n, u]$ is ($\varepsilon = \operatorname{div} u, \gamma = \operatorname{curl} u$)

$$\gamma^* - \mathcal{R}\gamma^* = \frac{-1}{2\pi} \left[n, \int_{\hat{G}} \operatorname{grad} \frac{1}{r} \varepsilon' \, dx' - \int_{\hat{G}} \operatorname{curl} \frac{1}{r} \gamma' \, dx' \right]$$

on ∂G, cf. Theorem I.8.1 in [8] p. 204 and (0.4). $\mathscr{R}$ is defined as in the proof of Theorem 2.1. By Hilfssatz I.3.5. in [8] p. 149 the null space of $I - \mathscr{R}$ has dimension $\hat{l} = 0$. Thus

$$\|\gamma^*\|_{L^p(\partial G)} \leqslant c \left(\left\| \int_{\hat{G}} \operatorname{grad} \frac{1}{r} \varepsilon' \, dx' \right\|_{L^p(\partial G)} + \left\| \int_{\hat{G}} \operatorname{curl} \frac{1}{r} \gamma' \, dx' \right\|_{L^p(\partial G)} \right)$$

and

$$\|\gamma^*\|_{W^{1-1/p,p}(\partial G)} \leqslant c \left(\left\| \int_{\hat{G}} \operatorname{grad} \frac{1}{r} \varepsilon' \, dx' \right\|_{W^{1-1/p,p}(\partial G)} + \left\| \int_{\hat{G}} \operatorname{curl} \frac{1}{r} \gamma' \, dx' \right\|_{W^{1-1/p,p}(\partial G)} \right)$$

by (2.4). Now we choose a boundary strip $\hat{G}_{\rho_0}$ of width ρ_0, lying in $\hat{G}$ and being adjacent to $\partial \hat{G} = \partial G$. The trace theorem furnishes

$$\left\| \int_{\hat{G}} \operatorname{grad} \frac{1}{r} \varepsilon' \, dx' \right\|_{W^{1-1/p,p}(\partial G)}$$
$$\leqslant c \left(\sum_{j=1}^{3} \left\| \frac{\partial}{\partial x_j} \int_{\hat{G}} \operatorname{grad} \frac{1}{r} \varepsilon' \, dx' \right\|_{L^p(\hat{G}_{\rho_0})} + \left\| \int_{\hat{G}} \operatorname{grad} \frac{1}{r} \varepsilon' \, dx' \right\|_{L^p(\hat{G}_{\rho_0})} \right), \, 3 > p > 1,$$

(2.5)

and

$$\left\| \int_{\hat{G}} \operatorname{grad} \frac{1}{r} \varepsilon' \, dx' \right\|_{W^{1-1/p,p}(\partial G)}$$
$$\leqslant c \left(\sum_{j=1}^{3} \left\| \frac{\partial}{\partial x_j} \int_{\hat{G}} \operatorname{grad} \frac{1}{r} \varepsilon' \, dx' \right\|_{L^p(\hat{G}_{\rho_0})} \right.$$
$$\left. + \left\| \int_{\hat{G}} \operatorname{grad} \frac{1}{r} \varepsilon' \, dx' \right\|_{L^{p_0}(\hat{G}_{\rho_0})} \right), \, + \infty > p \geqslant 3.$$

(2.6)

On applying the Hardy–Littlewood inequality we arrive at

$$\left\| \int_{\hat{G}} \operatorname{grad} \frac{1}{r} \varepsilon' \, dx' \right\|_{L^{\tilde{p}}(\hat{G}_{\rho_0})} \leqslant c \left\| \int_{\hat{G}} \operatorname{grad} \frac{1}{r} \varepsilon' \, dx' \right\|_{L^p(\hat{G}_{\rho_0})} \leqslant c \|\varepsilon\|_{L^p(\hat{G})},$$

$$\frac{1}{\tilde{p}} = \frac{1}{p} + \frac{2}{3} - 1, \quad 1 < p < 3,$$

$$\left\| \int_{\hat{G}} \operatorname{grad} \frac{1}{r} \varepsilon' \, dx' \right\|_{L^{\tilde{p}_0}(\hat{G}_{\rho_0})} \leqslant c \left\| \int_{\hat{G}} \operatorname{grad} \frac{1}{r} \varepsilon' \, dx' \right\|_{L^{p_0}(\hat{G}_{\rho_0})} \leqslant c \|\varepsilon\|_{L^{p_0}(\hat{G})},$$

$$\frac{1}{\tilde{p}_0} = \frac{1}{p_0} + \frac{2}{3} - 1, \quad + \infty > p \geqslant 3.$$

By the Calderón–Zygmund inequality the sums in (2.5) and (2.6) can be estimated by $\|\varepsilon\|_{L^p(\hat{G})}$ in any case. Then we estimate

$$\left\| \int_{\hat{G}} \operatorname{curl} \frac{1}{r} \gamma' \, dx' \right\|_{W^{1-1/p,p}(\partial G)}$$

in an analogous fashion and continue $u|\partial \hat{G} = -[n, \gamma^*]$ to $\hat{G}_{\rho_0}$ by means of the trace theorem. Thus the proof can be completed along similar lines to the proof of Theorem

2.1. We also see that the zero-order term on the right-hand sides of (2.5) and (2.6) cause difficulties in the case of an unbounded domain $\hat{G}$. $\square$

If $l \geqslant 1$ Theorem 2.1 becomes false. A counterexample is provided by the fact that the vector space

$$Z(G) = \{u \mid u \in (C^{1+\alpha}(\bar{G}))^3, \operatorname{curl} u = 0 \text{ in } \bar{G}, \operatorname{div} u = 0 \text{ in } \bar{G}, (n, u) = 0 \text{ on } \partial G\}$$

has dimension l. For this see [8] p. 94. It also follows in the same way $(l = \hat{l}!)$ that

$$Z(\hat{G}) = \left\{ u \mid u \in \bigcap_{R, R \geqslant R_0 > 0} (C^{1+\alpha}(\bar{\hat{G}} \cap \overline{K_R(0)}))^3, \right.$$

$$\operatorname{curl} u = 0 \text{ in } \bar{\hat{G}}, \operatorname{div} u = 0 \text{ in } \bar{\hat{G}},$$

$$\left. (n, u) = 0 \text{ on } \partial \hat{G} = \partial G, |u(x)| \to 0 \text{ as } |x| \to \infty \right\}$$

has dimension $\hat{l}$. Thus Theorem 2.2 becomes wrong if $\hat{l} \geqslant 1$. To express this in a more precise way: let $\hat{l} \geqslant 1$. Let w be an element of $Z(\hat{G})$. For w we have the representation formula (in $\bar{\hat{G}}$)

$$w = \frac{1}{4\pi} \operatorname{curl} \int_{\partial G} \frac{1}{r} [n, w]' \, d\Omega'.$$

It now follows that

$$\nabla w \in \bigcap_{p, +\infty > p > 1} L^p(\hat{G}).$$

In [8] it was not shown that the elements of $Z(G)$ or $Z(\hat{G})$ are in

$$(C^{1+\alpha}(\bar{G}))^3, \quad \bigcap_{R, R \geqslant R_0 > 0} (C^{1+\alpha}(\bar{\hat{G}} \cap \overline{K_R(0)}))^3,$$

respectively. However, if w is any vector field from $C^1(G) \cap C^\alpha(\bar{G})$ with $\operatorname{curl} w = 0$ in G, $\operatorname{div} w = 0$ in G, $(n, w) = 0$ on ∂G, then $(I + \mathscr{R})\gamma^* = 0$ with $\gamma^* = -[n, w]$. We can expand the kernel of $\mathscr{R}$ in the same way as we did for K in [7] p. 12. Thus we obtain $\gamma^* \in C^{1+\alpha}(\partial G)$. Using the formula (in $\bar{G}$)

$$w = -\frac{1}{4\pi} \operatorname{curl} \int_{\partial G} \frac{1}{r} \gamma^{*\prime} \, d\Omega'$$

we can conclude as in section 1 to show that $\nabla w \in C^{1+\alpha}(\bar{G})$. The case of the unbounded domain $\hat{G}$ is treated in an analogous way.

3. Final results

In this paragraph we extend the previous results to vector fields in $(W^{1,p}(G))^3$ or $(\hat{W}^{1,p}(\hat{G}))^3$ with

$$\hat{W}^{1,p}(\hat{G}) = \{u \mid u \in \bigcap_{R, R \geqslant R_0 > 0} (W^{1,p}(\hat{G} \cap K_R(0)))^3, \nabla u \in L^p(\hat{G})\}.$$

Our first result is

Theorem 3.1. *Let* $+ \infty > p > 1$, $u \in (W^{1,p}(G))^3$, $[n, u] = 0$ *on* ∂G *in the sense of the trace operator. The estimate* (c *independent on* u)

$$\| \nabla u \|_{L^p(G)} \leqslant c(\| \operatorname{curl} u \|_{L^p(G)} + \| \operatorname{div} u \|_{L^p(G)}) \tag{3.1}$$

is true for all u *as above if and only if* G *has second Betti number 0. Let* p *be as above,* $u \in (\hat{W}^{1,p}(\hat{G}))^3$, $[n, u] = 0$ *on* ∂G *in the sense of the trace operator. The estimate* (c *independent on* u)

$$\| \nabla u \|_{L^p(\hat{G})} \leqslant c(\| \operatorname{curl} u \|_{L^p(\hat{G})} + \| \operatorname{div} u \|_{L^p(\hat{G})}) \tag{3.2}$$

is true for all u *as above if and only if* $\hat{G}$ *has a second Betti number zero, that is* (3.2) *is false since* $\hat{G}$ *always has a second Betti number greater than or equal to one*

Proof. In view of section 1 the only thing we have to do is to approximate an element u of $(W^{1,p}(G))^3$ with $[n, u] = 0$ by elements u_ν as employed in section 1. Let q be the dual exponent to p. Let $h \in W^{1,q}(G)$. Then there is a sequence (h_ν) with $h_\nu \in C^2(\bar{G})$, $h_\nu \to h$, $\nu \to \infty$, with $W^{1,q}(G)$. We have

$$(\nabla h, \operatorname{curl} u) = \lim_{\nu \to \infty} (\nabla h_\nu, \operatorname{curl} u)$$

$$= \lim_{\nu \to \infty} \int_{\partial G} (\nabla h_\nu, [n, u]) \, d\Omega = 0.$$

Let $v \in (L^q(G))^3$. Taking its Helmholtz decomposition $v = \nabla h + P_q v$ with $h \in W^{1,q}(G)$ and with P_q being the projection of $(L^q(G))^3$ onto its divergence-free part we arrive at

$$(v, \operatorname{curl} u) = (\nabla h + P_q v, \operatorname{curl} u),$$

$$= (P_q v, \operatorname{curl} u),$$

$$= (v, P_q^* \operatorname{curl} u),$$

$$= (v, P_p \operatorname{curl} u),$$

$$\operatorname{curl} u = P_p \operatorname{curl} u$$

(cf. [7], Theorem 4.1, p. 18). According to [7], Theorem 5.1, p. 24 there is a sequence (γ_ν) with

$$\gamma_\nu \in (C_0^\infty(G))^3, \nu \in \mathbb{N},$$

$$\operatorname{div} \gamma_\nu = 0$$

$$\gamma_\nu \to \gamma = \operatorname{curl} u \text{ in } (L^p(G))^3, \nu \to \infty.$$

We also choose a sequence (ε_ν) with

$$\varepsilon_\nu \in C_0^\infty(G),$$

$$\varepsilon_\nu \to \varepsilon = \operatorname{div} u \text{ in } L^p(G), \nu \to \infty.$$

Let u_ν be the solution of the problem

$$\operatorname{curl} u_\nu = \gamma_\nu,$$

$$\operatorname{div} u_\nu = \varepsilon_\nu,$$

$$- [n, u_\nu] = 0.$$

Since G has a second Betti number $m = 0$ this problem has one and only one solution u_ν from $C^\alpha(\bar{G}) \cap C^{1+\alpha}(G)$ (cf. [8], p. 121, Theorem I.3.4]), $0 < \alpha < 1$. u_ν has the representation ($\varepsilon_\nu^* = -(n, u_\nu)$)

$$u_\nu = -\frac{1}{4\pi} \operatorname{grad} \int_G \frac{1}{r} \varepsilon_\nu' \, dx' - \frac{1}{4\pi} \operatorname{grad} \int_{\partial G} \frac{1}{r} \varepsilon_\nu^{*\prime} \, d\Omega' + \operatorname{curl} \frac{1}{4\pi} \int_G \frac{1}{r} \gamma_\nu' \, dx'.$$

For u we have ($\varepsilon^* = -(n, u)$)

$$u = -\frac{1}{4\pi} \operatorname{grad} \int_G \frac{1}{r} \varepsilon' \, dx' - \frac{1}{4\pi} \operatorname{grad} \int_{\partial G} \frac{1}{r} \varepsilon' \, d\Omega' + \operatorname{curl} \frac{1}{4\pi} \int_G \frac{1}{r} \gamma' \, dx'.$$

By Kellogg's theorem

$$\int_G \frac{1}{r} \varepsilon_\nu' \, dx' \in C^{2+\alpha}(\bar{G}),$$

$$\int_G \frac{1}{r} \gamma_\nu' \, dx' \in C^{2+\alpha}(\bar{G}).$$

For ε_ν^* we have the integral equation (1.1). Therefore $\varepsilon_\nu^* \in C^{1+\alpha}(\partial G)$ (cf. [7] pp. 9–13 and p. 2). As was pointed out at the end of section 1, the term

$$\int_G \frac{1}{r} \varepsilon_\nu^{*\prime} \, d\Omega'$$

is therefore in $C^{2+\alpha}(\bar{G})$. Consequently

$$u_\nu \in C^{1+\alpha}(\bar{G}).$$

By the Calderón–Zygmund and Hardy–Littlewood inequalities we obtain

$$\operatorname{grad} \int_G \frac{1}{r} \varepsilon_\nu' \, dx' \to \operatorname{grad} \int_G \frac{1}{r} \varepsilon' \, dx' \text{ in } W^{1,p}(G) \text{ as } \nu \to \infty,$$

$$\operatorname{curl} \int_G \frac{1}{r} \gamma_\nu' \, dx' \to \operatorname{curl} \int_G \frac{1}{r} \gamma' \, dx' \text{ in } W^{1,p}(G) \text{ as } \nu \to \infty.$$

The estimate for $\| \varepsilon^* \|_{W^{1-1/p,p}(\partial G)}$ in section 1 implies that

$$\varepsilon_\nu^* \to \varepsilon^* \text{ in } W^{1-1/p,p}(\partial G) \text{ as } \nu \to \infty;$$

namely, it is easily checked that ε^* fulfills the integral equation (1.1) by following the lines of the proof of Theorem I.3.3 in [8] p. 133 (Observe that

$$\operatorname{grad} \int_G \frac{1}{r} \varepsilon' \, dx', \quad \operatorname{curl} \int_G \frac{1}{r} \gamma' \, dx',$$

when restricted to ∂G, are in $W^{1-1/p,p}(\partial G)$). The expressions

$$\int_{\partial G} \frac{1}{r} \varepsilon_\nu^{*\prime} \, d\Omega', \quad \int_{\partial G} \frac{1}{r} \varepsilon^{*\prime} \, d\Omega',$$

when restricted to ∂G, are in $W^{2-1/p,p}(\partial G)$ and fulfill the estimate (cf. [7] pp. 9–13),

$$\left\| \int_{\partial G} \frac{1}{r} \varepsilon_\nu^{*\prime} \, d\Omega' - \int_{\partial G} \frac{1}{r} \varepsilon^{*\prime} \, d\Omega' \right\|_{W^{2-1/p,p}(\partial G)} \leqslant c \| \varepsilon_\nu^* - \varepsilon^* \|_{W^{1-1/p,p}(\partial G)}.$$

On using the well known estimate for harmonic functions in $W^{2,p}(G)$ we arrive at

$$\left\| \int_{\partial G} \frac{1}{r} \varepsilon_v^* \, d\Omega' - \int_{\partial G} \frac{1}{r} \varepsilon^{*\prime} \, d\Omega' \right\|_{W^{2,p}(G)} \leqslant c \| \varepsilon_v^* - \varepsilon^* \|_{W^{1-1/p,p}(\partial G)}.$$

The theorem in question is proved.   □

Next we deal with the case of vanishing normal component. Our result then is

Theorem 3.2. *Let* $+ \infty > p > 1$. *Let* $u \in (W^{1,p}(G))^3$, $(n, u) = 0$ *on* ∂G *in the sense of the trace operator. The estimate* (c *independent of* u)

$$\| \nabla u \|_{L^p(G)} \leqslant c(\| \operatorname{curl} u \|_{L^p(G)} + \| \operatorname{div} u \|_{L^p(G)}) \tag{3.3}$$

is true for all u *as above if and only if* G *has a first Betti number of zero. Let* $3 > p > 1$, $u \in (\hat{W}^{1,p}(\hat{G}))^3$, $u \in L^q(\hat{G})$, $q^{-1} = p^{-1} - \frac{1}{3}$, $(n, u) = 0$ *on* ∂G. *The estimate* (c *independent of* u)

$$\| \nabla u \|_{L^p(\hat{G})} \leqslant c(\| \operatorname{curl} u \|_{L^p(\hat{G})} + \| \operatorname{div} u \|_{L^p(\hat{G})}) \tag{3.4}$$

is true for all u *as above if and only if* $\hat{G}$ *has a first Betti number of zero. Let* $+ \infty > p \geqslant 3$,

$$u \in (\hat{W}^{1,p}(\hat{G}))^3 \cap (\hat{W}^{1,p_0}(\hat{G}))^3,$$

$u \in L^{q_0}(\hat{G})$, $q_0^{-1} = p_0^{-1} - \frac{1}{3}$, *for some* p_0, $1 < p_0 < 3$. *The estimate* (c *independent on* u)

$$\| \nabla u \|_{L^p(\hat{G})} + \| \nabla u \|_{L^{p_0}(G)} \leqslant c(\| \operatorname{curl} u \|_{L^p(\hat{G})} + \| \operatorname{curl} u \|_{L^{p_0}(\hat{G})}$$
$$+ \| \operatorname{div} u \|_{L^p(\hat{G})} + \| \operatorname{div} u \|_{L^{p_0}(\hat{G})}) \tag{3.5}$$

is true for all u *as above if and only if* $\hat{G}$ *has a first Betti number of zero.*

Proof. First we treat the case of the bounded domain G. We choose a sequence $(\hat{u}_v)$ with $\hat{u}_v \in (C^2(\bar{G}))^3$, $\hat{u}_v \to u$ in $(W^{1,p}(G))^3$ as $v \to \infty$. The vector fields $\operatorname{curl} \hat{u}_v$ are of class $C^1(\bar{G})$ and free of productivity. Set $\gamma_v = \operatorname{curl} \hat{u}_{v'}$, $\gamma = \operatorname{curl} u$. Next

$$\int_{G_i} \varepsilon \, dx = 0 \quad \text{with } \varepsilon = \operatorname{div} u, \quad i = 1, \ldots, \hat{m},$$

since $(n, u) = 0$ on ∂G. We choose a sequence (ε_v) with

$$\varepsilon_v \in C_0^\infty(G),$$

$$\int_G \varepsilon_v \, dx = 0, \quad v \in \mathbb{N},$$

$$\varepsilon_v \to \varepsilon \text{ in } L^p(G) \text{ as } v \to \infty.$$

Let u_v be the solution of the problem

$$\operatorname{curl} u_v = \gamma_v,$$

$$\operatorname{div} u_v = \varepsilon_v,$$

$$-(n, u_v) = 0.$$

Since G has first Betti number $l = 0$ this problem has one and only one solution u_ν from $C^\alpha(\bar{G}) \cap C^{1+\alpha}(G)$ (cf. [8] Satz I.3.8, p. 143), $0 < \alpha < 1$. u_ν has the representation $(\gamma_\nu^* = -[n, u_\nu])$

$$u_\nu = -\frac{1}{4\pi} \operatorname{grad} \int_G \frac{1}{r} \varepsilon_\nu' \, dx' + \operatorname{curl}\left(\frac{1}{4\pi} \int_G \frac{1}{r} \gamma_\nu' \, dx' + \frac{1}{4\pi} \int_{\partial G} \frac{1}{r} \gamma_\nu^{*\prime} \, d\Omega'\right).$$

For u we have $(\gamma^* = -[n, u])$

$$u = -\frac{1}{4\pi} \operatorname{grad} \int_G \frac{1}{r} \varepsilon' \, dx' + \operatorname{curl}\left(\frac{1}{4\pi} \int_G \frac{1}{r} \gamma' \, dx' + \frac{1}{4\pi} \int_{\partial G} \frac{1}{r} \gamma^{*\prime} \, d\Omega'\right).$$

Again by Kellogg's theorem

$$\int_G \frac{1}{r} \varepsilon_\nu' \, dx', \int_G \frac{1}{r} \gamma_\nu' \, dx' \in C^{2+\alpha}(\bar{G}).$$

For γ_ν^* we have the integral equation (2.1). Evidently

$$\mathcal{R}\gamma_\nu^* = \frac{1}{2\pi} \int_{\partial G} (n, \gamma^{*\prime}) \operatorname{grad} \frac{1}{r} \, d\Omega' + K\gamma_\nu^*, \tag{3.6}$$

where $K\gamma_\nu^*$ is taken componentwise with respect to γ_ν^*. This gives $K\gamma_\nu^* \in (C^{1+\alpha}(\partial G))^3$, cf. [7] pp. 9–13 and p. 2. As for the first integral we have

$$(n(\xi), \gamma^*(\xi'))\left(\operatorname{grad} \frac{1}{r}\right)(\xi, \xi')$$

$$= (n(\xi) - n(\xi'), \gamma^*(\xi'))\left(\operatorname{grad} \frac{1}{r}\right)(\xi, \xi'). \tag{3.7}$$

Now we expand $n(\xi) - n(\xi')$ in local coordinates into a Taylor polynomial with remainder term in integral form, just as we did for the kernels

$$\left(\frac{\partial}{\partial n} \frac{1}{r}\right)(\xi, \xi'), \left(\frac{\partial}{\partial n'} \frac{1}{r}\right)(\xi, \xi')$$

in [7] pp. 12–13 which gives (as in [7] pp. 9–13 and p. 2

$$\int_{\partial G} (n, \gamma^{*\prime}) \operatorname{grad} \frac{1}{r} \, d\Omega' \in (C^{1+\alpha}(\partial G))^3.$$

As in the proof of Theorem 3.1 we arrive at $u_\nu \in C^{1+\alpha}(\bar{G})$. It is easily checked that γ^* fulfills the integral equation (2.1). The estimate for the term $\|\gamma^*\|_{W^{1-1/p,p}(\partial G)}$ in section 2 shows in the same way as in the proof of Theorem 3.1 that

$$\gamma_\nu^* \to \gamma^* \text{ in } W^{1-1/p,p}(\partial G) \text{ as } \nu \to \infty.$$

For the remaining part of the proof of the first assertion of the present theorem we refer to the proof of Theorem 3.1. As for the second assertion we choose for $u \in (\hat{W}^{1,p}(\hat{G}))^3$ a sequence $(\hat{u}_\nu)$ with

$$\hat{u}_\nu \in \bigcap_{R, R \geqslant R_0 > 0} (C^2(\hat{G} \cap \overline{K_R(0)}))^3$$

$$\hat{u}_\nu(x) = 0, |x| \geqslant R_{\nu+1} \text{ for some } R_{\nu+1} > R_0 > 0,$$

$$\nabla \hat{u}_\nu \to \nabla u \text{ in } L^p(\hat{G}) \text{ as } \nu \to \infty.$$

This is done as follows: on the bounded components of connectedness $\hat{G}_i$, $1 \leqslant i \leqslant m$, this is no problem. On the single unbounded component of connectedness $\hat{G}_{m+1}$ we have

$$u \in L^q(\hat{G}_{m+1}) \text{ with } \frac{1}{q} = \frac{1}{p} - \frac{1}{3}$$

(observe that $1 < p < 3$). Next we choose a sequence (R_ν) of positive reals such that $R_0 \leqslant R_\nu < R_{\nu+1}$, $\nu \in \mathbb{N}$, $R_\nu \to \infty$ as $\nu \to +\infty$,

$$\frac{R_{\nu+1}^3 - R_\nu^3}{(R_{\nu+1} - R_\nu)^3} \leqslant \frac{(R_{\nu+1} + R_\nu)^2}{(R_{\nu+1} - R_\nu)^2} \leqslant c, \quad \nu \in \mathbb{N}. \tag{3.8}$$

For each $\nu \in \mathbb{N}$ we can choose a function ζ_ν with ζ_ν being continuously differentiable on $\overline{\hat{G}_{m+1}}$, $0 \leqslant \zeta_\nu \leqslant 1$,

$$\zeta_\nu|(K_{R_\nu}(0) \cap \hat{G}) \equiv 1, \zeta_\nu|(\hat{G}_{m+1} - \overline{K_{R_{\nu+1}}(0)}) \equiv 0,$$

$|\nabla\zeta_\nu| \leqslant c/(R_{\nu+1} - R_\nu)$ on

$$\hat{G}_{m+1} \cap K_{R_{\nu+1}}(0) - \hat{G}_{m+1} \cap \overline{K_{R_\nu}(0)},$$

$\nu \in \mathbb{N}$. Then $(i = 1, 2, 3)$ it follows that

$$\frac{\partial}{\partial x_i}(\zeta_\nu u) = \frac{\partial}{\partial x_i}\zeta_\nu u + \zeta_\nu \frac{\partial u}{\partial x_i}$$

on $\hat{G}_{m+1}$. Since u is in $L^q(\hat{G}_{m+1})$ we have

$$\int_{\hat{G}_{m+1}} \left|\frac{\partial}{\partial x_i}\zeta_\nu u\right|^p dx \leqslant \left(\int_{\mathscr{D}_\nu} |\nabla\zeta_\nu|^{pq_1} dx\right)^{1/q_1} \left(\int_{\mathscr{D}_\nu} |u|^{pq_2} dx\right)^{1/q_2}$$

with

$$q_2 = qp, q_1 = qp^{-1}/(qp^{-1} - 1), \mathscr{D}_\nu = \hat{G}_{m+1} \cap K_{R_{\nu+1}}(0) - \hat{G}_{m+1} \cap \overline{K_{R_\nu}(0)}.$$

Thus $pq_1 = 3$,

$$\int_{\mathscr{D}_\nu} |\nabla\zeta_\nu|^{pq_1} dx \leqslant c, \quad \nu \in \mathbb{N},$$

by (3.8). As

$$\int_{\mathscr{D}_\nu} |u|^q dx \to 0, \quad \nu \to \infty,$$

we see that

$$\nabla(\zeta_\nu u) \to \nabla u \text{ in } L^p(\hat{G}_{m+1}), \quad \nu \to \infty.$$

$\zeta_\nu u$ can be approximated in $W^{1,p}(\hat{G}_{m+1} \cap K_{R_{\nu+1}}(0))$ by

$$\overline{\hat{G}_{m+1} \cap K_{R_{\nu+1}}(0)}$$

vector fields $\hat{u}_\nu$, which are twice continuously differentiable in $\overline{\hat{G}_{m+1} \cap K_{R_{\nu+1}}(0)}$ and vanish near $\partial K_{R_{\nu+1}}(0)$. Let us set

$$\gamma_\nu = \operatorname{curl}\hat{u}_\nu \text{ in } \hat{G},$$

$$\varepsilon_\nu = \operatorname{div}\hat{u}_\nu \text{ in } \hat{G}_{m+1},$$

whereas on $\hat{G}_i$, $1 \leqslant i \leqslant m$, we choose a sequence (ε_ν) as in the first part of the proof. Let u_ν be the solution of the problem

$$\operatorname{curl} u_\nu = \gamma_\nu,$$

$$\operatorname{div} u_\nu = \varepsilon_\nu,$$

$$-(n, u_\nu) = 0.$$

Since $\hat{G}$ has first Betti number $\hat{l} = 0$ this problem has one and only one solution u_ν with

$$u_\nu \in \bigcap_{\substack{\delta', \delta_0 \geqslant \delta' > 0, \\ R, R \geqslant R_0 > 0}} (L^{3-\delta'}(\hat{G}))^3 \cap (C^\alpha(\bar{\hat{G}} \cap \overline{K_R(0)}))^3 \cap (C^{1+\alpha}(\hat{G}))^3 \cap (L^\infty(\hat{G}))^3,$$

and

$$\frac{\partial u_\nu}{\partial x_j} \in \bigcap_{\substack{\delta', \delta_0 \geqslant \delta' > 0, \\ R, R \geqslant R_0 > 0}} (L^{3/2-\delta}(\hat{G}))^3 \cap (C^\alpha(\hat{G} \cap K_R(0)))^3 \cap (L^\infty(\hat{G}))^3, \quad j = 1, 2, 3,$$

for every $\delta_0 \in (0, \tfrac{1}{2})$ and for every α, $0 < \alpha < 1$ (cf. [8], Theorem I.8.2, p. 205). For u_ν we have the formula

$$u_\nu = -\frac{1}{4\pi} \operatorname{grad} \int_{\hat{G}} \frac{1}{r} \varepsilon_\nu' \, dx' + \operatorname{curl}\left(\frac{1}{4\pi} \int_{\hat{G}} \frac{1}{r} \gamma_\nu' \, dx' + \frac{1}{4\pi} \int_{\partial\hat{G}} \frac{1}{r} \gamma_\nu^{*'} \, d\Omega \right)$$

with $\gamma_\nu^* = -[\hat{n}, u_\nu]$, and γ_ν^* satisfies the integral equation

$$\gamma_\nu^* - \mathscr{R}\gamma_\nu^* = \frac{1}{2\pi}\left[\hat{n}, \operatorname{grad} \int_{\hat{G}} \frac{1}{r} \varepsilon_\nu' \, dx' - \operatorname{curl} \int_{\hat{G}} \frac{1}{r} \gamma_\nu' \, dx' \right]$$

with $\mathscr{R}$ defined by (3.6). As before we arrive at

$$u_\nu \in \bigcap_{R, R \geqslant R_0 > 0} C^{1+\alpha}(\bar{\hat{G}} \cap \overline{K_R(0)}),$$

$\gamma^* = -[\hat{n}, u]$ that, owing to the restriction $3 > p > 1$, satisfies the integral equation

$$\gamma^* - \mathscr{R}\gamma^* = \frac{1}{2\pi}\left[\hat{n}, \int_{\hat{G}} \operatorname{grad}\frac{1}{r}\varepsilon' \, dx' - \int_{\hat{G}} \operatorname{curl}\frac{1}{r}\gamma' \, dx' \right] \tag{3.9}$$

(cf. [8] proof of Theorem I.8.1, p. 204) and, again owing to the restriction $3 > p > 1$ (cf. the proof of Theorem 2.2), $\gamma_\nu^* \to \gamma^*$ in $W^{1-1/p,p}(\partial G)$ as $\nu \to \infty$.

Since $3 > p > 1$ one can show that

$$u = -\frac{1}{4\pi} \int_{\hat{G}} \operatorname{grad}\frac{1}{r}\varepsilon' \, dx' + \frac{1}{4\pi} \operatorname{curl} \int_{\hat{G}} \frac{1}{r}\gamma' \, dx'$$

$$+ \frac{1}{4\pi} \operatorname{curl} \int_{\partial\hat{G}} \frac{1}{r}\gamma^{*'} \, d\Omega' \quad \text{with } \varepsilon = \operatorname{div} u, \, \gamma = \operatorname{curl} u \tag{3.10}$$

(cf. the proof of (0.4) in section 0). Employing the Calderón–Zygmund inequality we obtain ($i = 1, 2, 3$)

$$\frac{\partial}{\partial x_i} \operatorname{grad} \int_{\hat{G}} \frac{1}{r}\varepsilon_\nu' \, dx' \to \frac{\partial}{\partial x_i} \int_{\hat{G}} \operatorname{grad}\frac{1}{r}\varepsilon' \, dx' \text{ in } L^p(\hat{G}) \text{ as } \nu \to \infty,$$

$$\frac{\partial}{\partial x_i} \operatorname{curl} \int_{\hat{G}} \frac{1}{r}\gamma_\nu' \, dx' \to \frac{\partial}{\partial x_i} \int_{\hat{G}} \operatorname{curl}\frac{1}{r}\gamma' \, dx' \text{ in } L^p(\hat{G}) \text{ as } \nu \to \infty.$$

The expressions

$$\frac{1}{4\pi}\int_{\partial\hat{G}}\frac{1}{r}\gamma_\nu^{*\prime}\,d\Omega',\qquad\frac{1}{4\pi}\int_{\partial\hat{G}}\frac{1}{r}\gamma^{*\prime}\,d\Omega'$$

are in $W^{2-1/p,p}(\partial G)$ and (cf. [7] pp. 9–13)

$$\left\|\int_{\partial\hat{G}}\frac{1}{r}\gamma_\nu^{*\prime}\,d\Omega' - \int_{\partial\hat{G}}\frac{1}{r}\gamma^{*\prime}\,d\Omega'\right\|_{W^{2-1/p,p}(\partial G)} \leqslant c\,\|\gamma_\nu^* - \gamma^*\|_{W^{1-1/p,p}(\partial G)}.$$

Since the components of

$$\int_{\partial\hat{G}}\frac{1}{r}\gamma_\nu^{*\prime}\,d\Omega',\int_{\partial\hat{G}}\frac{1}{r}\gamma^{*\prime}\,d\Omega'$$

are harmonic functions we arrive at

$$\left\|\frac{\partial^2}{\partial x_i\,\partial x_j}\left(\int_{\partial\hat{G}}\frac{1}{r}\gamma_\nu^*\,d\Omega' - \int_{\partial\hat{G}}\frac{1}{r}\gamma^{*\prime}\,d\Omega'\right)\right\|_{L^p(\hat{G})}$$
$$\leqslant c\,\|\gamma_\nu^* - \gamma^*\|_{W^{1-1/p,p}(\partial G)},\quad 1\leqslant i,j\leqslant 3.$$

Thus

$$\nabla u_\nu \to \nabla u \text{ in } L^p(\hat{G}) \text{ as } \nu\to\infty.$$

Since for ∇u_ν the estimate in question holds, also the second case of the present theorem is settled. As for the third case, it is essentially a consequence of the second one, since (3.9), (3.10) are at our disposal. We can then proceed as in the proof of Theorem 2.2.   □

In connection with Theorem 3.2 it is natural to ask whether a vector field u with

$$u\in\bigcap_{R,R\geqslant R_0} W^{1,p}(\hat{G}\cap K_R(0)),$$

has its gradient in $L^p(\hat{G})$, provided that div $u\in L^p(\hat{G})$, curl $u\in L^p(\hat{G})$, $u+\tilde{c}\in L^q(\hat{G})$, $\tilde{c}=0$ on $\hat{G}_i$, $1\leqslant i\leqslant m$, $\tilde{c}$ constant on $\hat{G}_{m+1}$, $1<p<3$, $q^{-1}=p^{-1}-\frac{1}{3}$ (no restriction on the Betti numbers of $\hat{G}$). Setting $\varepsilon^* = -(\hat{n},u+\tilde{c})$, $\gamma^* = -[\hat{n},u+\tilde{c}]$ we have $\varepsilon^*\in W^{1-1/p,p}(\partial G)$, $\gamma^*\in W^{1-1/p,p}(\partial G)$. As in section 0 we derive the representation

$$u+\tilde{c} = -\left(\frac{1}{4\pi}\int_{\hat{G}}\text{grad}\,\frac{1}{r}\varepsilon'\,dx' + \frac{1}{4\pi}\text{grad}\int_{\partial\hat{G}}\frac{1}{r}\varepsilon^{*\prime}\,d\Omega'\right)$$
$$+\left(\frac{1}{4\pi}\int_{\hat{G}}\text{curl}\,\frac{1}{r}\gamma'\,dx' + \frac{1}{4\pi}\text{curl}\int_{\partial\hat{G}}\frac{1}{r}\gamma^{*\prime}\,d\Omega'\right).$$

Using the Calderón–Zygmund inequality we arrive at $\nabla u\in L^p(\hat{G})$.

References

1. Bègue, C., Conca, C., Murat, F. and Pironneau, O., 'Les équations de Stokes et de Navier–Stokes avec des conditions aux limites sur la pression', *Nonlinear Partial Differential Equations and their Applications. Collège de France Seminar IX*, Eds H. Brézis and J. L. Lions (*Pitman Research Notes in Mathematics* 181, pp. 179–264, Longman, Harlow, 1988).

2. Deuring, P., Wahl, W. von, Weidemaier, P., 'Das lineare Stokes-System in $\mathbb{R}^3$, I. Vorlesungen über das Innenraumproblem', *Bayr. Math. Schriften*, **27**, 1–250 (1988).

3. Duff, J. F. D. and Spencer, D. C., 'Harmonic tensors on Riemannian manifolds with boundary', *Ann. Math.*, **56**, 128–156 (1952).

4. Kress, R., 'Grundzüge einer Theorie der verallgemeinerten harmonischen Vektorfelder', *Math. Verf. Math. Phys.*, **2**, 49–83 (1969).

5. Ladyzhenskaja, O. A., *The Mathematical Theory of Viscous Incompressible Flow*, Gordon and Breach, New York, 1969.

6. Martensen, E., *Potentialtheorie*, Teubner, Stuttgart, 1968.

7. Wahl, W. von, 'Abschätzungen für das Neumann-Problem und die Helmholtz-Zerlegung von L^p', *Nachr. Ak. d. Wiss. Göttingen, II. Mathematisch-Physikalische Klasse*, **2**, 1–29, 1990.

8. Wahl, W. von, 'Vorlesung über das Außenraumproblem für die instationären Gleichungen von Navier–Stokes. Rudolph-Lipschitz-Vorlesung', *Sonderforschungsbereich 256 Nichtlineare Partielle Differentialgleichungen*, Bonn, 1989.