

A semigroup approach to the time dependent parabolic initial-boundary value problem

Alessandra Lunardi *

Dipartimento di Matematica
Università di Cagliari
Via Ospedale 72, 09124 Cagliari (Italy)

Eugenio Sinestrari*

Dipartimento di Matematica
Università di Roma "La Sapienza"
P. Aldo Moro 2, 00185 Roma (Italy)

Wolf von Wahl

Lehrstuhl für Angewandte Mathematik
Universität Bayreuth, P.O.Box 101251
8580 Bayreuth (Federal Republic of Germany)

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Abstract. In this paper a general initial-boundary value problem for a higher order linear parabolic equation with time dependent coefficients and non homogeneous data is studied by the abstract results obtained from a generalization of the classical theory of analytic semigroups of linear operators.

1 Introduction and notations

This paper gives a proof of existence, uniqueness, and optimal Hölder regularity of the solutions of the linear parabolic initial-boundary value problem of higher order:

$$(1.1) \quad \begin{cases} D_t u(t, x) = \sum_{|\gamma| \leq 2m} a_\gamma(t, x) D_x^\gamma u(t, x) + f(t, x), \\ (t, x) \in Q = [0, T] \times \bar{\Omega} \\ u(0, x) = u_0(x), \quad x \in \bar{\Omega} \\ \sum_{|\beta| \leq m_j} b_{j\beta}(t, x) D_x^\beta u(t, x) = g_j(t, x); \\ (t, x) \in S = [0, T] \times \Gamma, \quad j = 1, \dots, m. \end{cases}$$

Here Ω is a bounded open set in \mathbb{R}^n with smooth boundary Γ . For every multi-index $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}^n$ we set $|\beta| = \beta_1 + \dots + \beta_n$ and $D_x^\beta u = D_{x_1}^{\beta_1} \dots D_{x_n}^{\beta_n} u$. We assume that, for every $t \in [0, T]$, the operator

$$(1.2) \quad A(t) = \sum_{|\gamma| \leq 2m} a_\gamma(t, \cdot) D_x^\gamma$$

is elliptic and the boundary operators

$$(1.3) \quad B_j(t) = \sum_{|\beta| \leq m_j} b_{j\beta}(t, \cdot) D_x^\beta, \quad j = 1, \dots, m$$

of order $m_j < 2m$ satisfy the usual requirements which guarantee the existence of the resolvent in a sufficiently large set of the complex plane (see section 2 for precise conditions). In [S] V. A. Solonnikov considered a larger

class of parabolic problems using the methods of potential theory. He showed in particular, the unique solvability of (1.1) in the Hölder classes $C^{\frac{\alpha}{2m},\alpha}(Q)$ for any nonintegral $\alpha > 0$ if the $A(t)$ are uniformly strongly elliptic and if the B_j fulfill conditions being very similar to ours.

If the leading coefficients of $A(t)$ are real valued the conditions in [S] coincide with ours in the case of a single equation, which is considered here. The space $C^{\frac{\alpha}{2m},\alpha}(Q)$ is the set of the α -Hölder continuous functions with respect to the parabolic distance $d((t,x),(s,y)) = |t-s|^{\frac{1}{2m}} + |x-y|$, $t,s \in [0,T]; x,y \in \bar{\Omega}$. In section 1 we will show that a function $u : [0,T] \times \bar{\Omega} \rightarrow \mathbb{C}$ belongs to $C^{\frac{\alpha}{2m},\alpha}(Q)$ if and only if $t \rightarrow u(t,\cdot)$ belongs to $C^{\frac{\alpha}{2m}}([0,T]; C^0(\bar{\Omega}))$ and $t \rightarrow D_t^k u(t,\cdot)$ is bounded with values in $C^{\alpha-2mk}(\bar{\Omega})$ for $k = 0, \dots, [\frac{\alpha}{2m}]$ ($[r]$ denotes the integral part of the real number r). This result is important for the abstract approach to problem (1.1) that we will develop. In fact we reduce (1.1) to a problem for an ordinary differential equation in the Banach space X of continuous functions on $\bar{\Omega}$, namely

$$(1.4) \quad \begin{cases} u'(t) = A(t)u(t) + f(t), & t \in [0, T] \\ u(0) = u_0 \\ B_j(t)u(t) = g_j(t), & t \in [0, T], \quad j = 1, \dots, m \end{cases}$$

in the unknown $u : [0, T] \rightarrow X$, where we have set $u(t) = u(t, \cdot)$, $f(t) = f(t, \cdot)$ and $g_j(t) = g_j(t, \cdot)$. For each $t_0 \in [0, T]$ the linear operator

$$(1.5) \quad \begin{cases} \Lambda : D(\Lambda) \subseteq X \rightarrow X \\ D(\Lambda) = \{\varphi \in \bigcap_{p>1} W^{2m,p}(\Omega); A(t_0)\varphi \in X, B_j(t_0)\varphi = 0, \\ \hspace{15em} j = 1, \dots, m\} \\ \Lambda\varphi = A(t_0)\varphi, \varphi \in D(\Lambda) \end{cases}$$

generates an analytic semigroup in X (not necessary strongly continuous at $t = 0$) but the classical theory of semigroups cannot be directly used to solve (1.4) when $g_j \not\equiv 0$ for some $j = 1, \dots, m$ or when B_j has order $m_j = 0$ for some $j = 1, \dots, m$: in this latter case $\overline{D(\Lambda)} \neq X$, so that we must employ

the results of the theory of analytic semigroups with non densely defined generators (see [Si]). We study problem (1.4) in two steps: first we consider the autonomous case in an arbitrary interval $[t_0, t_1]$ contained in $[0, T]$:

$$(1.6) \quad \begin{cases} v'(t) = A(t_0)v(t) + f(t), & t \in [t_0, t_1] \\ v(t_0) = v_0 \\ B_j(t_0)v(t) = g_j(t), & t \in [t_0, t_1], \quad j = 1, \dots, m. \end{cases}$$

Then we use a perturbation method to solve (1.4) in a suitable interval $[0, \tau]$ and by means of a sharp estimate of the solution we repeat the same procedure in $[\tau, 2\tau]$ so that we can extend our solution on the whole of $[0, T]$. To overcome the above mentioned problem caused by the non homogeneous boundary data g_j we adapt to our situation a device introduced by A. Balakrishnan ([B]): we construct a function $n : [0, T] \rightarrow X$ such that

$$(1.7) \quad B_j(t_0)(n(t)) = g_j(t), \quad t \in [t_0, t_1], \quad j = 1, \dots, m.$$

If v is a solution of 1.6 and n is sufficiently regular then, $w(t) = v(t) - n(t)$ satisfies

$$(1.8) \quad \begin{cases} w'(t) = A(t_0)w(t) + A(t_0)n(t) - n'(t) + f(t), & t \in [t_0, t_1] \\ w(t_0) = v_0 - n(t_0) \\ B_j(t_0)w(t) = 0, & t \in [t_0, t_1] \end{cases}$$

so that w is given by the variation of constants formula

$$(1.9) \quad w(t) = e^{(t-t_0)\Lambda} [v_0 - n(t_0)] + \int_{t_0}^t e^{(t-s)\Lambda} [A(t_0)n(s) - n'(s) + f(s)] ds, \\ t \in [t_0, t_1]$$

where $e^{t\Lambda}$ is the semigroup generated by Λ in X . Integrating by parts we get a representation formula for the solution of (1.6):

$$(1.10) \quad v(t) = e^{(t-t_0)\Lambda} v_0 + \int_{t_0}^t e^{(t-s)\Lambda} [A(t_0)n(s) + f(s)] ds -$$

$$-\Lambda \int_{t_0}^t e^{(t-s)\Lambda} n(s) ds, \quad t \in [t_0, t_1]$$

which makes sense even if n is merely Hölder continuous. In fact we will choose n in such a way that v defined by (1.10) is the unique solution of (1.6) in the mentioned Hölder classes. In [L] for each $t \in [t_0, t_1]$, $n(t) = n(t, \cdot)$ is defined as the solution of the elliptic problem

$$(1.11) \quad \begin{cases} A(t_0)n(t, \cdot) = \omega n(t, \cdot) \\ B_j(t_0)n(t, \cdot) = g_j(t), \quad j = 1, \dots, m \end{cases}$$

where $\omega \in \rho(\Lambda)$, the resolvent set of Λ . But this choice is not useful in our case and thus we are led to construct explicitly a linear operator N such that

$$(1.12) \quad \begin{cases} \varphi_j \in C^{2m-m_j}(\Gamma), \quad j = 1, \dots, m \Rightarrow N(\varphi_1, \dots, \varphi_m) \in C^{2m}(\bar{\Omega}) \\ \varphi_j \in C^{2m+\alpha-m_j}(\Gamma), \quad j = 1, \dots, m \Rightarrow N(\varphi_1, \dots, \varphi_m) \in C^{2m+\alpha}(\bar{\Omega}) \\ B_j(t_0)N(\varphi_1, \dots, \varphi_m) = \varphi_j, \quad j = 1, \dots, m \end{cases}$$

and then set $n(t) = N(\varphi_1(t), \dots, \varphi_m(t))$, $t \in [t_0, t_1]$.

In this paper we use the following notations: If X is a Banach space, $k \in \mathbb{N}$, $\sigma \in]0, 1[$ then $C^k([t_0, t_1]; X)$ is the space of the k times continuously differentiable functions from $[t_0, t_1]$ to X and $C^{k+\sigma}([t_0, t_1]; X)$ is the space of functions whose k -th derivative is σ -Hölder continuous. These spaces are endowed with their usual norms. $B([t_0, t_1]; X)$ denotes the space of all bounded functions from $[t_0, t_1]$ to X with the sup-norm. If Ω is a bounded open set of \mathbb{R}^n and $\alpha > 0$, $\alpha \notin \mathbb{N}$ then $C^\alpha(\bar{\Omega})$ is the space of the α -Hölder continuous functions from $\bar{\Omega}$ to \mathbb{C} with the usual norm. If Γ is the boundary of Ω we say that Γ is of class α if there exist a finite number of closed balls B_1, \dots, B_N such that $\Gamma \subseteq \bigcup_{i=1}^N \overset{\circ}{B}_i$ and a C^α -diffeomorphism φ_i from B_i into the closed unit ball B of \mathbb{R}^n such that $\varphi_i(B_i \cap \Omega) = B_+ = \{(y_1, \dots, y_n) \in B; y_n > 0\}$ and $\varphi_i(B_i \cap \Gamma) = \Sigma = \{(y_1, \dots, y_n) \in B; y_n = 0\}$. The space $C^\beta(\Gamma)$ with $\beta \in [0, \alpha]$ is the set of all $g : \Gamma \rightarrow \mathbb{C}$ such that

$g \circ \varphi_i^{-1} \in C^\beta(\Sigma)$ for $i = 1, \dots, N$. If $\alpha \in]0, 1[$ and $u : \bar{\Omega} \rightarrow \mathbb{C}$ we set $[u]_{C^\alpha(\bar{\Omega})} = \sup\{|u(x) - u(y)|/|x - y|^\alpha; x, y \in \bar{\Omega}, x \neq y\}$; an analogous definition is given for $[u]_{C^\alpha([t_0, t_1])}$.

The structure of the paper is the following: in section 2 we study the spaces $C^{\frac{\alpha}{2m}, \alpha}([0, T] \times \bar{\Omega})$ and in section 3 we prove some new maximal regularity results for abstract parabolic equations: they are the basic tools of our proofs. Sections 4, 5, 7, 8, concern the autonomous problem (1.8): they can be considered as the Hölder counterpart of [L] where several L^2 and H^α regularity results are given for the function v defined by (1.10). In section 6 we construct the extension operator N from Γ to $\bar{\Omega}$ satisfying (1.12). Finally in section 9 we treat the non autonomous problem: in this case the regularity theorem is obtained from the autonomous case by perturbation techniques and by sharp estimates. The homogeneous Dirichlet problem was treated by Sinestrari and von Wahl (see [W1]).

2 Hölder spaces in cylindrical domains

Let Ω be a bounded set of \mathbb{R}^n with boundary Γ of class $2m + \alpha$, where $m \in \mathbb{N}$ and $\alpha > 0$ is not integer. Setting $Q = [t_0, t_1] \times \bar{\Omega}$, we recall the following (see [S]).

Definition 2.1 $C^{\frac{\alpha}{2m}, \alpha}(Q)$ is the Banach space of the functions $u : Q \rightarrow \mathbb{C}$ such that:

$$(2.1) \quad \text{There exist continuous } D_t^k D_x^\beta u \text{ in } Q \text{ if } 2mk + |\beta| \leq [\alpha].$$

$$(2.2) \quad \begin{aligned} I_1 &= \sum_{2mk + |\beta| = [\alpha]} \sup_{t \in [t_0, t_1]} [D_t^k D_x^\beta u(t, \cdot)]_{C^{\alpha - [\alpha]}(\bar{\Omega})} \\ &= \sum_{k=0}^{[\frac{\alpha}{2m}]} \sum_{|\beta| = [\alpha] - 2mk} \sup_{t \in [t_0, t_1]} [D_t^k D_x^\beta u(t, \cdot)]_{C^{\alpha - [\alpha]}(\bar{\Omega})} < \infty \end{aligned}$$

$$(2.3) \quad \begin{aligned} I_2 &= \sum_{\alpha - 2m < 2mk + |\beta| < \alpha} \sup_{x \in \bar{\Omega}} [D_t^k D_x^\beta u(\cdot, x)]_{C^{\frac{\alpha - |\beta| - 2mk}{2m}}([t_0, t_1])} \\ &= \sum_{0 \leq |\beta| \leq [\alpha]} \sup_{x \in \bar{\Omega}} [D_t^{[\frac{\alpha - |\beta|}{2m}]} D_x^\beta u(\cdot, x)]_{C^{\frac{\alpha - |\beta|}{2m} - [\frac{\alpha - |\beta|}{2m}]([t_0, t_1])} < \infty \end{aligned}$$

with the norm

$$(2.4) \quad \|u\|_{C^{\frac{\alpha}{2m}, \alpha}(Q)} = I_0 + I_1 + I_2$$

where

$$(2.5) \quad \begin{aligned} I_0 &= \sum_{2mk + |\beta| \leq [\alpha]} \sup_{(t,x) \in Q} |D_t^k D_x^\beta u(t,x)| \\ &= \sum_{k=0}^{[\frac{\alpha}{2m}]} \sum_{|\beta| \leq [\alpha] - 2mk} \sup_{(t,x) \in Q} |D_t^k D_x^\beta u(t,x)| \end{aligned}$$

We prove now a result which gives a characterization of the space $C^{\frac{\alpha}{2m}, \alpha}(Q)$.

Theorem 2.2 $u \in C^{\frac{\alpha}{2m}, \alpha}(Q)$ if and only if, setting $u(t, \cdot) = u(t)$ for $t \in [t_0, t_1]$, we have

$$(2.6) \quad u \in C^{\frac{\alpha}{2m}}([t_0, t_1]; C^0(\bar{\Omega}))$$

$$(2.7) \quad u^{(k)} \in B([t_0, t_1]; C^{\alpha-2mk}(\bar{\Omega})), \quad k = 0, \dots, [\frac{\alpha}{2m}]$$

Moreover defining

$$(2.8) \quad |||u|||_{C^{\frac{\alpha}{2m}, \alpha}(Q)} = J_1 + J_2$$

with

$$(2.9) \quad J_1 = \|u\|_{C^{\frac{\alpha}{2m}}([t_0, t_1]; C(\bar{\Omega}))}$$

$$(2.10) \quad J_2 = \sum_{k=0}^{[\frac{\alpha}{2m}]} \|u^{(k)}\|_{B([t_0, t_1]; C^{\alpha-2mk}(\bar{\Omega}))}$$

there exists $c_1 = c_1(\alpha, m, \Omega)$ such that

$$(2.11) \quad c_1 \|u\|_{C^{\frac{\alpha}{2m}, \alpha}(Q)} \leq |||u|||_{C^{\frac{\alpha}{2m}, \alpha}(Q)} \leq 2 \|u\|_{C^{\frac{\alpha}{2m}, \alpha}(Q)}.$$

Finally if $u \in C^{\frac{\alpha}{2m}, \alpha}(Q)$ then

$$(2.12) \quad u \in C^{\frac{\alpha-h}{2m}}([t_0, t_1]; C^h(\bar{\Omega})), \quad h = 0, \dots, [\alpha]$$

and there exists $c_2 = c_2(\alpha, m, \Omega)$ such that

$$(2.13) \quad \sum_{h=0}^{[\alpha]} \|u\|_{C^{\frac{\alpha-h}{2m}}([t_0, t_1]; C^h(\bar{\Omega}))} \leq c_2 \|u\|_{C^{\frac{\alpha}{2m}, \alpha}(Q)}.$$

Proof. Conditions (2.1) and (2.2) are equivalent to

$$(2.14) \quad u \in C^k([t_0, t_1]; C^{[\alpha]-2mk}(\bar{\Omega})), \quad k = 0, \dots, [\frac{\alpha}{2m}]$$

$$(2.15) \quad u^{(k)} \in B([t_0, t_1]; C^{\alpha-2mk}(\bar{\Omega})), \quad k = 0, \dots, [\frac{\alpha}{2m}].$$

Moreover there exists $c_{11} = c_{11}(\alpha, m)$ such that

$$(2.16) \quad J_2 \leq I_0 + I_1 \leq c_{11} J_2.$$

In addition conditions (2.1) and (2.3) imply that

$$(2.17) \quad u \in C^{\frac{\alpha}{2m}}([t_0, t_1]; C^0(\bar{\Omega}))$$

and we have

$$(2.18) \quad J_1 \leq I_0 + I_2.$$

In conclusion, if $u \in C^{\frac{\alpha}{2m}, \alpha}(Q)$ then (2.6) and (2.7) are true and

$$(2.19) \quad \|u\|_{C^{\frac{\alpha}{2m}, \alpha}(Q)} \leq 2 \|u\|_{C^{\frac{\alpha}{2m}, \alpha}(Q)}.$$

Before proving the converse let us show that (2.6) and (2.7) imply (2.12): we can suppose $[\alpha] > 0$. From (2.7) we first deduce for $h = 1, \dots, [\alpha]$

$$(2.20) \quad u^{(r)} \in B([t_0, t_1]; C^h(\bar{\Omega})), \quad r = 0, \dots, \left[\frac{\alpha - h}{2m} \right]$$

then setting

$$(2.21) \quad k = \left[\frac{\alpha - h}{2m} \right], \quad \theta = \frac{\alpha - h}{2m} - \left[\frac{\alpha - h}{2m} \right]$$

we have

$$(2.22) \quad u^{(k)} \in C^\theta([t_0, t_1]; C^h(\bar{\Omega})).$$

Actually, by using interpolatory estimates (see e.g. [ADN], p. 657, [W2], p. 255) we obtain when $k = \left[\frac{\alpha}{2m} \right]$

$$(2.23) \quad \begin{cases} \|u^{(k)}(t'') - u^{(k)}(t')\|_{C^h(\bar{\Omega})} \\ \leq c_{12} \|u^{(k)}(t'') - u^{(k)}(t')\|_{C^{\alpha-2mk}(\bar{\Omega})}^{\frac{h}{\alpha-2mk}} \|u^{(k)}(t'') - u^{(k)}(t')\|_{C^0(\bar{\Omega})}^{1-\frac{h}{\alpha-2mk}} \\ \leq 2c_{12}(J_1 + J_2) |t'' - t'|^{\frac{\alpha-h}{2m}-k}; \quad t', t'' \in [t_0, t_1] \end{cases}$$

and when $k < \left[\frac{\alpha}{2m} \right]$

$$(2.24) \quad \begin{cases} \|u^{(k)}(t'') - u^{(k)}(t')\|_{C^h(\bar{\Omega})} \\ \leq c_{13} \|u^{(k)}(t'') - u^{(k)}(t')\|_{C^{\frac{h-\alpha}{2m}+k+1}(\bar{\Omega})} \\ \quad \cdot \|u^{(k)}(t'') - u^{(k)}(t')\|_{C^{\frac{\alpha-h}{2m}-k}(\bar{\Omega})} \\ \leq 2c_{13} J_1 |t'' - t'|^{\frac{\alpha-h}{2m}-k}; \quad t', t'' \in [t_0, t_1] \end{cases}$$

where c_{12} and c_{13} depend on α, m and Ω . Now (2.20) -(2.24) prove (2.12) and (2.13). But (2.12) implies (2.14) and (2.7) is (2.15): hence from what we have seen at the beginning of the proof we can say that (2.6) and (2.7) imply (2.1) and (2.2); but also (2.3) is verified because from (2.23) and (2.24) we deduce (setting $\gamma = \frac{\alpha-h}{2m}$)

$$(2.25) \quad \begin{cases} I_2 \leq \sum_{h=0}^{[\alpha]} \sup_{t', t'' \in [t_0, t_1]} \frac{\|u^{[\gamma]}(t'') - u^{[\gamma]}(t')\|_{C^h(\bar{\Omega})}}{|t'' - t'|^{\gamma-[\gamma]}} \\ \leq 2(c_{12} + c_{13})(J_1 + J_2) + J_2 \end{cases}$$

which (together with (2.16)) gives also the first of (2.11). \square

Remark: From definition 1.1 it follows that if $u \in C^{\frac{\alpha}{2m}, \alpha}(Q)$ and $\gamma \in \mathbb{N}^n$ with $1 \leq |\gamma| \leq [\alpha]$ then $D_x^\gamma u \in C^{\frac{\alpha'}{2m}, \alpha'}(Q)$ with $\alpha' = \alpha - |\gamma|$ and if $h \in \mathbb{N}$ with $1 \leq h \leq [\frac{\alpha}{2m}]$ then $D_t^h u \in C^{\frac{\alpha''}{2m}, \alpha''}(Q)$ with $\alpha'' = \alpha - 2mh$.

In addition, from theorem 1.2 it follows that $C^{\frac{\alpha''}{2m}, \alpha''}(Q) \hookrightarrow C^{\frac{\alpha'}{2m}, \alpha'}(Q)$ for $\alpha' < \alpha''$.

3 A maximal regularity result for abstract parabolic equations

Let X be a Banach space with norm $\|\cdot\|$ and let $\Lambda : D(\Lambda) \subseteq X \rightarrow X$ be a linear operator satisfying the following condition:

$$(3.1) \quad \left\{ \begin{array}{l} \text{There exist } \omega \in \mathbb{R}, \eta \in]\frac{\pi}{2}, \pi[\text{ and } M > 0 \text{ such that if } \lambda \in \mathbb{C}, \\ \lambda \neq \omega \text{ and } |\arg(\lambda - \omega)| < \eta \text{ then } \lambda \text{ is in the resolvent} \\ \text{set of } \Lambda \text{ and } \|(\lambda - \omega)(\lambda - \Lambda)^{-1}\|_{L(X)} \leq M. \end{array} \right.$$

In this case Λ generates an analytic semigroup $e^{t\Lambda}$ which is not strongly continuous at $t = 0$ if $\overline{D(\Lambda)} \neq X$ (for analytic semigroups with non dense domain we refer to ([Si]).

A family of intermediate spaces between $D(\Lambda)$ and X can be defined by

$$(3.2) \quad D_\Lambda(\beta, \infty) = \left\{ x \in X; [x]_\beta = \sup_{0 < t \leq 1} \|t^{1-\beta} \Lambda e^{t\Lambda} x\| < \infty \right\}, \quad 0 < \beta < 1.$$

They are Banach spaces under the norm

$$(3.3) \quad \|x\|_{D_\Lambda(\beta, \infty)} = \|x\| + [x]_\beta$$

and given $k \in \mathbb{N}$ and $\beta \in]0, 1[$ there exists a continuous and increasing function $M_{k, \beta} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that

$$(3.4) \quad \|t^{k-\beta}\Lambda^k e^{t\Lambda}\|_{L(D_\Lambda(\beta,\infty),X)} \leq M_{k,\beta}(T), \quad 0 < t \leq T.$$

The spaces $D_\Lambda(\beta, \infty)$ are necessary to study the maximal Hölder regularity in time up to $t = t_0$ of the initial value problem

$$(3.5) \quad \begin{cases} v'(t) = \Lambda v(t) + \varphi(t), & t \in [t_0, t_1] \\ v(t_0) = v_0. \end{cases}$$

Here we recall first a sharp regularity result proved in [Si], Theorem 4.5 in the case $\omega = 0$: the case $\omega \neq 0$ is obtained by changing the unknown $v(t)$ with $e^{\omega t}v(t)$.

Theorem 3.1 *Let (3.1) hold and let $\varphi \in C^\theta([t_0, t_1]; X)$, $v_0 \in D(\Lambda)$, $\Lambda v_0 + \varphi(t_0) \in D_\Lambda(\theta, \infty)$, $0 < \theta < 1$. Then the unique solution of problem (3.5) is given by*

$$(3.6) \quad v(t) = e^{(t-t_0)\Lambda}v_0 + \int_{t_0}^t e^{(t-s)\Lambda}\varphi(s)ds, \quad t \in [t_0, t_1].$$

Moreover $v', \Lambda v \in C^\theta([t_0, t_1]; X)$, $v' \in B([t_0, t_1]; D_\Lambda(\theta, \infty))$ and there exists a continuous and increasing function $c_3 : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that

$$(3.7) \quad \begin{cases} \|v\|_{C^{1+\theta}([t_0, t_1]; X)} + \|v'\|_{B([t_0, t_1]; D_\Lambda(\theta, \infty))} \\ \leq c_3(t_1 - t_0) \{[\varphi]_{C^\theta([t_0, t_1]; X)} + \|v_0\| + \|\Lambda v_0 + \varphi(t_0)\|_{D_\Lambda(\theta, \infty)}\}. \end{cases}$$

The function c_3 depends also on M, ω, η and θ (see (3.1)).

By using this theorem we can prove a further result of maximal regularity under the assumption that φ has values in some intermediate space: this will be used in the application to the parabolic problem (1.1).

Theorem 3.2 *Let (3.1) hold and let $\varphi \in C^\theta([t_0, t_1], D_\Lambda(\beta, \infty))$ with $\theta, \beta \in]0, 1[$. If $\varphi(t_0) = v_0 = 0$ then the solution v of (3.5) is such that*

$$(3.8) \quad v', \Lambda v \in C^\theta([t_0, t_1]; D_\Lambda(\beta, \infty)).$$

$$(3.14) \quad \|v\|_{C^{1+\theta}([t_0, t_1]; D_\Lambda(\beta, \infty))} \leq c[\varphi]_{C^\theta}.$$

Suppose now $0 < \theta + \beta < 1$. From (4.1) of [Si] we get

$$(3.15) \quad \Lambda v(t) = \Lambda \int_{t_0}^t e^{(t-s)\Lambda} [\varphi(s) - \varphi(t)] ds + [e^{(t-t_0)\Lambda} - I]\varphi(t), \quad t \in [t_0, t_1]$$

and so when $t_0 \leq r \leq t \leq t_1$

$$(3.16) \quad \left\{ \begin{array}{l} \Lambda v(t) - \Lambda v(r) \\ = \Lambda \int_{t_0}^r [e^{(t-s)\Lambda} - e^{(r-s)\Lambda}] [\varphi(s) - \varphi(r)] ds + \\ \Lambda \int_r^t e^{(t-s)\Lambda} [\varphi(s) - \varphi(t)] ds + [e^{(t-t_0)\Lambda} - e^{(r-t_0)\Lambda}] \varphi(r) + \\ [I - e^{(t-r)\Lambda}] [\varphi(r) - \varphi(t)]. \end{array} \right.$$

By using (3.4) we have for $t_0 \leq s \leq r \leq t \leq t_1$:

$$(3.17) \quad \left\{ \begin{array}{l} \|\Lambda [e^{(t-s)\Lambda} - e^{(r-s)\Lambda}] [\varphi(s) - \varphi(r)]\| \\ = \|\Lambda \int_{r-s}^{t-s} \Lambda e^{\xi\Lambda} [\varphi(s) - \varphi(r)] d\xi\| \\ \leq c[\varphi]_{C^\theta} \int_{r-s}^{t-s} (r-s)^\theta \xi^{\beta-2} d\xi. \end{array} \right.$$

Moreover, as $\varphi(t_0) = 0$:

$$(3.18) \quad \left\{ \begin{array}{l} \|[e^{(t-t_0)\Lambda} - e^{(r-t_0)\Lambda}] \varphi(r)\| = \|\int_{r-t_0}^{t-t_0} \Lambda e^{\xi\Lambda} [\varphi(r) - \varphi(t_0)] d\xi\| \\ \leq c[\varphi]_{C^\theta} \int_{r-t_0}^{t-t_0} (r-t_0)^\theta \xi^{\beta-1} d\xi \end{array} \right.$$

and

$$(3.19) \quad \left\{ \begin{array}{l} \|[I - e^{(t-r)\Lambda}] [\varphi(r) - \varphi(t)]\| = \|\int_0^{t-r} \Lambda e^{\xi\Lambda} [\varphi(r) - \varphi(t)] d\xi\| \\ \leq c[\varphi]_{C^\theta} \int_0^{t-r} (t-r)^\theta \xi^{\beta-1} d\xi. \end{array} \right.$$

Now we can deduce from (3.16) for $t_0 \leq r \leq t \leq t_1$:

$$(3.20) \quad \left\{ \begin{array}{l} \|\Lambda v(t) - \Lambda v(r)\| \\ \leq c[\varphi]_{C^\theta} \left\{ \int_{t_0}^r (r-s)^\theta ds \int_{r-s}^{t-s} \xi^{\beta-2} d\xi + \int_r^t (t-s)^{\theta+\beta-1} ds + \right. \\ \left. (r-t_0)^\theta \int_{r-t_0}^{t-t_0} \xi^{\beta-1} d\xi + (t-r)^\theta \int_0^{t-r} \xi^{\beta-1} d\xi \right\} \end{array} \right.$$

As $\beta + \theta < 1$ we get

$$(3.21) \quad \left\{ \begin{array}{l} \int_{t_0}^r (r-s)^\theta ds \int_{r-s}^{t-s} \xi^{\beta-2} d\xi \\ \leq \int_{t_0}^r ds \int_{r-s}^{t-s} \xi^{\beta+\theta-2} d\xi \\ = (1-\beta-\theta)^{-1}(\beta+\theta)^{-1}[(t-r)^{\beta+\theta} - (t-t_0)^{\beta+\theta} \\ + (r-t_0)^{\beta+\theta}] \\ \leq (1-\beta-\theta)^{-1}(\beta+\theta)^{-1}(t-r)^{\beta+\theta} \end{array} \right.$$

and also

$$(3.22) \quad \left\{ \begin{array}{l} (r-t_0)^\theta \int_{r-t_0}^{t-t_0} \xi^{\beta-1} d\xi \\ \leq \int_{r-t_0}^{t-t_0} \xi^{\theta+\beta-1} d\xi = (\theta+\beta)^{-1}[(t-t_0)^{\theta+\beta} - (r-t_0)^{\theta+\beta}] \\ \leq (\theta+\beta)^{-1}(t-r)^{\theta+\beta} \end{array} \right.$$

and so from (3.20) we get

$$\|\Lambda v(t) - \Lambda v(r)\| \leq c[\varphi]_{C^\theta} (t-r)^{\theta+\beta}, \quad t_0 \leq r \leq t \leq t_1$$

hence $\Lambda v \in C^{\theta+\beta}([t_0, t_1]; X)$, i.e. (3.9) holds. Moreover

$$(3.23) \quad [\Lambda v]_{C^{\theta+\beta}([t_0, t_1]; X)} \leq c[\varphi]_{C^\theta}.$$

From the previous theorem we deduce

$$(3.24) \quad \|v'\|_{B([t_0, t_1]; D_A(\theta, \infty))} \leq c[\varphi]_{C^\theta}$$

and by using (3.2) and (3.4) we get for $t \in [t_0, t_1]$

$$(3.25) \quad \left\{ \begin{array}{l} \|v'(t)\|_{D_\Lambda(\theta+\beta, \infty)} = \|v'(t)\| + \sup_{0 < s \leq 1} \|s^{1-\theta-\beta} \Lambda e^{s\Lambda} v'(t)\| \\ \leq \|v'(t)\| + c \sup_{0 < s \leq 1} \|s^{1-\theta} \Lambda e^{s\Lambda} v'(t)\|_{D_\Lambda(\beta, \infty)} \\ \leq c \|v'(t)\|_{D_A(\theta, \infty)}. \end{array} \right.$$

Hence from (3.23) we obtain (3.10) and

$$(3.26) \quad \|v'\|_{B([t_0, t_1]; D_\Lambda(\theta+\beta, \infty))} \leq c[\varphi]_{C^\theta}.$$

To prove (3.11) we will use (3.15) and (3.4) to get a first estimate which holds also when $\theta + \beta > 1$:

$$(3.27) \quad \left\{ \begin{array}{l} \|\Lambda v(t)\| \leq \int_{t_0}^t \|\Lambda e^{(t-s)\Lambda}\|_{L(D_\Lambda(\beta, \infty), X)} \|\varphi(s) - \varphi(t)\|_{D_\Lambda(\beta, \infty)} ds + \\ \quad \left\| \int_0^{t-t_0} \Lambda e^{\xi\Lambda} [\varphi(t) - \varphi(t_0)] d\xi \right\| \\ \leq c \left[\int_{t_0}^t (t-s)^{\theta+\beta-1} ds + \int_0^{t-t_0} (t-t_0)^\theta \xi^{\beta-1} d\xi \right] [\varphi]_{C^\theta} \\ \leq c[(\theta + \beta)^{-1} + \beta^{-1}](t_1 - t_0)^{\theta+\beta} [\varphi]_{C^\theta}. \end{array} \right.$$

Hence by using also (3.23) we obtain

$$(3.28) \quad \|\Lambda v\|_{C^{\theta+\beta}([t_0, t_1]; X)} \leq c[\varphi]_{C^\theta}.$$

Now (3.11) is a consequence of (3.14), (3.26), and (3.28).

Let us examine the case $1 < \theta + \beta < 2$, setting $\gamma = \theta + \beta$. From (4.2) of [Si] we have

$$(3.29) \quad v'(t) = \int_{t_0}^t \Lambda e^{(t-s)\Lambda} [\varphi(s) - \varphi(t)] ds + e^{(t-t_0)\Lambda} \varphi(t), \quad t \in [t_0, t_1]$$

hence by using (3.4) we obtain for $t \in [t_0, t_1]$:

$$(3.30) \quad \left\{ \begin{aligned} \|\Lambda v'(t)\| &= \left\| \int_{t_0}^t \Lambda^2 e^{(t-s)\Lambda} [\varphi(s) - \varphi(t)] ds \right. \\ &\quad \left. + \Lambda e^{(t-t_0)\Lambda} [\varphi(t) - \varphi(t_0)] \right\| \\ &\leq \int_{t_0}^t \|\Lambda^2 e^{(t-s)\Lambda}\|_{L(D_\Lambda(\beta, \infty), X)} \|\varphi(s) - \varphi(t)\|_{D_\Lambda(\beta, \infty)} ds \\ &\quad + \|\Lambda e^{(t-t_0)\Lambda}\|_{L(D_\Lambda(\beta, \infty), X)} \|\varphi(t) - \varphi(t_0)\|_{D_\Lambda(\beta, \infty)} \\ &\leq c[\varphi]_{C^\theta} \left[\int_{t_0}^t (t-s)^{\gamma-2} ds + (t-t_0)^{\gamma-1} \right] \leq c[\varphi]_{C^\theta} \end{aligned} \right.$$

whereas for $t_0 \leq s \leq t \leq t_1$ we have

$$(3.31) \quad \left\{ \begin{aligned} \Lambda v'(t) - \Lambda v'(s) &= \int_{t_0}^s \Lambda^2 [e^{(t-\sigma)\Lambda} - e^{(s-\sigma)\Lambda}] [\varphi(\sigma) - \varphi(s)] d\sigma \\ &\quad + \int_s^t \Lambda^2 e^{(t-\sigma)\Lambda} [\varphi(\sigma) - \varphi(t)] d\sigma \\ &\quad + \Lambda [e^{(t-t_0)\Lambda} - e^{(s-t_0)\Lambda}] \varphi(s) \\ &\quad + \Lambda e^{(t-s)\Lambda} [\varphi(t) - \varphi(s)]. \end{aligned} \right.$$

Now from (3.4) we get

$$(3.32) \quad \left\{ \begin{aligned} &\left\| \int_{t_0}^s \Lambda^2 [e^{(t-\sigma)\Lambda} - e^{(s-\sigma)\Lambda}] [\varphi(\sigma) - \varphi(s)] d\sigma \right\| \\ &= \left\| \int_{t_0}^s \Lambda^2 \left[\Lambda \int_{s-\sigma}^{t-\sigma} e^{\tau\Lambda} [\varphi(\sigma) - \varphi(s)] d\tau \right] d\sigma \right\| \\ &\leq \int_{t_0}^s d\sigma \int_{s-\sigma}^{t-\sigma} \|\Lambda^3 e^{\tau\Lambda}\|_{L(D_\Lambda(\beta, \infty), X)} \|\varphi(\sigma) - \varphi(s)\|_{D_\Lambda(\beta, \infty)} d\tau \\ &\leq c[\varphi]_{C^\theta} \int_{t_0}^s (s-\sigma)^\theta d\sigma \int_{s-\sigma}^{t-\sigma} \tau^{\beta-3} d\tau \\ &\leq c[\varphi]_{C^\theta} \int_{t_0}^s d\sigma \int_{s-\sigma}^{t-\sigma} \tau^{\gamma-3} d\tau \\ &= c[\varphi]_{C^\theta} (2-\gamma)^{-1} (\gamma-1)^{-1} \\ &\quad [(t-s)^{\gamma-1} - (t-t_0)^{\gamma-1} + (s-t_0)^{\gamma-1}] \\ &\leq c[\varphi]_{C^\theta} (2-\gamma)^{-1} (\gamma-1)^{-1} (t-s)^{\gamma-1} \end{aligned} \right.$$

and also

$$(3.33) \quad \left\{ \begin{aligned} \left\| \int_s^t \Lambda^2 e^{(t-\sigma)\Lambda} [\varphi(\sigma) - \varphi(t)] d\sigma \right\| &\leq c[\varphi]_{C^\theta} \int_s^t (t-\sigma)^{\gamma-2} d\sigma \\ &= c(2-\gamma)^{-1} [\varphi]_{C^\theta} (t-s)^{\gamma-1} \end{aligned} \right.$$

$$(3.34) \quad \left\{ \begin{aligned} &\| \Lambda [e^{(t-t_0)\Lambda} - e^{(s-t_0)\Lambda}] \varphi(s) \| \\ &= \left\| \int_{s-t_0}^{t-t_0} \Lambda^2 e^{\sigma\Lambda} [\varphi(s) - \varphi(t_0)] d\sigma \right\| \\ &\leq c[\varphi]_{C^\theta} \int_{s-t_0}^{t-t_0} \sigma^{\beta-2} (s-t_0)^\theta d\sigma \\ &\leq c[\varphi]_{C^\theta} \int_{s-t_0}^{t-t_0} \sigma^{\gamma-2} d\sigma \\ &\leq c[\varphi]_{C^\theta} (\gamma-1)^{-1} (t-s)^{\gamma-1} \end{aligned} \right.$$

$$(3.35) \quad \left\{ \begin{aligned} \| \Lambda e^{(t-s)\Lambda} [\varphi(t) - \varphi(s)] \| &\leq \| \Lambda e^{(t-s)\Lambda} \|_{L(D_\Lambda(\beta, \infty); X)} [\varphi]_{C^\theta} (t-s)^\theta \\ &\leq c[\varphi]_{C^\theta} (t-s)^{\gamma-1}. \end{aligned} \right.$$

From (3.31), by virtue of (3.32) - (3.35), we obtain

$$(3.36) \quad \Lambda v' \in C^{\gamma-1}([t_0, t_1]; X).$$

But for $t, t+h \in [t_0, t_1]$, $h \neq 0$ we have:

$$(3.37) \quad \left\| \frac{\Lambda v(t+h) - \Lambda v(t)}{h} - \Lambda v'(t) \right\| = \left\| \frac{1}{h} \int_t^{t+h} [\Lambda v'(s) - \Lambda v'(t)] ds \right\|$$

and so the continuity of $\Lambda v'(\cdot)$ implies that there exists

$$(3.38) \quad (\Lambda v)'(t) = \Lambda v'(t), \quad t \in [t_0, t_1].$$

Then from (3.36) we deduce that $\Lambda v \in C^\gamma([t_0, t_1]; X)$, i.e. (3.9) holds; by using (3.27), (3.30), (3.31) and (3.32)- (3.35) we obtain also

$$(3.39) \quad \|\Lambda v\|_{C^{\theta+\beta}([t_0, t_1]; X)} \leq c[\varphi]_{C^\theta}.$$

Let us prove (3.12). For $t \in [t_0, t_1]$ and $\xi \in]0, 1[$ we get from (3.29) and (3.4)

$$(3.40) \quad \left\{ \begin{array}{l} \|\xi^{2-\gamma} \Lambda e^{\xi \Lambda} \Lambda v'(t)\| \\ = \|\xi^{2-\gamma} \int_{t_0}^t \Lambda^3 e^{(t-s+\xi)\Lambda} [\varphi(s) - \varphi(t)] ds \\ \quad + \xi^{2-\gamma} \Lambda^2 e^{(t-t_0+\xi)\Lambda} [\varphi(t) - \varphi(t_0)] \| \\ \leq c[\varphi]_{C^\theta} [\xi^{2-\gamma} \int_{t_0}^t (t-s)^\theta (t-s+\xi)^{\beta-3} ds \\ \quad + \xi^{2-\gamma} (t-t_0+\xi)^{\beta-2} (t-t_0)^\theta] \\ \leq c[\varphi]_{C^\theta} [\xi^{2-\gamma} \int_{t_0}^t (t-s+\xi)^{\gamma-3} ds + \xi^{2-\gamma} (t-t_0+\xi)^{\gamma-2}] \\ = c[\varphi]_{C^\theta} \left\{ (2-\gamma)^{-1} \left[1 - \left(\frac{\xi}{\xi+t-t_0} \right)^{2-\gamma} \right] + \left(\frac{\xi}{t-t_0+\xi} \right)^{2-\gamma} \right\} \\ \leq c[\varphi]_{C^\theta} [(2-\gamma)^{-1} + 1]. \end{array} \right.$$

This proves that $\Lambda v' \in B([t_0, t_1]; D_\Lambda(\gamma-1, \infty))$, i.e. (3.12) holds. Moreover from (3.30) and (3.40) we obtain

$$(3.41) \quad \left\{ \begin{array}{l} \|\Lambda v'\|_{B([t_0, t_1]; D_\Lambda(\gamma-1, \infty))} \\ = \sup_{t \in [t_0, t_1]} [\|\Lambda v'(t)\| + \sup_{0 < \xi \leq 1} \|\xi^{2-\gamma} \Lambda e^{\xi \Lambda} \Lambda v'(t)\|] \\ \leq c[\varphi]_{C^\theta} \end{array} \right.$$

and also estimate (3.13) as a consequence of (3.14), (3.39), and (3.41). The last assertion of the theorem follows from the fact that (3.8) implies $v \in C^{\theta+\beta}([t_0, t_1]; X)$ and $\Lambda v \in C^{\theta+\beta}([t_0, t_1]; X)$ (see (3.9)) when $\theta + \beta \neq 1$. \square

4 The case of time independent coefficients

Let Ω be a bounded open set in \mathbb{R}^n with $C^{2m+\alpha}$ boundary Γ where $m, \alpha > 0$ with $m \in \mathbb{N}$ and $\alpha \notin \mathbb{N}$. For each $x \in \Gamma$, we denote by $\nu(x) =$

$(\nu_1(x), \dots, \nu_n(x))$ the unit exterior normal vector to Γ at the point x . Given $[t_0, t_1] \subseteq \mathbb{R}$ we set again $Q = [t_0, t_1] \times \bar{\Omega}$ and $S = [t_0, t_1] \times \Gamma$.

We consider the linear parabolic initial-boundary value problem (1.1) when the coefficients are independent on time, i.e.:

$$(4.1) \quad \begin{cases} D_t u(t, x) = \sum_{|\gamma| \leq 2m} a_\gamma(x) D_x^\gamma u(t, x) + f(t, x), & (t, x) \in Q \\ u(t_0, x) = u_0(x), & x \in \bar{\Omega} \\ \sum_{|\beta| \leq m_j} b_{j\beta}(x) D_x^\beta u(t, x) = g_j(t, x), & j = 1, \dots, m, \quad (t, x) \in S. \end{cases}$$

The coefficients a_γ and $b_{j\beta}$ are subject to the following assumptions:

$$(4.2) \quad \left\{ \begin{array}{l} \text{(a) (regularity) } a_\gamma \in C^\alpha(\bar{\Omega}) \text{ for each } \gamma \in \mathbb{N}^n \text{ such that } |\gamma| \leq 2m. \\ \text{(b) (ellipticity) There exist } \mu > 0, \eta_0 \in]\frac{\pi}{2}, \pi[\text{ such that for each } \\ x \in \bar{\Omega}, \eta \in [-\eta_0, \eta_0], \xi \in \mathbb{R}^n, r \in \mathbb{R} \text{ with } |\xi|^2 + r^2 \neq 0 \text{ we have} \\ \left| \sum_{|\gamma|=2m} a_\gamma(x) \xi^\gamma - (-1)^m r^{2m} e^{i\eta} \right| \geq \mu(|\xi|^{2m} + r^{2m}) \\ \text{(c) (roots condition) For each } x \in \Gamma, \eta \in [-\eta_0, \eta_0], \xi \in \mathbb{R}^n, r \in \mathbb{R} \\ \text{with } |\xi|^2 + r^2 \neq 0 \text{ and } \langle \xi, \nu(x) \rangle = 0, \text{ the polynomial} \\ p(z) = \sum_{|\gamma|=2m} a_\gamma(x) [\xi + z\nu(x)]^\gamma - (-1)^m r^{2m} e^{i\eta} \\ \text{has exactly } m \text{ roots } z_j^+(x, \xi, r, \eta), j = 1, \dots, m \text{ with positive} \\ \text{imaginary part.} \end{array} \right.$$

$$(4.3) \left\{ \begin{array}{l} \text{(a) (regularity) } b_{j\beta} \in C^{2m+\alpha-m_j}(\Gamma) \text{ for each } j = 1, \dots, m \\ \text{and for each } \beta \in \mathbb{N}^n \text{ such that } |\beta| \leq m_j \\ \text{(b) (normality) } 0 \leq m_1 < m_2 < \dots < m_m \leq 2m - 1 \text{ and for each} \\ j = 1, \dots, m, \sum_{|\beta|=m_j} b_{j\beta}(x)(\nu(x))^\beta \neq 0 \text{ for } x \in \Gamma. \\ \text{(c) (complementing condition) For each } x \in \Gamma, \xi \in \mathbb{R}^n, r \in \mathbb{R} \\ \text{with } |\xi|^2 + r^2 \neq 0 \text{ and } \langle \xi, \nu(x) \rangle = 0, \text{ the polynomials} \\ p_j(z) = \sum_{|\beta|=m_j} b_{j\beta}(x)(\xi + z\nu(x))^\beta, j = 1, \dots, m \text{ are linearly} \\ \text{independent modulo the polynomial} \\ q(z) = \prod_{j=1}^m (z - z_j^+(x, \xi, r, \eta)) \text{ where } z_j^+ \text{ are defined in 4.2(c).} \end{array} \right.$$

Let us observe that the last condition of (b) is a consequence of (c).

Concerning the data f, u_0 and g_j ($j = 1, \dots, m$) we assume that

$$(4.4) \quad \begin{cases} f \in C^{\frac{\alpha}{2m}, \alpha}(Q), u_0 \in C^{2m+\alpha}(\bar{\Omega}), g_j \in C^{\frac{2m+\alpha-m_j}{2m}, 2m+\alpha-m_j}(S), \\ j = 1, \dots, m. \end{cases}$$

We want to prove the existence of a solution u of (4.1) such that

$$(4.5) \quad u \in C^{\frac{2m+\alpha}{2m}, 2m+\alpha}(Q).$$

Then it is known (see [S]) that the following compatibility conditions must hold

5 The abstract setting

We shall use the following notation

$$(5.1) \quad \begin{cases} A = \sum_{|\gamma| \leq 2m} a_\gamma(\cdot) D_x^\gamma, \quad B_j = \sum_{|\beta| \leq m_j} b_{j\beta}(\cdot) D_x^\beta, \quad j = 1, \dots, m \\ f(t) = f(t, \cdot), \quad g_j(t) = g_j(t, \cdot), \quad t \in [t_0, t_1], \quad j = 1, \dots, m \\ u^{(0)} = u_0, \quad u^{(k)} = Au^{(k-1)} + f^{(k-1)}(t_0), \quad k = 1, \dots, [\frac{2m+\alpha}{2m}] \end{cases}$$

where $f^{(k)}(t) = D_t^k f(t, \cdot)$, so that problem (4.1) can be written as

$$(5.2) \quad \begin{cases} u'(t) = Au(t) + f(t), \quad t \in [t_0, t_1] \\ u(t_0) = u_0 \\ B_j u(t) = g_j(t), \quad j = 1, \dots, m; \quad t \in [t_0, t_1] \end{cases}$$

and the compatibility conditions as

$$(5.3) \quad B_j u^{(k)} = g_j^{(k)}(t_0), \quad j = 1, \dots, m; \quad k = 0, \dots, [\frac{2m+\alpha-m_j}{2m}].$$

By virtue of theorem 1.2, theorem 3.1 is equivalent to

Theorem 5.1 *Let assumptions (4.2)-(4.3) hold. Assume that*

$$(5.4) \quad \begin{cases} f \in C^{\frac{\alpha}{2m}}([t_0, t_1]; C(\bar{\Omega})) \text{ is such that} \\ f^{(k)} \in B([t_0, t_1]; C^{\alpha-2mk}(\bar{\Omega})), \quad k = 0, \dots, [\frac{\alpha}{2m}] \end{cases}$$

$$(5.5) \quad u_0 \in C^{2m+\alpha}(\bar{\Omega})$$

and assume that for each $j = 1, \dots, m$

$$(5.6) \quad \begin{cases} g_j \in C^{\frac{2m+\alpha-m_j}{2m}, 2m+\alpha-m_j}([t_0, t_1]; C(\Gamma)) \text{ is such that} \\ g_j^{(k)} \in B([t_0, t_1]; C^{2m+\alpha-m_j-2mk}(\Gamma)), \quad k = 0, \dots, [\frac{2m+\alpha-m_j}{2m}]. \end{cases}$$

Moreover suppose that, defining $u^{(k)}$ by (5.1), conditions (5.3) hold. Then problem (5.2) has a unique solution

$$(5.7) \quad \begin{cases} u \in C^{\frac{2m+\alpha}{2m}, 2m+\alpha}([t_0, t_1]; C(\bar{\Omega})) \text{ such that} \\ u^{(k)} \in B([t_0, t_1]; C^{2m+\alpha-2mk}(\bar{\Omega})), \quad k = 0, \dots, [\frac{2m+\alpha}{2m}] \end{cases}$$

and there exists a continuous and increasing function $c : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ (depending also on bounds of the norms of $a_\gamma, b_{j\beta}$ and on Ω, η_0) such that

$$(5.8) \quad \begin{cases} \|u\|_{C^{\frac{2m+\alpha}{2m}, 2m+\alpha}(Q)} \leq c(t_1 - t_0) \{ \|f\|_{C^{\frac{\alpha}{2m}, \alpha}(Q)} + \|u_0\|_{C^{2m+\alpha}(\bar{\Omega})} \\ + \sum_{j=1}^m \|g_j\|_{C^{\frac{2m+\alpha-m_j}{2m}, 2m+\alpha-m_j}(S)} \}. \end{cases}$$

In the proof of this theorem we shall use also known results about generators of analytic semigroups, characterization of interpolation spaces and Hölder regularity for elliptic equations which we collect in the following theorem.

Theorem 5.2 *Let (4.2)-(4.3) hold and let A and B_j ($j = 1, \dots, m$) be defined by (5.1).*

(i) *If $X = C(\bar{\Omega})$ is given the sup-norm and*

$$(5.9) \quad \begin{cases} D(\Lambda) = \{ \varphi \in \bigcap_{p>1} W^{2m,p}(\Omega); A\varphi \in C(\bar{\Omega}); B_j\varphi = 0, \\ \quad \quad \quad j = 1, \dots, m \} \\ \Lambda\varphi = A\varphi \end{cases}$$

then $\Lambda : D(\Lambda) \subseteq X \rightarrow X$ satisfies (3.1), where ω, η, M depend on bounds of the norms of $a_\gamma, b_{j\beta}$, and on Ω, η_0 .

(ii) *For each $\theta \in]0, 1[$ such that $2m\theta \notin \mathbb{N}$ we have*

$$(5.10) \quad D_\Lambda(\theta, \infty) = \{ \varphi \in C^{2m\theta}(\bar{\Omega}); B_j\varphi = 0 \text{ if } m_j \leq [2m\theta] \}$$

and the $C^{2m\theta}$ -norm is equivalent to the $D_\Lambda(\theta, \infty)$ -norm.

(iii) For each $k = 1, \dots, 2m - 1$ we have

$$(5.11) \quad C_B^k(\bar{\Omega}) = \{\varphi \in C^k(\bar{\Omega}); B_j\varphi = 0 \text{ if } m_j < k\} \hookrightarrow D_\Lambda\left(\frac{k}{2m}, \infty\right)$$

where $C_B^k(\bar{\Omega})$ is given the norm of $C^k(\bar{\Omega})$.

(iv) $D = \{\varphi \in \bigcap_{p>1} W^{2m,p}(\Omega); A\varphi \in C^\alpha(\bar{\Omega}); B_j\varphi \in C^{2m+\alpha-m_j}(\Gamma); j = 1, \dots, m\} \subset C^{2m+\alpha}(\bar{\Omega})$ and there exists $c_6 > 0$ such that

$$(5.12) \quad \|\varphi\|_{C^{2m+\alpha}(\bar{\Omega})} \leq c_6 \{ \|A\varphi\|_{C^\alpha(\bar{\Omega})} + \|\varphi\|_{C(\bar{\Omega})} + \sum_{j=1}^m \|B_j\varphi\|_{C^{2m+\alpha-m_j}(\Gamma)} \}.$$

Proof. (i) is a consequence of the results on the resolvent of Λ proved in [GG] and on the estimate of $\|(\lambda - \Lambda)^{-1}\|$ demonstrated in [St] and [AT]. (ii) is a result of [AT] and (iii) follows from the characterization of $D_\Lambda(\frac{k}{2m}, \infty)$ proved in [A]. Concerning (iv) one must observe that the results of [ADN] prove estimate (5.12) whereas the inclusion of the set D in $C^{2m+\alpha}(\bar{\Omega})$ is a consequence of the existence theorems proved in [GG]. \square

6 An extension operator

As mentioned in the introduction the theory of semigroups of linear operators cannot be applied to problem (1.1) when $g_j \neq 0$ for some $j = 1, \dots, m$. For this reason we need an operator N satisfying (1.12): its existence will be proved in three steps. We can suppose that $n > 1$ because when $n = 1$ the operator N can be constructed by using suitable polynomials.

The first step is essentially lemma 10 of [Se]:

Theorem 6.1 *Given $\varphi \in C_0^\infty(\mathbb{R}^{n-1})$ such that $\int_{\mathbb{R}^{n-1}} \varphi(t_1, \dots, t_{n-1}) dt_1 \cdots dt_{n-1} = 1$, set for each $k \in \mathbb{N}$ and $f \in C^0(\mathbb{R}^{n-1})$:*

$$(6.1) \quad \begin{cases} F_k(y_1, \dots, y_n) \\ = \frac{y_n^k}{k!} \int_{\mathbf{R}^{n-1}} \varphi(t_1, \dots, t_{n-1}) \\ f(y_1 + t_1 y_n, \dots, y_{n-1} + t_{n-1} y_n) dt_1 \cdots dt_{n-1}. \end{cases}$$

Then we have:

$$(6.2) \quad F_k \in C^k(\mathbb{R}^n)$$

and

$$(6.3) \quad \begin{cases} D_y^l F_k(y_1, \dots, y_{n-1}, 0) = \begin{cases} 0 & \text{if } |l| \leq k \text{ and } l \neq (0, \dots, 0, k) \\ f(y_1, \dots, y_{n-1}) & \text{if } l = (0, \dots, 0, k) \end{cases} \\ \text{for } (y_1, \dots, y_{n-1}) \in \mathbb{R}^{n-1}. \end{cases}$$

Moreover for each $m \in \mathbb{N}$ and $\theta \in [0, 1[$:

$$(6.4) \quad f \in C^{m+\theta}(\mathbb{R}^{n-1}) \Rightarrow F_k \in C^{k+m+\theta}(\mathbb{R}^n).$$

As second step we prove the following result.

Theorem 6.2 For each $j = 1, \dots, m$ there exists $N_j \in L(C^0(\Gamma); C^{m_j}(\bar{\Omega}))$ such that setting $v_j = N_j \psi$ for $\psi \in C^0(\Gamma)$, we have

$$(6.5) \quad D_x^l v_j(x) = 0, \quad x \in \Gamma, \quad l \in \mathbb{N}^n, \quad |l| < m_j$$

$$(6.6) \quad (B_j v_j)(x) = \psi(x), \quad x \in \Gamma.$$

Moreover

$$(6.7) \quad N_j \in L(C^r(\Gamma); C^{r+m_j}(\bar{\Omega})), \quad \forall r \in [0, 2m + \alpha - m_j].$$

Proof. If $\bigcup_{i=1}^N \mathring{B}_i$ is an open covering of Γ (see the notation of section 0) let $\varphi_i = \{\varphi_{i_1}, \dots, \varphi_{i_n}\} (i = 1, \dots, N)$ be the $C^{2m+\alpha}$ -diffeomorphism of B_i onto B . Then setting for $i = 1, \dots, N$ and $j = 1, \dots, m$

$$(6.8) \quad \gamma_{ij}(x) = \sum_{\substack{\beta=(\beta_1 \dots \beta_n), \\ |\beta|=m_j}} b_{j\beta}(x) \left(\frac{\partial \varphi_{i_n}(x)}{\partial x_1}\right)^{\beta_1} \dots \left(\frac{\partial \varphi_{i_n}(x)}{\partial x_n}\right)^{\beta_n}, \quad x \in \Gamma \cap B_i$$

we have $\gamma_{ij} \in C^{2m+\alpha-m_j}(\Gamma \cap B_i)$. Given $\psi \in C^r(\Gamma)$ with $r \in [0, 2m + \alpha - m_j]$ we set

$$(6.9) \quad f_{ij}(y_1, \dots, y_{n-1}, 0) = \frac{\psi[\varphi_i^{-1}(y_1, \dots, y_{n-1}, 0)]}{\gamma_{ij}[\varphi_i^{-1}(y_1, \dots, y_{n-1}, 0)]}, \quad (y_1, \dots, y_{n-1}, 0) \in \Sigma.$$

As $f_{ij} \in C^r(\Sigma)$, we can define (by virtue of theorem 5.1) $u_{ij}^* \in C^{r+m_j}(\mathbb{R}^n)$ such that for each $(y_1, \dots, y_{n-1}, 0) \in \Sigma$ we have:

$$(6.10) \quad \begin{cases} D_y^l u_{ij}^*(y_1, \dots, y_{n-1}, 0) \\ = 0 \text{ if } |l| \leq m_j \text{ and } l \neq (0, \dots, 0, m_j) \\ f_{ij}(y_1, \dots, y_{n-1}, 0) \text{ if } l = (0, \dots, 0, m_j). \end{cases}$$

Let φ_i^* and ξ_i^* be the extensions of φ_i and ξ_i obtained by setting $\varphi_i^*(x) = (0, \dots, 0)$ and $\xi_i^*(x) = 0$ for $x \in \bar{\Omega} \setminus B_i$. We will prove that

$$(6.11) \quad v_j(x) = \sum_{i=1}^N \xi_i^*(x) u_{ij}^*(\varphi_i^*(x)), \quad x \in \bar{\Omega}$$

satisfies (6.5)-(6.7). Let us fix $x_0 \in \Gamma$ and let $N_0 = \{i \in \{1, \dots, N\}; x_0 \in \mathring{B}_i\}$: there exists $r > 0$ such that $B(x_0, r) = \{x \in \mathbb{R}^n, |x - x_0| < r\} \subseteq \bigcap_{i \in N_0} \mathring{B}_i$,

$$(6.12) \quad v_j(x) = \sum_{i \in N_0} \xi_i(x) u_{ij}^*(\varphi_i(x)), \quad x \in B(x_0, r) \cap \bar{\Omega}.$$

Hence for $l \in \mathbb{N}^n$ such that $|l| < m_j$ we deduce (6.5) from the first of (6.10), while for $|l| = m_j$ we obtain

$$(6.13) \quad \left\{ \begin{aligned} (B_j v_j)(x_0) &= \sum_{|\beta|=m_j} b_{j\beta}(x_0) \sum_{i \in N_0} D_x^\beta [\xi_i(x) u_{ij}^*(\varphi_i(x))]_{x=x_0} \\ &= \sum_{i \in N_0} \xi_i(x_0) \sum_{|\beta|=m_j} b_{j\beta}(x_0) \left(\frac{\partial^{m_j} u_{ij}^*(y)}{\partial y_{m_j}^{m_j}} \right)_{y=\varphi_i(x_0)} \\ &\quad \left(\frac{\partial \varphi_{i_1}(x)}{\partial x_1} \right)_{x=x_0}^{l_1} \cdots \left(\frac{\partial \varphi_{i_n}(x)}{\partial x_n} \right)_{x=x_0}^{l_n}. \end{aligned} \right.$$

Hence (6.6) follows from (6.9) and (6.10). (6.7) is a consequence of the definition of u_{ij}^* and the regularity assumptions on Γ . \square

We can prove now the existence of the extension operator N verifying (1.12).

Theorem 6.3 *Given $s = 1, \dots, m$ there exists*

$$(6.14) \quad \left\{ \begin{aligned} M_s &\in L\left(\prod_{j=1}^s C^{\theta-m_j}(\Gamma), C^{\theta-m_s}(\Gamma)\right), \\ \forall \theta &\in [m_s, 2m + \alpha] \end{aligned} \right.$$

and such that setting

$$(6.15) \quad N(\psi_1, \dots, \psi_m) = \sum_{s=1}^m N_s M_s(\psi_1, \dots, \psi_s)$$

we have

$$(6.16) \quad \left\{ \begin{aligned} N &\in L\left(\prod_{j=1}^m C^{2m+\theta'-m_j}(\Gamma), C^{2m+\theta'}(\bar{\Omega})\right), \\ \forall \theta' &\in [0, \alpha] \end{aligned} \right.$$

and

$$(6.17) \quad B_j(N(\psi_1, \dots, \psi_m))(x) = \psi_j(x), \quad x \in \Gamma, \quad j = 1, \dots, m.$$

Proof. Let us first observe that by virtue of (4.3) (a) we have for each $j = 1, \dots, m$:

$$(6.18) \quad B_j \in L(C^\theta(\bar{\Omega}); C^{\theta-m_j}(\Gamma)), \quad \theta \in [m_j, 2m + \alpha].$$

For each $(r, s) \in \mathbb{N}^2$ with $1 \leq r \leq s \leq m$ we define the operator $H_{s,r}$ by induction:

$$(6.19) \quad H_{r,r} = I_\Gamma, \quad 1 \leq r \leq m$$

(where I_Γ is the identity in the set of functions defined in Γ) and

$$(6.20) \quad H_{r+h,r} = -B_{r+h} \sum_{j=0}^{h-1} N_{r+j} H_{r+j,r}.$$

From (6.7) and (6.18) we deduce, for $1 \leq r \leq s \leq m$:

$$(6.21) \quad H_{s,r} \in L(C^{\theta-m_r}(\Gamma); C^{\theta-m_s}(\Gamma)) \text{ if } \theta \in [m_s, 2m + \alpha].$$

Let us define now the following operators

$$(6.22) \quad \begin{cases} M_1(\psi_1) & = H_{1,1}(\psi_1) \\ M_2(\psi_1, \psi_2) & = H_{2,1}(\psi_1) + H_{2,2}(\psi_2) \\ \dots & \\ M_m(\psi_1, \dots, \psi_m) & = H_{m,1}(\psi_1) + \dots + H_{m,m}(\psi_m). \end{cases}$$

From (6.21) we obtain (6.14) which implies (6.16). To prove (6.17) let us fix

$$(6.23) \quad \psi_j \in C^{2m+\theta'-m_j}(\Gamma), \quad j = 1, \dots, m, \quad \theta' \in [0, \alpha]$$

and set

$$(6.24) \quad u_k = \sum_{s=1}^k N_s M_s(\psi_1, \dots, \psi_s), \quad k = 1, \dots, m.$$

By using properties (6.5)-(6.6) one proves by induction that

$$(6.25) \quad B_j u_k = \psi_j \text{ for } 1 \leq j \leq k$$

which, for $k = m$, coincides with (6.17). □

7 Proof of theorem 4.1 for $\alpha < 2m$

From (5.6) and (2.12) we deduce

$$(7.1) \quad \begin{cases} g_j \in C^{\frac{2m+\alpha-m_j-h}{2m}}([t_0, t_1]; C^h(\Gamma)), \\ h \in \mathbb{N}, 0 \leq h \leq [\alpha] + 2m - m_j \end{cases}$$

so that from (6.7), (6.14), and (7.1) we obtain (for $s = 1, \dots, m$):

$$(7.2) \quad \begin{aligned} N_s M_s(g_1(\cdot), \dots, g_s(\cdot)) &\in B([t_0, t_1]; C^{2m+\alpha}(\bar{\Omega})) \\ &\cap C^{\frac{\alpha}{2m}}([t_0, t_1]; C^{2m}(\bar{\Omega})) \cap C^{\frac{2m+\alpha-m_s}{2m}}([t_0, t_1]; C^{m_s}(\bar{\Omega})). \end{aligned}$$

Hence if

$$(7.3) \quad m_1 < \dots < m_s < \alpha$$

there exists, for $t \in [t_0, t_1]$

$$(7.4) \quad \frac{d}{dt} N_s M_s(g_1(t), \dots, g_s(t)) = N_s M_s(g'_1(t), \dots, g'_s(t))$$

and

$$(7.5) \quad \begin{cases} N_1 M_1(g'_1(\cdot)) \in C^{\frac{\alpha}{2m}}([t_0, t_1]; C^0(\bar{\Omega})) \text{ if } m_1 = 0 \\ N_s M_s(g'_1(\cdot), \dots, g'_s(\cdot)) \in C^{\frac{\alpha-m_s}{2m}}([t_0, t_1]; D_\Lambda(\frac{m_s}{2m}, \infty)) \\ \text{if } 1 \leq m_s < \alpha. \end{cases}$$

Actually, from (5.11) we get that the subspace of $C^{m_s}(\bar{\Omega})$ defined as

$$(7.6) \quad \{\varphi \in C^{m_s}(\bar{\Omega}); B_j \varphi = 0 \text{ if } m_j < m_s\}$$

is continuously embedded in $D_\Lambda(\frac{m_s}{2m}, \infty)$ if $m_s \geq 1$. Moreover from (6.5) we deduce that for each $u \in C^0(\Gamma)$:

$$(7.7) \quad (B_j N_s u)(x) \equiv 0 \text{ on } \bar{\Omega} \text{ for } j < s$$

and from (5.10) we obtain

$$(7.8) \quad D_\Lambda\left(\frac{m_s}{2m}, \infty\right) = \{u \in C^{m_s}(\bar{\Omega}); B_j u = 0 \text{ if } m_j \leq m_s\} \text{ for } 0 < m_s < 2m$$

which implies (7.5)(2); while (7.5)(1) is a consequence of (7.2) with $s = 1$.

Let us define for each $t \in [t_0, t_1]$:

$$(7.9) \quad n(t) = N(g_1(t), \dots, g_m(t)) = \sum_{s=1}^m N_s M_s(g_1(t), \dots, g_s(t))$$

$$(7.10) \quad \begin{cases} n_1(t) = \begin{cases} 0 & \text{if } \alpha < m_1 \\ \sum_{s=1}^p N_s M_s(g_1(t), \dots, g_s(t)) & \text{if } m_1 < \alpha \end{cases} \\ n_2(t) = n(t) - n_1(t), \end{cases}$$

where p is the greatest integer $j \leq m$ such that $m_j < \alpha$. From (7.2) and (7.4) it follows that

$$(7.11) \quad \begin{cases} n & \in B([t_0, t_1]; C^{2m+\alpha}(\bar{\Omega})) \cap C^{\frac{\alpha}{2m}}([t_0, t_1]; C^{2m}(\bar{\Omega})) \\ An & \in B([t_0, t_1]; C^\alpha(\bar{\Omega})) \cap C^{\frac{\alpha}{2m}}([t_0, t_1]; C^0(\bar{\Omega})) \end{cases}$$

$$(7.12) \quad \begin{cases} n_1 & \in B([t_0, t_1]; C^{2m+\alpha}(\bar{\Omega})) \\ An_1 & \in B([t_0, t_1]; C^\alpha(\bar{\Omega})) \end{cases}$$

$$(7.13) \quad \begin{cases} n_1 & \in C^1([t_0, t_1]; C^0(\bar{\Omega})) \\ n'_1 & \in B([t_0, t_1]; C^\alpha(\bar{\Omega})) \\ n'_1(t) & = \sum_{s=1}^p N_s M_s(g'_1(t), \dots, g'_s(t)), \quad t \in [t_0, t_1], \text{ if } m_1 < \alpha. \end{cases}$$

As we mentioned in the introduction, we will prove that a solution of problem (5.2) can be obtained by the representation formula (1.10) with some adaptations due to the compatibility conditions that must be satisfied to get solutions having the required regularity. More precisely we will solve the three problems

$$(7.14) \quad \begin{cases} v'(t) = Av(t) + f(t) + An(t) - n_1'(t_0), & t \in [t_0, t_1] \\ v(t_0) = u_0 - n(t_0) \\ B_j v(t) = 0, \quad j = 1, \dots, m, & t \in [t_0, t_1] \end{cases}$$

$$(7.15) \quad \begin{cases} w'(t) = Aw(t) - An_1(t) + n_1'(t_0), & t \in [t_0, t_1] \\ w(t_0) = n_1(t_0) \\ B_j w(t) = B_j n_1(t), \quad j = 1, \dots, m, & t \in [t_0, t_1] \end{cases}$$

$$(7.16) \quad \begin{cases} z'(t) = Az(t) - An_2(t), & t \in [t_0, t_1] \\ z(t_0) = n_2(t_0) \\ B_j z(t) = B_j n_2(t), \quad j = 1, \dots, m, & t \in [t_0, t_1] \end{cases}$$

and will prove that

$$(7.17) \quad u(t) = v(t) + w(t) + z(t), \quad t \in [t_0, t_1]$$

is the solution of problem (5.2). Uniqueness of the solution is a consequence of theorem 3.1.

To solve the three problems we will apply the abstract results of section 3 with X and Λ defined in (i) of theorem 5.2.

Let us first consider problem (7.14). To apply theorem 3.1 with $\theta = \frac{\alpha}{2m}$ we must show that

$$(7.18) \quad f(\cdot) + An(\cdot) - n_1'(t_0) \in C^{\frac{\alpha}{2m}}([t_0, t_1]; C(\bar{\Omega}))$$

$$(7.19) \quad u_0 - n(t_0) \in D(\Lambda)$$

and

$$(7.20) \quad \Lambda(u_0 - n(t_0)) + f(t_0) + An(t_0) - n'_1(t_0) \in D_\Lambda\left(\frac{\alpha}{2m}, \infty\right)$$

where

$$(7.21) \quad D_\Lambda\left(\frac{\alpha}{2m}, \infty\right) = \{\varphi \in C^\alpha(\bar{\Omega}); B_j\varphi = 0 \text{ if } m_j < \alpha\}.$$

Now (7.18) follows from (7.11) and (7.13); (7.20) is a consequence of (7.12), (6.17) and (5.3). To prove (7.20) we can use (7.11)-(7.13) and the fact that due to (6.25) and (5.3) we have, for $m_j < \alpha$

$$(7.22) \quad \begin{cases} B_j(Au_0 + f(t_0) - n'_1(t_0)) \\ = B_j(Au_0 + f(t_0)) - B_j \sum_{s=1}^p N_s M_s(g'_1(t_0), \dots, g'_s(t_0)) \\ = B_j(Au_0 + f(t_0)) - g'_j(t_0) = 0. \end{cases}$$

Theorem 3.1 gives now a solution v of (7.14) such that

$$(7.23) \quad \begin{cases} v \in C^{1+\frac{\alpha}{2m}}([t_0, t_1]; C(\bar{\Omega})), v' \in B([t_0, t_1]; C^\alpha(\bar{\Omega})) \\ v(t) \in \bigcap_{p>1} W^{2m,p}(\bar{\Omega}), Av(t) \in C^\alpha(\bar{\Omega}), t \in [t_0, t_1] \end{cases}$$

where we have used the fact that $D_\Lambda\left(\frac{\alpha}{2m}, \infty\right) \simeq C^\alpha(\bar{\Omega})$ (see theorem 5.2). To solve problem (7.15) we write

$$(7.24) \quad n'_1(t) - n'_1(t_0) = \sum_{s=1}^p \varphi_s(t), \quad t \in [t_0, t_1]$$

with

$$(7.25) \quad \varphi_s(t) = N_s M_s(g'_1(t) - g'_1(t_0), \dots, g'_s(t) - g'_s(t_0)), \quad t \in [t_0, t_1]$$

and for each $s = 1, \dots, p$, we consider the problem

$$(7.26) \quad \begin{cases} w'_s(t) &= Aw_s(t) - \varphi_s(t), \quad t \in [t_0, t_1] \\ w_s(t_0) &= 0 \\ B_j w_s(t) &= 0, \quad j = 1, \dots, m, \quad t \in [t_0, t_1]. \end{cases}$$

If $m_1 = 0$ we have $\varphi_1 \in C^{\frac{\alpha}{2m}}([t_0, t_1]; C^0(\bar{\Omega}))$ by virtue of (7.5): hence from theorem 2.1 we deduce the existence of a solution w_1 to (7.26)(1) such that

$$(7.27) \quad \begin{cases} w_1 &\in C^1([t_0, t_1]; X), \\ \Lambda w_1 &\in C^{\frac{\alpha}{2m}}([t_0, t_1]; X), \\ w'_1 &\in B([t_0, t_1]; D_\Lambda(\frac{\alpha}{2m}, \infty)). \end{cases}$$

If $m_1 > 0$ we get from (7.5) that $\varphi_s \in C^{\frac{\alpha-m_s}{2m}}([t_0, t_1]; D_\Lambda(\frac{m_s}{2m}, \infty))$ for each $s = 1, \dots, p$. Hence from theorem 3.2 we obtain

$$(7.28) \quad \begin{cases} w_s &\in C^1([t_0, t_1]; X), \\ \Lambda w_s &\in C^{\frac{\alpha}{2m}}([t_0, t_1]; X), \\ w'_s &\in B([t_0, t_1]; D_\Lambda(\frac{\alpha}{2m}, \infty)) \end{cases}$$

where w_s is the solution of (7.26). Setting

$$(7.29) \quad w(t) = \sum_{s=1}^p w_s(t) + n_1(t), \quad t \in [t_0, t_1]$$

we obtain a solution of (7.15) verifying the same properties as v (see (7.23)).

We can consider problem (7.16) when $n_2(t) \not\equiv 0$ i.e. when $p < m$. In this case we set for each $s = p+1, \dots, m$:

$$(7.30) \quad \psi_s(t) = N_s M_s (g_1(t) - g_1(t_0), \dots, g_s(t) - g_s(t_0)), \quad t \in [t_0, t_1]$$

and first we solve problem

$$(7.31) \quad \begin{cases} y'_s(t) &= \Lambda y_s(t) + \psi_s(t), \quad t \in [t_0, t_1] \\ y_s(t_0) &= 0. \end{cases}$$

We have

$$(7.32) \quad \psi_s \in C^{\frac{2m+\alpha-m_s}{2m}}([t_0, t_1]; D_\Lambda(\frac{m_s}{2m}, \infty))$$

by virtue of (5.11), (7.2) and the fact that $B_j N_s = 0$ for $j < s$. Hence theorem 5.2 gives a solution y_s of (7.31) such that

$$(7.33) \quad \begin{cases} \Lambda y_s &\in C^{1+\frac{\alpha}{2m}}([t_0, t_1]; X), \\ \Lambda y'_s &= (\Lambda y_s)' \in B([t_0, t_1]; D_\Lambda(\frac{\alpha}{2m}, \infty)). \end{cases}$$

Then, setting

$$(7.34) \quad z(t) = -\Lambda \left(\sum_{s=p+1}^m y_s(t) \right) + n_2(t_0), \quad t \in [t_0, t_1]$$

we obtain a solution of (7.16) verifying the same properties of v (see (7.23)). We conclude that $u = v + w + z$ is a solution of (5.2) because $B_j n(t) = g_j(t)$, $j = 1, \dots, m$, $t \in [t_0, t_1]$. Moreover we have:

$$(7.35) \quad \begin{cases} u \in C^{1+\frac{\alpha}{2m}}([t_0, t_1]; C(\bar{\Omega})), \quad u' \in B([t_0, t_1]; C^\alpha(\bar{\Omega})) \\ u(t) \in \bigcap_{p>1} W^{2m,p}(\Omega), \quad Au(t) \in C^\alpha(\bar{\Omega}), \quad t \in [t_0, t_1]. \end{cases}$$

As $\alpha < 2m$, to prove (5.7) we only need to show that:

$$(7.36) \quad u \in B([t_0, t_1]; C^{2m+\alpha}(\bar{\Omega}))$$

and this can be done by using (iv) of theorem 5.2, by virtue of (7.35) and the fact that $B_j u(t) = g_j(t) \in C^{2m+\alpha-m_j}(\Gamma)$, $t \in [t_0, t_1]$.

Estimate (5.8) is a consequence of estimates (2.11), (2.13), (3.7), (3.13), and (5.12). \square

Remark: From (3.5) it follows that the following representation formulas hold for each $t \in [t_0, t_1]$:

$$(7.37) \quad v(t) = e^{(t-t_0)\Lambda}(u_0 - n(t_0)) + \int_{t_0}^t e^{(t-s)\Lambda}(f(s) + An(s) - n'_1(t_0))ds$$

$$(7.38) \quad w(t) = n_1(t) - \int_{t_0}^t e^{(t-s)\Lambda}(n'_1(s) - n'_1(t_0))ds$$

$$(7.39) \quad z(t) = n_2(t_0) - \Lambda \int_{t_0}^t e^{(t-s)\Lambda}(n_2(s) - n_2(t_0))ds$$

and so, since $n_1 \in C^1([t_0, t_1]; X)$, we deduce that (1.10) holds, i.e.

$$(7.40) \quad \begin{cases} u(t) = e^{(t-t_0)\Lambda}u_0 + \int_{t_0}^t e^{(t-s)\Lambda}(f(s) + An(s))ds \\ \quad - \Lambda \int_{t_0}^t e^{(t-s)\Lambda}n(s)ds, \quad t \in [t_0, t_1]. \end{cases}$$

8 Proof of theorem 4.1 for $\alpha > 2m$

We use the notations of section 6 and we set

$$(8.1) \quad \alpha = 2mk + \sigma, \quad k \in \mathbb{N}, \quad 0 < \sigma < 2m.$$

The result will be proved by recurrence on k . Suppose $k = 1$ and consider the problem obtained differentiating formally (5.2):

$$(8.2) \quad \begin{cases} y'(t) &= Ay(t) + f'(t), \quad t \in [t_0, t_1] \\ y(t_0) &= Au_0 + f(t_0) \\ B_j y(t) &= g'_j(t), \quad j = 1, \dots, m, \quad t \in [t_0, t_1] \end{cases}$$

Since the conditions of the case $\alpha < 2m$ are verified, we deduce the existence of a unique solution y such that

$$(8.3) \quad \begin{cases} y \in C^{1+\frac{\sigma}{2m}}([t_0, t_1]; C(\bar{\Omega})) \cap B([t_0, t_1]; C^{2m+\sigma}(\bar{\Omega})), \\ y' \in B([t_0, t_1]; C^\sigma(\bar{\Omega})) \end{cases}$$

and the estimate

$$(8.4) \quad \begin{cases} \|y\|_{C^{1+\frac{\sigma}{2m}}([t_0, t_1]; C(\bar{\Omega}))} + \|y\|_{B([t_0, t_1]; C^{2m+\sigma}(\bar{\Omega}))} \\ \quad + \|y'\|_{B([t_0, t_1]; C^\sigma(\bar{\Omega}))} \\ \leq c_\sigma(t_1 - t_0) \{ \|Au_0 + f(t_0)\|_{C^{2m+\sigma}(\bar{\Omega})} + \|f'\|_{C^{\frac{\sigma}{2m}, \sigma}(Q)} \\ \quad + \sum_{j=1}^m \|g'_j\|_{C^{\frac{2m+\sigma-m_j}{2m}, 2m+\sigma-m_j}(S)} \} \end{cases}$$

holds, where c_σ has the same properties as c in (5.8).

We want to show that $y = u'$, where u is the solution of (5.2). To this end we fix $\bar{t} \in]t_0, t_1[$, we define for each $h \in]0, t_1 - \bar{t}[$

$$(8.5) \quad u_h(t) = h^{-1}(u(t+h) - u(t)), \quad t \in [t_0, \bar{t}]$$

and we define similarly f_h and g_{jh} . Setting $u_h(t_0) = u_{0h}$ we have for each h :

$$(8.6) \quad \begin{cases} u'_h(t) &= Au_h(t) + f_h(t), \quad t \in [t_0, \bar{t}] \\ u_h(t_0) &= u_{0h} \\ B_j u_h(t) &= g_{jh}(t), \quad j = 1, \dots, m; \quad t \in [t_0, \bar{t}]. \end{cases}$$

Taking into account the representation formula (7.40) we obtain:

$$(8.7) \quad \begin{cases} u_h(t) - y(t) = e^{(t-t_0)\Lambda}(u_{0h} - u'(t_0)) \\ \quad + \int_{t_0}^t e^{(t-s)\Lambda}[f_h(s) - f'(s) + A(n_h(s) - n'(s))]ds \\ \quad - \Lambda \int_{t_0}^t e^{(t-s)\Lambda}(n_h(s) - n'(s))ds \end{cases}$$

where

$$(8.8) \quad \begin{cases} n_h(t) = N(g_{1h}(t), \dots, g_{mh}(t)) = h^{-1}(n(t+h) - n(t)), \\ t \in [t_0, \bar{t}]. \end{cases}$$

As u, f and An are in $C^1([t_0, t_1]; X)$ (see (7.11)) we have $\lim_{h \rightarrow 0} u_{0h} = u'(t_0)$ and $\lim_{h \rightarrow 0} f_h(t) = f(t)$ uniformly in $[t_0, \bar{t}]$. Moreover from (7.11) we deduce $n \in C^{1+\frac{\sigma}{2m}}([t_0, t_1]; X)$. Then for $\varepsilon \in]0, \sigma[$ we have

$$(8.9) \quad \lim_{h \rightarrow 0} \|n_h - n'\|_{C^{\frac{\varepsilon}{2m}}([t_0, \bar{t}]; X)}.$$

Now for $t \in [t_0, \bar{t}]$ we have

$$(8.10) \quad \begin{cases} \Lambda \int_{t_0}^t e^{(t-s)\Lambda}(n_h(s) - n'(s))ds \\ = \int_{t_0}^t \Lambda e^{(t-s)\Lambda}(n_h(s) - n'(s) - n_h(t) + n'(t))ds \\ \quad + (e^{(t-t_0)\Lambda} - 1)(n_h(t) - n'(t)) \end{cases}$$

and so from (8.7) we obtain

$$(8.11) \quad \begin{cases} \|u_h(t) - y(t)\|_X \leq M\{\|u_{0h} - u'(t_0)\|_X \\ \quad + (\bar{t} - t_0)\|f_h - f'\|_{B([t_0, \bar{t}]; X)} \\ \quad + \|A(n_h - n')\|_{B([t_0, \bar{t}]; X)}\} \\ \quad + 2m\varepsilon^{-1}M_1[n_h - n']_{C^{\frac{\varepsilon}{2m}}([t_0, \bar{t}]; X)} \\ \quad \cdot (\bar{t} - t_0)^{\frac{\varepsilon}{2m}} + (M + 1)\|n_h - n'\|_{B([t_0, \bar{t}]; X)} \end{cases}$$

where $M = \sup_{t_0 \leq t \leq t_1} \|e^{t\Lambda}\|_{L(X)}$ and $M_1 = \sup_{t_0 \leq t \leq t_1} \|t\Lambda e^{t\Lambda}\|_{L(X)}$.

This shows that $\lim_{h \rightarrow 0} u_h(t) = y(t)$ uniformly in $[t_0, \bar{t}]$: as \bar{t} is arbitrary in $]t_0, t_1[$ we have $u'(t) = y(t)$ for $t_0 \leq t < t_1$ and, by continuity, also for $t = t_1$. From (8.3) it follows that

$$(8.12) \quad \begin{cases} u & \in C^{2+\frac{\sigma}{2m}}([t_0, t_1], C(\bar{\Omega})), \\ u' & \in B([t_0, t_1]; C^{2m+\sigma}(\bar{\Omega})), \\ u'' & \in B([t_0, t_1]; C^\sigma(\bar{\Omega})). \end{cases}$$

Moreover, since $Au = u' - f = y - f$, from (8.3) we deduce also

$$(8.13) \quad Au \in B([t_0, t_1]; C^{2m+\sigma}(\bar{\Omega}))$$

and $B_j u = g_j \in B([t_0, t_1]; C^{4m+\sigma-m_j}(\Gamma))$: now from (iv) of theorem 5.2 we get $u \in B([t_0, t_1]; C^{4m+\sigma}(\bar{\Omega}))$; this is sufficient to conclude that $u \in C^{\frac{2m+\alpha}{2m}, 2m+\alpha}(Q)$ by virtue of theorem 2.2. Estimate (5.8) follows from (5.12) and (8.4), and the statement is proved for $k = 1$.

Assume now that the statement holds for $k = 0, 1, \dots, \bar{k} - 1$ with $\bar{k} \geq 2$: we will prove that it holds also for $k = \bar{k}$. We know that (8.2) is satisfied by $y = u'$ and (also by virtue of theorem 2.2) that

$$(8.14) \quad \begin{cases} f' \in C^{\bar{k}-1+\frac{\sigma}{2m}}([t_0, t_1]; C(\bar{\Omega})), \\ \frac{d^k f'}{dt^k} \in B([t_0, t_1]; C^{2m(\bar{k}-1-k)+\sigma}(\bar{\Omega})), \quad k = 1, \dots, \bar{k} - 1 \\ Au_0 + f(t_0) \in C^{2m\bar{k}+\sigma}(\bar{\Omega}) \\ g'_j \in C^{\bar{k}+\frac{\sigma-m_j}{2m}}([t_0, t_1]; C(\Gamma)), \\ \frac{d^k g'_j}{dt^k} \in B([t_0, t_1]; C^{2m(\bar{k}-k)+\sigma-m_j}(\Gamma)), \quad k = 1, \dots, [\bar{k} + \frac{\sigma-m_j}{2m}]. \end{cases}$$

Moreover, due to (5.3), the compatibility conditions are satisfied by f' , $Au_0 + f(t_0)$ and g'_j . By assumption the statement is true when $k = \bar{k} - 1$ and so $u' \in C^{\bar{k}+\frac{\sigma}{2m}}([t_0, t_1]; C(\bar{\Omega}))$ and $\frac{d^k u'}{dt^k} \in B([t_0, t_1]; C^{2m(\bar{k}-k)+\sigma}(\bar{\Omega}))$ for $k = 1, \dots, \bar{k}$,

so that $u \in C^{\bar{k}+1+\frac{\sigma}{2m}}([t_0, t_1]; C(\bar{\Omega}))$ and $u^{(k)} \in B([t_0, t_1]; C^{2m(\bar{k}+1-k)+\sigma}(\bar{\Omega}))$ for $k = 1, \dots, \bar{k}+1$. As in the case $k = 1$ one proves that $u \in B([t_0, t_1]; C^{2m(\bar{k}+1)+\sigma}(\bar{\Omega}))$. Estimate (5.8) with $\alpha = 2m\bar{k} + \sigma$ follows now easily, and the statement is proved for $k = \bar{k}$. \square

9 The time dependent coefficient case

In what follows Ω is again a bounded open set of \mathbb{R}^n with $C^{2m+\alpha}$ boundary Γ with $m \in \mathbb{N}$ and $\alpha > 0, \alpha \notin \mathbb{N}$. Given $T > 0$ we set $Q_0 = [0, T] \times \bar{\Omega}$ and $S_0 = [0, T] \times \Gamma$ and we consider the problem:

$$(9.1) \quad \begin{cases} D_t u(t, x) = \sum_{|\gamma| \leq 2m} a_\gamma(t, x) D_x^\gamma u(t, x) + f(t, x), & (t, x) \in Q_0 \\ u(0, x) = u_0(x), & x \in \bar{\Omega} \\ \sum_{|\beta| \leq m_j} b_{j\beta}(t, x) D_x^\beta u(t, x) = g_j(t, x), & j = 1, \dots, m, (t, x) \in S_0. \end{cases}$$

We assume that

$$(9.2) \quad \begin{cases} f, a_\gamma \in C^{\frac{\alpha}{2m}, \alpha}(Q_0) \text{ for each } \gamma \in \mathbb{N}^n \text{ with } |\gamma| \leq 2m, \\ g_j, b_{j\beta} \in C^{\frac{2m+\alpha-m_j}{2m}, 2m+\alpha-m_j}(S_0) \\ \text{for each } \beta \in \mathbb{N}^n \text{ with } |\beta| \leq m_j, j = 1, \dots, m, \\ u_0 \in C^{2m+\alpha}(\bar{\Omega}) \end{cases}$$

and moreover that

$$(9.3) \quad \begin{cases} \text{for each } t \in [0, T], a_\gamma(t, \cdot) \text{ satisfy (4.2)(b)(c)} \\ \text{and } b_{j\beta}(t, \cdot) \text{ satisfy (4.3)(b)(c) with } \eta_0 \text{ independent on } t. \end{cases}$$

Also in this case we look for solutions $u \in C^{\frac{2m+\alpha}{2m}, 2m+\alpha}(Q_0)$: this implies (see [S]) that the following compatibility conditions hold:

$$(9.4) \left\{ \begin{array}{l} \sum_{r=0}^k \binom{k}{r} \sum_{|\beta| \leq m_j} D_t^{k-r} b_{j\beta}(0, x) D_x^\beta u^{(r)}(x) = D_t^k g_j(0, x), \\ x \in \Gamma; k = 0, \dots, [\frac{2m+\alpha-m_j}{2m}]; j = 1, \dots, m \\ \text{where we have set for each } x \in \bar{\Omega} \\ u^{(0)}(x) = u_0(x) \\ u^{(k)}(x) = \sum_{s=0}^{k-1} \binom{k-1}{s} \sum_{|\gamma| \leq 2m} D_t^{k-1-s} a_\gamma(0, x) D_x^\gamma u^{(s)}(x) + D_t^{k-1} f(0, x), \\ k = 1, \dots, [\frac{2m+\alpha}{2m}]. \end{array} \right.$$

We have also

$$(9.5) \quad u^{(k)}(x) = D_t^k u(0, x), \quad k = 1, \dots, [\frac{2m+\alpha}{2m}], \quad x \in \bar{\Omega}.$$

In this section we will prove the following result:

Theorem 9.1 *Let assumptions (9.2)-(9.4) hold. Then problem (9.1) has a unique solution $u \in C^{\frac{2m+\alpha}{2m}, 2m+\alpha}(Q_0)$ and there exists a continuous and increasing function $\tilde{c} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ (depending also on bounds of the norms of $a_\gamma, b_{j\beta}$ and on Ω, η_0) such that*

$$(9.6) \quad \left\{ \begin{array}{l} \|u\|_{C^{\frac{2m+\alpha}{2m}, 2m+\alpha}(Q_0)} \\ \leq \tilde{c}(T) \{ \|f\|_{C^{\frac{\alpha}{2m}, \alpha}(Q_0)} + \|u_0\|_{C^{2m+\alpha}(\bar{\Omega})} \\ \quad + \sum_{j=1}^m \|g_j\|_{C^{\frac{2m+\alpha-m_j}{2m}, 2m+\alpha-m_j}(S_0)} \}. \end{array} \right.$$

Proof. We will prove the existence of $\delta > 0$ such that if $v_0 \in C^{2m+\alpha}(\bar{\Omega})$, $[t_0, t_1] \subseteq [0, T]$ and $t_1 - t_0 \leq \delta$, then problem

$$(9.7) \quad \left\{ \begin{array}{l} D_t v(t, x) = \sum_{|\gamma| \leq 2m} a_\gamma(t, x) D_x^\gamma v(t, x) + f(t, x), \\ (t, x) \in Q = [t_0, t_1] \times \bar{\Omega} \\ v(t_0, x) = v_0(x), \quad x \in \bar{\Omega} \\ \sum_{|\beta| \leq m_j} b_{j\beta}(t, x) D_x^\beta v(t, x) = g_j(t, x), \quad j = 1, \dots, m, \\ (t, x) \in S = [t_0, t_1] \times \Gamma \end{array} \right.$$

has a unique solution $v \in C^{\frac{2m+\alpha}{2m}, 2m+\alpha}(Q)$, depending on f, v_0 and g_j in the sense of estimate (9.6), provided the following compatibility conditions hold

$$(9.8) \quad \left\{ \begin{array}{l} \sum_{r=0}^k \binom{k}{r} \sum_{|\beta| \leq m_j} D_t^{k-r} b_{j\beta}(t_0, x) D_x^\beta v_r(x) = D_t^k g_j(t_0, x); \\ x \in \Gamma; \quad k = 0, \dots, \lfloor \frac{2m+\alpha-m_j}{2m} \rfloor; \quad j = 1, \dots, m \\ \text{where we have set for each } x \in \bar{\Omega} \\ v_k(x) = \sum_{s=0}^{k-1} \binom{k-1}{s} \sum_{|\gamma| \leq 2m} D_t^{k-1-s} a_\gamma(t_0, x) D_x^\gamma v_s(x) + D_t^{k-1} f(t_0, x); \\ k = 1, \dots, \lfloor \frac{2m+\alpha}{2m} \rfloor. \end{array} \right.$$

This will imply the statement because we can first choose $v_0 = u_0$ and $t_0 = 0$ and obtain a solution in $[0, \delta] \times \bar{\Omega}$: if $\delta < T$, taking $v_0 = u(\delta)$ and $t_0 = \delta$ conditions (9.8) are satisfied so that we can extend the solution to $([0, 2\delta] \cap [0, T]) \times \bar{\Omega}$: after a finite number of steps we obtain the conclusion. As in section 5 we write (9.7) in an abstract form

$$(9.9) \quad \left\{ \begin{array}{l} v'(t) = A(t)v(t) + f(t), \quad t \in [t_0, t_1] \\ v(t_0) = v_0 \\ B_j v(t) = g_j(t), \quad j = 1, \dots, m; \quad t \in [t_0, t_1] \end{array} \right.$$

where for each $t \in [t_0, t_1]$ and $j = 1, \dots, m$ we have set:

$$(9.10) \quad \begin{cases} A(t) = \sum_{|\gamma| \leq 2m} a_\gamma(t, \cdot) D_x^\gamma, & B_j(t) = \sum_{|\beta| \leq m_j} b_{j\beta}(t, \cdot) D_x^\beta \\ f(t) = f(t, \cdot), g_j(t) = g_j(t, \cdot), v(t) = v(t, \cdot). \end{cases}$$

We will solve (9.9) by a perturbation method by means of the existence results and the sharp estimates of the corresponding time independent case. Setting

$$(9.11) \quad \alpha = 2m\bar{k} + \sigma, \quad \bar{k} \in \mathbb{N}, \quad \sigma \in]0, 2m[$$

we list some consequences of assumptions (9.2).

There exist $\hat{a} > 0$ depending on Ω, T and the norms of a_γ such that:

$$(9.12) \quad \left\{ \begin{array}{l} \text{(i) } A(\cdot) \in C^{\frac{\alpha}{2m}}([0, T]; L(C^{2m}(\bar{\Omega}), C^0(\bar{\Omega}))) \text{ and} \\ \quad \|A\|_{C^{\frac{\alpha}{2m}}([0, T]; L(C^{2m}(\bar{\Omega}), C^0(\bar{\Omega})))} \leq \hat{a} \\ \text{(ii) for } k = 0, \dots, \bar{k} \text{ and } t \in [0, T], \\ \quad A^{(k)}(t) \in L(C^{\alpha+2m-2mk}(\bar{\Omega}); C^{\alpha-2mk}(\bar{\Omega})) \text{ and} \\ \quad \|[A^{(k)}(t) - A^{(k)}(r)]\varphi\|_{C^{\alpha-2mk}(\bar{\Omega})} \\ \quad \leq \hat{a} \{ \|\varphi\|_{C^{2m(\bar{k}-k+1)+[\sigma]}(\bar{\Omega})} + (t-r)^{\frac{\sigma}{2m}} [\varphi]_{C^{\alpha+2m-2mk}(\bar{\Omega})} \} \\ \quad \text{for each } \varphi \in C^{\alpha+2m-2mk}(\bar{\Omega}) \text{ and } 0 \leq r < t \leq T \\ \text{(iii) for } 0 \leq k \leq h \leq \bar{k} \text{ and } t \in [0, T], \\ \quad A^{(k)}(t) \in L(C^{\alpha+2m-2mh}(\bar{\Omega}), C^{\alpha-2mh}(\bar{\Omega})) \text{ and} \\ \quad \|A^{(k)}(t)\|_{L(C^{\alpha+2m-2mh}(\bar{\Omega}), C^{\alpha-2mh}(\bar{\Omega}))} \leq \hat{a} \end{array} \right.$$

and for each $j = 1, \dots, m$ there is $\hat{b}_j > 0$, depending on Ω, T and the norms of $b_{j\beta}$ such that

$$(9.13) \left\{ \begin{array}{l} \text{(i) } B_j(\cdot) \in C^{\frac{2m+\alpha-m_j}{2m}}([0, T]; L(C^{m_j}(\bar{\Omega}), C^0(\Gamma))) \text{ and} \\ \quad \|B_j\|_{C^{\frac{2m+\alpha-m_j}{2m}}([0, T]; L(C^{m_j}(\bar{\Omega}), C^0(\Gamma)))} \leq \hat{b}_j \\ \text{(ii) for } k = 0, \dots, [\frac{2m+\alpha-m_j}{2m}] = \bar{k} + 1 + [\frac{\sigma-m_j}{2m}], \\ \quad B_j^{(k)}(t) \in L(C^{\alpha+2m-2mk+m_j}(\bar{\Omega}), C^{\alpha+2m-2mk}(\Gamma)) \\ \quad \cap L(C^{\alpha-2m(k-1)}(\bar{\Omega}), C^{\alpha-2m(k-1)-m_j}(\Gamma)) \text{ and} \\ \quad \| [B_j^{(k)}(t) - B_j^{(k)}(r)] \varphi \|_{C^{\alpha-m_j-2m(k-1)}(\Gamma)} \\ \quad \leq \hat{b}_j \{ \|\varphi\|_{C^{2m(\bar{k}-k+1)+[\sigma]}(\bar{\Omega})} + (t-r)^{\frac{\sigma}{2m}} [\varphi]_{C^{\alpha-2m(k-1)}(\bar{\Omega})} \} \\ \quad \text{for each } \varphi \in C^{\alpha-2m(k-1)}(\bar{\Omega}) \text{ and } 0 \leq r < t \leq T \\ \text{(iii) for } 0 \leq k \leq h \leq [\frac{2m+\alpha-m_j}{2m}] = \bar{k} + 1 + [\frac{\sigma-m_j}{2m}] \text{ and } t \in [0, T], \\ \quad B_j^{(k)}(t) \in L(C^{\alpha+2m-2mh}(\bar{\Omega}), C^{\alpha+2m-2mh-m_j}(\Gamma)) \text{ and} \\ \quad \|B_j^{(k)}(t)\|_{L(C^{\alpha+2m-2mh}(\bar{\Omega}), C^{\alpha+2m-2mh-m_j}(\Gamma))} \leq b_j. \end{array} \right.$$

Let us set

$$(9.14) \left\{ \begin{array}{l} Y = \{v \in C^{\frac{2m+\alpha}{2m}}([t_0, t_1]; C(\bar{\Omega})), \\ \quad v^{(k)} \in B([t_0, t_1]; C^{2m+\alpha-2mk}(\bar{\Omega})), \\ \quad v^{(k)}(t_0) = v_k, \quad k = 0, \dots, \bar{k} + 1 \} \end{array} \right.$$

where v_k is defined in (9.8). For each $v \in Y$ let us consider the perturbed problem

$$(9.15) \left\{ \begin{array}{l} u'(t) = A(t)u(t) + [A(t) - A(t_0)]v(t) + f(t), \quad t \in [t_0, t_1] \\ u(t_0) = v_0 \\ B_j(t_0)u(t) = [B_j(t_0) - B_j(t)]v(t) + g_j(t), \quad t \in [t_0, t_1]. \end{array} \right.$$

Setting for $t \in [t_0, t_1]$:

$$(9.16) \left\{ \begin{array}{l} \varphi(t) = [A(t) - A(t_0)]v(t) + f(t) \\ \psi_j(t) = [B_j(t_0) - B_j(t)]v(t) + g_j(t), \quad j = 1, \dots, m \end{array} \right.$$

and recalling that for $j = 1, \dots, m$ we have

$$(9.17) \quad Y \subseteq C^{\frac{\alpha}{2m}}([t_0, t_1]; C^{2m}(\bar{\Omega})) \cap C^{\frac{2m+\alpha-m_j}{2m}, 2m+\alpha-m_j}([t_0, t_1]; C^{m_j}(\bar{\Omega}))$$

we get

$$(9.18) \quad \begin{cases} \varphi & \in C^{\frac{\alpha}{2m}}([t_0, t_1]; C^0(\bar{\Omega})), \\ \varphi^{(k)} & \in B([t_0, t_1]; C^{\alpha-2mk}(\bar{\Omega})) \text{ for } k = 0, \dots, \bar{k} \end{cases}$$

$$(9.19) \quad \begin{cases} \psi_j & \in C^{\frac{2m+\alpha-m_j}{2m}}([t_0, t_1]; C^0(\Gamma)), \\ \psi_j^{(k)} & \in B([t_0, t_1]; C^{\alpha-m_j-2m(k-1)}(\Gamma)) \\ & \text{for } k = 0, \dots, \lfloor \frac{1+\alpha-m_j}{2m} \rfloor \text{ and } j = 1, \dots, m. \end{cases}$$

Therefore the assumptions of theorem 4.1 are satisfied by the data φ, ψ_j, v_0 , because also the compatibility conditions (5.3) are verified: in fact we have

$$(9.20) \quad \begin{cases} \varphi^{(k)}(t_0) = \sum_{r=1}^k \binom{k}{r} A^{(k-r)}(t_0)v_r + f^{(k)}(t_0), & k = 1, \dots, \lfloor \frac{\alpha}{2m} \rfloor \\ \psi_j^{(k)}(t_0) = - \sum_{r=1}^k \binom{k}{r} B^{(k-r)}(t_0)v_r + g_j^{(k)}(t_0), & j = 1, \dots, m; \\ & k = 1, \dots, \lfloor \frac{2m+\alpha-m_j}{2m} \rfloor \end{cases}$$

so that setting

$$(9.21) \quad \begin{cases} u^{(0)} = v_0 \\ u^{(k)} = A(t_0)u^{(k-1)} + \varphi^{(k-1)}(t_0), & k = 1, \dots, \lfloor \frac{2m+\alpha}{2m} \rfloor \end{cases}$$

we get from (9.20) $u^{(k)} = v_k, k = 0, \dots, \lfloor \frac{2m+\alpha}{2m} \rfloor$, and so from (9.8) and (9.20) we deduce, for $j = 1, \dots, m$ and $k = 0, \dots, \lfloor \frac{2m+\alpha-m_j}{2m} \rfloor$:

$$B_j(t_0)u^{(k)} - \psi_j^{(k)}(t_0) = B_j(t_0)v_k + \sum_{r=1}^k \binom{k}{r} B_j^{k-r}(t_0)v_r - g_j^{(k)}(t_0) = 0$$

so that (5.3) holds. We are now able to apply theorem 5.1 and deduce the existence of a solution $u \in C^{\frac{2m+\alpha}{2m}, 2m+\alpha}(Q)$ of problem (9.15). Let us define

$$(9.22) \quad \begin{cases} \mathcal{T} : Y \rightarrow Y \\ \mathcal{T}v = u, \text{ where } u \text{ is the solution of (9.15)}. \end{cases}$$

We will prove that \mathcal{T} is a contraction on Y in the norm $||| \cdot |||_{C^{\frac{2m+\alpha}{2m}, 2m+\alpha}(Q)}$ provided $t_1 - t_0$ is sufficiently small. Let us set $||| \cdot |||_{C^{\frac{\alpha}{2m}, \alpha}(Q)} = ||| \cdot |||_{\alpha}$ and $||| \cdot |||_{C^{\frac{\alpha}{2m}, \alpha}(S)} = ||| \cdot |||_{\alpha}$.

First from (2.13) we get

$$(9.23) \quad \sum_{h=0}^{[2m+\alpha]} \|v\|_{C^{\frac{2m+\alpha-h}{2m}}([t_0, t_1]; C^h(\bar{\Omega}))} \leq c_2 |||v|||.$$

Now let $w_1, w_2 \in Y$, and set for $i = 1, 2$, $j = 1, \dots, m$ and $t \in [t_0, t_1]$:

$$(9.24) \quad \begin{cases} \varphi_i(t) &= [A(t) - A(t_0)]w_i(t) \\ \psi_{ij}(t) &= [B_j(t_0) - B_j(t)]w_i(t). \end{cases}$$

From estimate (5.8) we obtain

$$(9.25) \quad \begin{cases} |||\mathcal{T}w_1 - \mathcal{T}w_2|||_{2m+\alpha} \\ \leq c(T) \{ |||\varphi_1 - \varphi_2|||_{\alpha} + \sum_{j=1}^m |||\psi_{1j} - \psi_{2j}|||_{2m+\alpha-m_j} \}. \end{cases}$$

Recalling that

$$(9.26) \quad \begin{cases} |||\varphi|||_{\alpha} &= \sum_{k=0}^{\bar{k}} \|\varphi^{(k)}\|_{B([t_0, t_1]; C^{\alpha-2mk}(\bar{\Omega}))} \\ &+ [\varphi^{(k)}]_{C^{\frac{\alpha}{2m}}([t_0, t_1]; C^0(\bar{\Omega}))} \end{cases}$$

we consider first the case $\bar{k} = 0$ i.e. $\alpha < 2m$. From now on we denote by c a generic function independent on $[t_0, t_1] \subseteq [0, T]$ and on $w_1, w_2 \in Y$, and by $\rho : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ a continuous function with $\rho(0) = 0$.

From (9.12)(i)(ii), (9.13)(i)(ii), and (9.14)(i)(ii) we get

$$(9.27) \quad |||\varphi_1 - \varphi_2|||_\alpha \leq c \cdot \rho(t_1 - t_0) |||w_1 - w_2|||_{2m+\alpha}$$

$$(9.28) \quad |||\psi_{1j} - \psi_{2j}|||_{2m+\alpha-m_j} \leq c \cdot \rho(t_1 - t_0) |||w_1 - w_2|||_{2m+\alpha}, \quad j = 1, \dots, m$$

and so from (9.25) we deduce that for $t_1 - t_0$ small \mathcal{T} is a contraction; hence it has a unique fixed point $v \in Y$ which is the unique solution v of (9.9) such that $v \in C^{\frac{2m+\alpha}{2m}}([t_0, t_1]; C(\bar{\Omega}))$ and $v^{(k)} \in B([t_0, t_1]; C^{2m+\alpha-2mk}(\bar{\Omega}))$ for $k = 0, \dots, 1 + [\frac{\alpha}{2m}]$. By using again (iv) of theorem 5.2 we get $u \in C^{\frac{2m+\alpha}{2m}, 2m+\alpha}(Q)$, and the conclusion follows. \square

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