Generation of Centres by Adding Higher Order Terms in $y' = -\frac{x^{2n-1} + P(x,y)}{y^{2n-1} + Q(x,y)}$

Wolf von Wahl
Universität Bayreuth
Department of Mathematics
D-95440 Bayreuth, GERMANY

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Synopsis:
We systematically study the question how to convert a focus into a centre. This question was first raised by Frommer [1].

1 Introduction

Let $n \in \mathbb{N}$. $P(x,y), Q(x,y)$ are polynomials in $x, y$ starting with terms of order $2n$ at least. If

$$y' = -\frac{x^{2n-1} + P(x,y)}{y^{2n-1} + Q(x,y)} = -\frac{A(x,y)}{B(x,y)} \quad (1.1)$$

has a focus at the critical point $(0,0)$ it is sometimes possible to convert $(0,0)$ into a centre by adding higher order polynomials in the numerator and denominator. Frommer [1] was the first to study the influence of higher order terms on the question whether (1.1) can be made a centre or not. Our work is motivated by his contributions.

2 Systematic Approach

As announced we intend to convert a focus $y' = -\frac{A(x,y)}{B(x,y)}$ into a centre by replacing the preceding equation by $y' = -\frac{A(x,y) + Z_1(x,y)}{B(x,y) + Z_2(x,y)}$. The additional terms $Z_1(x,y), Z_2(x,y)$ vanish faster than $A(x,y), B(x,y)$ at $(0,0)$.

We are needing a Eulerian multiplier. Since such a multiplier does not vanish it is close by to try it with an expression $\mu = ce^p \ (c$ constant $\neq 0$, $\tilde{p}$ an appropriate function). We start with $n \in \mathbb{N},$

$$A(x,y) = x^{2n-1} + P(x,y), \quad (2.1)$$

$$B(x,y) = y^{2n-1} + Q(x,y), \quad (2.2)$$

$P, Q$ homogeneous polynomials of one and the same degree $p \geq n$. With still unknown polynomials $\tilde{q}, \tilde{p}$ we try the ansatz

$$\frac{x^{2n-1} + P + \frac{1}{2n}\tilde{q}x}{y^{2n-1} + Q + \frac{1}{2n}\tilde{q}y} = \frac{2n(x^{2n-1} + P + \frac{1}{2n}\tilde{q}x)}{2n(y^{2n-1} + Q + \frac{1}{2n}\tilde{q}y)}$$

$$= \frac{(2nx^{2n-1} + \tilde{q}x) + (x^{2n} + y^{2n} + \tilde{q})\tilde{p}x}{(2ny^{2n-1} + \tilde{q}y) + (x^{2n} + y^{2n} + \tilde{q})\tilde{p}y}$$

$$= \frac{\partial_x F}{\partial_y F} \text{ with } F = (x^{2n} + y^{2n} + \tilde{q})e^\tilde{p}, \quad \mu = 2ne^\tilde{p}. \quad (2.3)$$
If
\[ grad \tilde{q} \geq 2n+1 \] (2.4)
the level lines of \( F \) are closed and the origin is a centre for \( y' = -\frac{A+\frac{\sqrt{\mu}}{\sqrt{q}}}{B+\frac{\sqrt{\mu}}{\sqrt{q}}} \) provided
\[
\begin{align*}
2nP &= \tilde{q}_x + (x^{2n} + y^{2n})\tilde{p}_x, \\
2nQ &= \tilde{q}_y + (x^{2n} + y^{2n})\tilde{p}_y.
\end{align*}
\] (2.5)
(2.6)
The additional term in the denominator is \( \frac{1}{2n}\tilde{p}_y \) and in the numerator it is \( \frac{1}{2n}\tilde{p}_x \). Let us compare the degrees. We have
\[
\deg P = \deg Q = p \geq 2n.
\] (2.7)
Consequently
\[
\begin{align*}
deg \tilde{q} &= p + 1, \\
2n - 1 + deg \tilde{p} &= p, \ \deg \tilde{p} = p - (2n-1).
\end{align*}
\] (2.8)
(2.9)
For the given \( 2(p+1) \) coefficients of \( P,Q \) we have at our disposal \( 2p+2+2-2n = 2p+4-2n \) coefficients of \( \tilde{q} \) and \( \tilde{p} \). If the coefficient vectors of \( P,Q,\tilde{q},\tilde{p} \) are \( a,b,c,d \) respectively we arrive at a linear system
\[
\begin{pmatrix}
a \\ b
\end{pmatrix} = \mathcal{C} \begin{pmatrix} f \\ g \end{pmatrix}.
\] (2.10)
Here \( a,b \) have \( p+1 \) rows each. \( (a\ b)^T \) is a column, \( \mathcal{C} \) has \( 2(p+1) \) rows and \( 2(p+1) + 2-2n \) columns, \( c,d \) have \( p+2, p-(2n-1)+1 = p-2n+2 \) rows respectively and \( (c\ d)^T \) is a column. For \( n=1 \) the matrix \( \mathcal{C} \) is quadratic. At most in the case (2.10) has a solution \( (c\ d)^T \) for any right hand side \( (a\ b)^T \). For \( n \geq 2 \) the system (2.10) is overdetermined. \( \mathcal{C} \) has only nonnegative integer entries.

We achieve a considerable simplification if we exploit the structure of (2.5,6).

**Theorem 2.1** Let \( n \in \mathbb{N} \) and \( P,Q \) homogeneous polynomials in \( x,y \) of degree \( p \geq 2n \). Let \( \tilde{p} \) a homogeneous polynomial of degree \( p-(2n-1) \). Let
\[
y^{2n-1}\tilde{p}_x x^{2n-1}\tilde{p}_y = \tilde{p}_x - \tilde{p}_y.
\] (2.11)
Then there is a homogeneous polynomial \( \tilde{q} \) of degree \( p+1 \) such that (2.5,6) are valid. This means
\[
\begin{align*}
2nP &= \tilde{q}_x + (x^{2n} + y^{2n})\tilde{p}_x, \\
2nQ &= \tilde{q}_y + (x^{2n} + y^{2n})\tilde{p}_y.
\end{align*}
\] (2.12)
**Proof:** (2.11) implies that \( (2nP - (x^{2n} + y^{2n})\tilde{p}_x, 2nQ - (x^{2n} + y^{2n})\tilde{p}_y) \) is a gradient. Set for instance
\[
\begin{align*}
\partial_x f &= \sum_{\nu+\mu=p} \tilde{p}_{\mu}\cdot x^\mu y^{\nu} = 2nP - (x^{2n} + y^{2n})\tilde{p}_x, \\
\partial_y f &= \sum_{\nu+\mu=p} \tilde{q}_{\mu}\cdot x^\mu y^{\nu} = 2nQ - (x^{2n} + y^{2n})\tilde{p}_y.
\end{align*}
\]
Since we have on the right hand side the Taylor expansions for \( \partial_x f, \partial_y f \) around the origin we obtain that \( f \) is a polynomial with degree \( p+1 \). Employing principal functions in \( x,y \) respectively we get
\[
\begin{align*}
f &= \sum_{\mu+\nu=p+1, \mu \geq 1, \nu \geq 1} \tilde{p}_{\mu-1}\nu\cdot x^\mu y^{\nu} + \sum_{\mu+\nu=p+1, \mu \geq 1, \nu \geq 1} \tilde{q}_{\mu-1}\nu\cdot x^\mu y^{\nu} + \psi(y), \\
&= \sum_{\mu+\nu=p+1, \mu \geq 1, \nu \geq 1} \tilde{p}_{\mu-1}\nu\cdot x^\mu y^{\nu} + \sum_{\nu=\lambda+1, \nu \geq 1} \tilde{q}_{\nu-1}\mu\cdot y^{\nu} + \psi(x).
\end{align*}
\]
with \( n_{\mu-1,\nu} = \mu n_{\mu-1,\nu} \) for \( \mu + \nu = p + 1, \mu \geq 1, \nu \geq 1 \). Setting \( \varphi = \sum^I, \psi = \sum^I \) we arrive at

\[
\tilde{q} = f = \sum + \sum + \sum
\]

as the desired homogeneous polynomial of degree \( p + 1 \).

Evidently (2.12) implies (2.11). (2.11) furnishes a linear system for the \( p + 2 - 2n \) coefficients of \( \tilde{p} \). The right hand side \( \eta \) of this system consists of the \( p \) coefficients of \( P_y - Q_x \). We arrive at

\[
\mathcal{D} \mathbf{d} = \eta.
\]

\( \eta \) is a column with \( p \) rows, \( \mathcal{D} \) is a matrix with \( p \) rows and \( p + 2 - 2n \) columns. \( \mathcal{D} \) has only integer entries. We see that by (2.13) the Matrix \( \mathcal{C} \) in (2.10) is diminished. A detailed discussion of (2.11) can be found in [2].

### 3 Examples First Part

In our first example we deal with \( n = 1 \). Then \( \mathcal{D} \) in (2.13) is quadratic and we are interested in \( \det \mathcal{D} \).

Let \( p = 4 \), thus \( \tilde{p} \) has degree 3. Set

\[
\tilde{p}(x, y) = ax^3 + bx^2 y + cxy^2 + dy^3.
\]

Then

\[
y\tilde{p}_x - x\tilde{p}_y = (y(ax^3 + bx^2 y + cxy^2 + dy^3)_x - x(ax^3 + bx^2 y + cxy^2 + dy^3)_y,
\]

\[
= y(3ax^2 + 2bxy + cy^2) - x(3bx^2 + 2cxy + 3dy^2),
\]

\[
= 3ayx^2 + 2bxy^2 + cy^3 - bx^3 - 2cxy^2 - 3dxy^2,
\]

\[
= -bx^3 + (3a - 2c)x^2 y + (2b - 3d)xy^2 + cy^3.
\]

We can easily satisfy \( P_y - Q_x = y\tilde{p}_x - x\tilde{p}_y \) by choosing a suitable \( \tilde{p} \) if for any column \( \eta = (\alpha, \beta, \gamma, \delta)^T \) the system

\[
0 - b + 0 + 0 = \alpha,
\]

\[
3a + 0 - 2c + 0 = \beta,
\]

\[
0 + 2b + 0 - 3d = \gamma,
\]

\[
0 + 0 + c + 0 = \delta,
\]

which means \( \mathcal{D} \mathbf{d} = \eta \), is solvable in the unknowns \( \mathbf{d} = (a, b, c, d)^T \). Since

\[
\det \mathcal{D} = \det \begin{pmatrix}
0 & -1 & 0 & 0 \\
3 & 0 & -2 & 0 \\
0 & 2 & 0 & -3 \\
0 & 0 & 1 & 0
\end{pmatrix} = -\det \begin{pmatrix}
0 & -1 & 0 \\
3 & 0 & 0 \\
0 & 2 & -3
\end{pmatrix} = -\det \begin{pmatrix}
3 & 0 \\
0 & -3
\end{pmatrix} = 9
\]

this in fact the case. Our second example treats \( n = 1, p = 5 \), this is degree \( \tilde{p} = 4 \) and exhibits a characteristic difference between the cases "\( \tilde{p} \) has odd degree, this is \( p \) is even" and "\( \tilde{p} \) has even degree, this is \( p \) is odd " which has already been observed by Frommer [1]. The reason is that in case \( p \) odd the system (2.13) may not be solvable. For details cf. [2]. If it is solvable however then the first focal value \( d_1 \) in the expansion

\[
\det \begin{pmatrix}
F_x & F_y \\
\mathcal{A} & \mathcal{B}
\end{pmatrix} = \sum_{j=1}^{\infty} d_j(x^{2j+2} + y^{2j+2})
\]

(3.1)
vanishes. $F = x^2 + y^2 + f_2(x, y) + f_3(x, y) + \ldots$ is a formal power series whose construction goes back to Poincaré. The observation on the disappearance of $d_1$ was already made by Frommer [1, p. 406]. We obtain

$$y\tilde{p}_x - x\tilde{p}_y = y(ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4) -$$

$$-x(ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4)y,$$

$$= y(4ax^3 + 3bx^2y + 2cxy^2 + dy^3) -$$

$$-x(bx^3 + 2cx^2y + 3dxy^2 + 4ey^3),$$

$$= 4ax^3y + 3bx^2y^2 + 2cxy^3 + dy^4 -$$

$$-bx^4 - 2cx^3y - 3dxy^3 + 4xy^3,$$

$$= -bx^4 + (4a - 2c)x^3y + (3b - 3d)x^2y^2 + (2c - 4e)xy^3 + dy^4.$$

For arbitrary $h = (\alpha, \beta, \gamma, \delta, \epsilon)^T$ we consider the system $h = Dd$, this is

$$0 + -b + 0 + 0 + 0 = \alpha,$$

$$4a + 0 - 2c + 0 + 0 = \beta,$$

$$0 + 3b + 0 - 3d + 0 = \gamma,$$

$$0 + 0 + 2c + 0 - 4c = \delta,$$

$$0 + 0 + 0 + d + 0 = \epsilon$$

in the variables $d = (a, b, c, d, e)^T$. Its determinant vanishes since

$$-\frac{1}{2} (\text{first column of } D) - \frac{1}{2} (\text{fifth column of } D) = \text{third column of } D.$$ 

Since

$$\det D' = 12, D' = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 4 & 0 & -2 & 0 \\ 0 & 3 & 0 & -3 \\ 0 & 0 & 2 & 0 \end{pmatrix}$$

$D'$ has rank 4.

**Third example** Let $n = 2$. We consider

$$y' = -x^3 + P(x, y) + \frac{1}{y^3 + Q(x, y)}$$

with homogeneous polynomials $P, Q$ of degree 4 and look for a homogeneous polynomial $\tilde{p}$ of degree 1; this means

$$\tilde{p}(x, y) = ax + by$$

and

$$P_y - Q_x = ay^3 - bx^3.$$ 

In the simplest case we have

$$P(x, y) = \frac{a}{4}y^4, \quad Q(x, y) = \frac{b}{4}x^4$$

and there exists a homogeneous polynomial $\tilde{q}$ of degree 5 such that the origin is a centre for $y' = -\frac{x^3 + P(x, y) + \frac{1}{y^3 + Q(x, y)}}{y^3 + Q(x, y)}$. Interest in this statement could be increased by showing that the origin is a focus for $y' = -\frac{x^3 + P(x, y)}{y^3 + Q(x, y)}$. To discuss this question is more difficult than in the case $n = 1$. We need to find a
Now $r' = \frac{dr}{d\varphi} = r \frac{A(r \cos \varphi, r \sin \varphi)}{A(r \cos \varphi, r \sin \varphi)} \sin \varphi - B(r \cos \varphi, r \sin \varphi) \cos \varphi = \frac{Z(\varphi, r)}{N(\varphi, r)}$

Now $r' = \frac{Z}{\mathcal{N}}$ is compared with $r' = -\frac{\partial \varphi}{\partial \mathcal{F}}/\partial \mathcal{F}$ where $\mathcal{F}$ is a formal power series $\mathcal{F}(\varphi, r) = \sum_{\lambda \geq 1} f_\lambda(\varphi)r^\lambda$ in $r$ with coefficient functions $f_\lambda : [0, 2\pi] \to \mathbb{R}$. The result corresponds to (3.1) and reads as follows.

**Theorem 2:** There is a unique formal power series $\mathcal{F}(\varphi, r) = \sum_{\lambda \geq 2n} f_\lambda(\varphi)r^\lambda$ in $r$ with continuously differentiable $2\pi$-periodic functions $f_\lambda$, $f_\lambda(0) = 1$, and a unique sequence $c_{2n}, c_{2n+1}, \ldots$ such that

$$\det \begin{pmatrix} \partial_\varphi \mathcal{F} & \partial_r \mathcal{F} \\ -Z & \mathcal{N} \end{pmatrix} = \sum_{j=4n}^\infty c_j r^j$$

**Proof:** Set $Z(\varphi, r) = \sum_{\lambda \geq 2n} \mathcal{Z}_\lambda(\varphi)r^\lambda$, $\mathcal{N}(\varphi, r) = \sum_{\lambda \geq 2n-1} \mathcal{N}_\lambda(\varphi)r^\lambda$. If we compare the coefficients of the $r$-powers in

$$Z \partial_r \mathcal{F} = -N \partial_\varphi \mathcal{F} + \sum_{j=2n}^\infty c_j r^j$$

we arrive at

$$\sum_{\lambda \geq 2n} \mathcal{Z}_\lambda(\varphi)r^\lambda \sum_{\lambda \geq 2n} \lambda f_\lambda(\varphi)r^{\lambda-1} = \sum_{\lambda \geq 2n} \mathcal{Z}_\lambda(\varphi)r^\lambda \sum_{\lambda \geq 2n-1} (\lambda + 1)f_{\lambda+1}(\varphi)r^\lambda$$

$$= (\sum_{\lambda \geq 0} \mathcal{Z}_{\lambda+2n}(\varphi)r^\lambda)(\sum_{\lambda \geq 0} (\lambda + 2n)f_{\lambda+2n}(\varphi)r^\lambda)r^{n-1} - \sum_{\lambda \geq 2n-1} \mathcal{N}_\lambda(\varphi)r^\lambda \sum_{\lambda \geq 2n} f_\lambda'(\varphi)r^\lambda + \sum_{j=2n}^\infty c_j r^j$$

$$= -(\sum_{\lambda \geq 0} \mathcal{N}_{\lambda+2n-1}(\varphi)r^\lambda)(\sum_{\lambda \geq 0} f_{\lambda+2n}(\varphi)r^\lambda)r^{n-1} + \sum_{j=2n}^\infty c_j r^j$$

$$= -\sum_{\lambda \geq 0} \mathcal{N}_{\lambda+2n-1}(\varphi)f_{\lambda+2n}(\varphi)r^{\lambda+4n-1} + \sum_{\lambda \geq 0} c_{\lambda+4n-1} r^{\lambda+4n-1}$$

with $c_{2n}, c_{2n+1}, \ldots, c_{4n-2} = 0$,

$$\sum_{\lambda \geq 0} \mathcal{N}_{\lambda+2n-1}(\varphi)f_{\lambda+2n} + \sum_{\lambda \geq 0} \mathcal{Z}_{\lambda+2n}(\varphi)(\lambda + 2n)f_{\lambda+2n} = c_{\lambda+4n-1}$$

Now $\mathcal{N}_{2n-1}(\varphi) = \cos^{2n} \varphi + \sin^{2n} \varphi$ is positive definite and we arrive at

$$f_\lambda' + \frac{(\lambda + 2n)\mathcal{Z}_{2n}}{\mathcal{N}_{2n-1}(\varphi)} = f_{\lambda+2n} + \sum_{\lambda \geq 0} \frac{1}{\mathcal{N}_{2n-1}(\varphi)} (\mathcal{Z}_{\lambda+2n-1}(\varphi)(\lambda + 2n)f_{\lambda+2n} + \mathcal{N}_{\lambda+2n-1}(\varphi)f_{\lambda+2n}(\varphi))$$

(3.2)

for $\lambda \geq 1$ and

$$f_\lambda' + \frac{2n\mathcal{Z}_{2n}(\varphi)}{\mathcal{N}_{2n-1}(\varphi)} f_{2n} = c_{4n-1}$$

for $\lambda = 0$

(3.3)

Since $\mathcal{Z}_{2n}(\varphi) = \cos^{2n-1} \varphi \sin \varphi - \sin^{2n-1} \varphi \cos \varphi$ the coefficient $\mathcal{Z}_{2n}/\mathcal{N}_{2n-1}$ is $2\pi$-periodic. Moreover

$$\int_0^{2\pi} \frac{\mathcal{Z}_{2n}(\varphi)}{\mathcal{N}_{2n-1}(\varphi)} d\varphi = 0$$

(3.4)
\[ f'_{2n+1} + \frac{(1+2n)Z_{2n}(\varphi)}{N_{2n-1}(\varphi)} f_{2n} + \frac{1}{N_{2n-1}(\varphi)}(2nZ_{1+2n}f_{2n} + N_{1+2n}f'_{2n}) = c_{4n} \]  

(3.5)

If (3.5) has a $2\pi$-periodic solution then $c_{4n}$ is uniquely determined and every solution of (3.5) is $2\pi$-periodic. This follows from (3.4). In general the situation is as follows: If $h, f : \mathbb{R} \rightarrow \mathbb{R}$ are continuous and $2\pi$ periodic with $\int_0^{2\pi} h \, d\varphi = 0$ and if

\[ y' + hy + f = c, \quad c = \text{constant}, \]  

(3.6)

has a $2\pi$-periodic solution then

\[ c = \frac{\int_0^{2\pi} e^{iP} h \, d\varphi f \, d\varphi}{\int_0^{2\pi} e^{iP} h \, d\varphi} \]  

(3.7)

and every solution is $2\pi$-periodic. On the other hand, $c$ in (3.7) is the only constant such that every solution of $y' + hy + f = c$ is $2\pi$ periodic. In view of (3.2) it is now easy to prove the assertion by induction over $\lambda$. 

According to Frommer [1, p. 412] the constants $c_j$ play the rôle of the focal values $d_j$ in (1.1). Cf. also section 4 to follow. As for our example $y' = -\frac{x^3 + P(x,y)}{y + q(x,y)} = -\frac{x^3 + (4/5)y^4}{y + (6/4)y^4}$ we set

\[ P_3(x, y) = -x^3, \quad Q_3 = y^3, \quad P_4 = \sin \varphi P_3(\cos \varphi, \sin \varphi) + \cos \varphi Q_3(\cos \varphi, \sin \varphi), \]  

\[ P_4(x, y) = -\frac{a}{4} y^4, \quad Q_4 = \frac{b}{4} x^4, \quad P_5 = \sin \varphi P_4(\cos \varphi, \sin \varphi) + \cos \varphi Q_4(\cos \varphi, \sin \varphi), \]  

\[ q_4 = \cos \varphi P_3(\cos \varphi, \sin \varphi) - \sin \varphi Q_3(\cos \varphi, \sin \varphi), \]  

\[ q_5 = \cos \varphi P_4(\cos \varphi, \sin \varphi) - \sin \varphi Q_4(\cos \varphi, \sin \varphi). \]

Then

\[ Z(\varphi, r) = -r(\sin \varphi(-P_3)r^3 + \sin \varphi(-P_4)r^4) - r(\cos \varphi Q_3 r^3 + \sin \varphi Q_4 r^4), \]  

\[ = -r(-p_4r^3 - p_5r^5) = p_4r^4 + p_5r^5, \]  

\[ N(\varphi, r) = -(\cos \varphi(-P_3)r^3 + \cos \varphi(-P_4)r^4 + \cos \varphi Q_3 r^3 + \cos \varphi Q_4 r^4), \]  

\[ = -(q_4r^3 + q_5r^4) = q_4r^3 + q_5r^4. \]

\[ f'_{4} + \frac{4Z_4}{N_3} f_4 = 0, \quad f'_{5} + \frac{5Z_4}{N_4} f_5 + \frac{1}{N_3} (4Z_5 f_4 + N_4 f'_4) = c_{4n} \]  

\[ f'_{6} + \frac{6Z_4}{N_3} f_6 + \frac{1}{N_3} (5Z_5 f_5 + N_4 f'_5) = c_{4n+1}. \]

We intend to show that $c_{4n+1} \neq 0$ if $a, b$ are chosen appropriately. In terms of the trigonometric polynomials $p_i, q_i$ we have

\[ f'_{4} + \frac{4q_4}{q_4} f_4 = 0, \]  

(3.8)

\[ f'_{5} + \frac{5q_4}{q_4} f_5 + \frac{1}{q_4} (4p_5 f_4 + q_5 f'_4) = c_{4n}, \]  

(3.9)

\[ f'_{6} + \frac{6q_4}{q_4} f_6 + \frac{1}{q_4} (5p_5 f_5 + q_5 f'_5) = c_{4n+1}, \]  

(3.10)
We now turn to a sharpened version of Theorem 2. It is due to Frommer [1, p. 413].

Thus the origin is a focus.

\[ f_4 = \exp(-\int_0^\varphi (4p_4/q_4)d\psi)(f(0) = 1). \]

\( f_4 \) is even, \( p_4, q_4 \) have period \( \pi \), \( p_4/q_4 \) is odd. Then \( \int_{-\pi}^{\pi} (4p_4/q_4)d\psi = \int_0^{\pi} (4p_4/p_4)d\psi = 0 \) and \( f_4 \) is \( \pi \)-periodic. Since \( p_5, q_5 \) have degree 5 as polynomials in \( \sin \varphi, \cos \varphi \) we have \( p_5(\varphi + \pi) = -p_5(\varphi) \), \( q_5(\varphi + \pi) = -q_5(\varphi) \). Every solution to (3.9) with \( c_{4n} = 0 \) is \( 2\pi \)-periodic (cf. section 4). We thus remain with (3.10). It now turns out, after some tedious calculations, that \( c_{4n+1} = 0 \) for any choice of \( a, b \). As we will show in the next section we have a focus if there is a coefficient \( c_{\lambda+4n-1} \neq 0 \). If on the contrary all \( c_{\lambda+4n-1} \) vanish it should be conjectured that \( (0,0) \) is a center. The proof in [1] is not complete however since the lack of convergence of the \( F \)-series requires a more detailed discussion. Thus a decision if at \( (0,0) \) there is a focus in our particular example is not yet possible. We are going to take up this question in the next section.

4 Examples Second Part

If in Theorem 2 the first nonvanishing constant amongst \( c_{4n}, c_{4n+1}, \ldots \) is \( c_{\lambda_0+4n-1} \) for some \( \lambda_0 \geq 1 \) we obtain with \( F = \sum_{\mu=0}^{\lambda_0} f_{\mu+2n}r^{\mu+2n} \)

\[ r' - r_1' = \frac{Z}{N} + \frac{\partial_2 F}{\partial_2 F} = \frac{Z\partial_1 F + N\partial_2 F}{N\partial_1 F} = \frac{1}{N\partial_1 F} \left\{ c_{\lambda_0+4n-1}r^{\lambda_0+4n-1} + \sum_{\lambda \geq \lambda_0+1} \left( \sum_{n=0}^{\lambda_0} Z_{\lambda+2n-\kappa}(\kappa + 2n)f_{\kappa+2n} + N_{\lambda+2n-\kappa}(\kappa + 2n)f_{\kappa+2n} \right) r^{\lambda+4n} \right\} \]

\[ = \frac{1}{N\partial_1 F} (c_{\lambda_0+4n-1}r^{\lambda_0+4n-1} + O(r^{\lambda_0+4n})). \]

Since \( f_{2n}(\varphi) = e^{-\frac{f_0}{2} \varphi^{2n-1} + \varphi}, \partial_1 F = 2nf_{2n}r^{2n-1} + \ldots, N = N_{2n}r^{2n-1} \) the functions \( \partial_1 F, N \) have positive resp. negative definite lowest order coefficients and we obtain

\[ r' - r_1' = \frac{c_{\lambda_0+4n-1}}{2nf_{2n}N_{2n-1}} r^{\lambda_0+1} + \ldots \]

Thus the origin is a focus.

We now turn to a sharpened version of Theorem 2. It is due to Frommer [1, p. 413]. A remark on trigonometric polynomials

\[ p_l(\varphi) = \sum_{\alpha_1, \alpha_2} c_{\alpha_1, \alpha_2} \cos^{\alpha_1} \varphi \sin^{\alpha_2} \varphi, \text{ with } c_{\alpha_1, \alpha_2} \text{ constant}, \]

of degree \( l \) is in order. We have

\[ p_l(\varphi + \pi) = p_l(\varphi), \text{ if even, } p_l(\varphi + \pi) = -p_l(\varphi), \text{ if odd}. \quad (4.1) \]

Let \( l \) be odd, \( f, h : \mathbb{R} \to \mathbb{R} \) continuous and \( \pi \)-periodic with \( \int_0^\pi h d\psi = 0 \). Then every solution of \((*)g' + hy + p_l f = 0 \) is \( 2\pi \)-periodic. This is seen as follows: We have
$$y'(\varphi + \pi) + hy(\varphi + \pi) + p_1 f(\varphi + \pi) = 0,$$

$$y'(\varphi + \pi) + h(\varphi)y(\varphi + \pi) - p_1(\varphi)f(\varphi) = 0,$$

$$-y'(\varphi) - h(\varphi)y(\varphi) - p_1(\varphi)f(\varphi) = 0.$$  

Thus $y(\varphi + \pi) + y(\varphi)$ solves the homogeneous problem. We obtain

$$y(\varphi + \pi) + y(\varphi) = (y(\pi) + y(0))(\exp(-\int_{-\pi}^{\pi} h d\psi)),$$

$$y(\varphi + 2\pi) + y(\varphi + \pi) = (y(\pi) + y(0))(\exp(-\int_{-\pi}^{\pi} h d\psi)),$$

Thus this clearly implies $y(\varphi + 2\pi) = y(\varphi)$. Next we show that there is one and only one solution of (19) with initial value $y(-\frac{\pi}{2})$ is

$$y(\varphi) = y(-\frac{\pi}{2})\exp(-\int_{-\frac{\pi}{2}}^{\varphi} h d\psi) - \int_{-\frac{\pi}{2}}^{\varphi} \exp(-\int_{-\frac{\pi}{2}}^{\varphi} h d\psi)p_1 f d\tilde{\psi}.$$  

Thus the desired solution has initial value

$$y(-\frac{\pi}{2}) = (1 + \exp(-\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} h d\psi))^{-1} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \exp(-\int_{-\frac{\pi}{2}}^{\varphi} h d\psi)p_1 f d\tilde{\psi}.$$  

(4.2)

It is clearly uniquely determined by the requirement $y(\varphi + \pi) + y(\varphi) = 0$. Let $l$ be even. $f, h$ as above. Then $p_1 f(\varphi + \pi) = p_1 f(\varphi)$ and any $2\pi$-periodic solution of (19)$y' + hy + p_1 f = 0$ is $\pi$-periodic. Namely, we have for any solution $y$ the relations

$$y(\varphi) - y(\varphi + \pi)$$

is $\pi$-periodic, thus

$$y(\varphi + \pi) - y(\varphi + 2\pi) = y(\varphi) - y(\varphi + \pi)$$

whence by $y(\varphi) = y(\varphi + 2\pi)$ it follows

$$y(\varphi) = y(\varphi + \pi)$$

(19) holds correspondingly.

**Theorem 3:** There are a uniquely determined even $\Lambda \in \mathbb{N} \cap \{0, +\infty\}$, uniquely determined continuously differentiable functions $\hat{f}_{2n}, \ldots, \hat{f}_{2n+\Lambda-2}, \hat{f}_{2n+\Lambda-1}, \hat{f}_{2n+1}, \hat{f}_{2n+\Lambda+1}, \ldots: \mathbb{R} \rightarrow \mathbb{R}$ and uniquely determined numbers $\hat{a}_{4n-1} = 0, \ldots, \hat{a}_{4n+\Lambda-3} = 0, \hat{a}_{4n+\Lambda-2} = 0, \hat{a}_{4n+\Lambda} \neq 0, \hat{a}_{4n+\Lambda+1}, \ldots$ such that
\[ \hat{f}_{2n} \text{ is } \pi\text{-periodic}, \hat{f}_{2n}(-\frac{\pi}{2}) = 1 \]  
(4.3)

\[ \hat{f}_{2n+1}(\varphi + \pi) + \hat{f}_{2n+1}(\varphi) = 0, \hat{f}_{2n+1} \text{ is } 2\pi\text{-periodic}, \]  
(4.4)

\[ \vdots \]

\[ \hat{f}_{2n+\Lambda+2} \text{ is } \pi\text{-periodic}, \hat{f}_{2n+\Lambda+2}(-\frac{\pi}{2}) = 1, \]  
(4.5)

\[ \hat{f}_{2n+\Lambda-1}(\varphi + \pi) + \hat{f}_{2n+\Lambda-1}(\varphi) = 0, \hat{f}_{2n+\Lambda-1} \text{ is } 2\pi\text{-periodic}, \]  
(4.6)

\[ \hat{f}_{2n+\Lambda} \text{ is } 2\pi\text{-periodic with } \hat{d}_{4n+\Lambda-1} \neq 0, \hat{f}_{2n+\Lambda}(-\frac{\pi}{2}) = 1, \]  
(4.7)

\[ \hat{f}_{2n+\Lambda+j} \text{ is } 2\pi\text{-periodic with } \hat{f}_{2n+\Lambda+j}(-\frac{\pi}{2}) = 1, j \geq 1, \]  
(4.8)

the formal power series \( \hat{F}(\varphi, r) = \sum_{\lambda \geq 2n} \hat{f}_\lambda(\varphi) r^\lambda \) satisfies

\[ \det \left( \begin{array}{cc} \partial_\varphi \hat{F} & \partial_r \hat{F} \\ -Z & N \end{array} \right) = \sum_{j=4n+\Lambda-1}^{\infty} \hat{d}_j r^j \]  
(4.9)

**Proof:** We employ (3.2) with \( \hat{d}_{4n+\Lambda-1}, \hat{f}_{2n+\Lambda} \) instead of \( c_{4n+\Lambda-1}, f_{2n+\Lambda}, \mathcal{Z}_{\Lambda+2n-\kappa}, N_{\Lambda+2n-\kappa} \) are homogeneous polynomials in \( \cos \varphi \) and \( \sin \varphi \) of degree \( \lambda + 2n - \kappa \). Then \( \mathcal{R}(\varphi) = (N_{2n-1}(\varphi))^{-1} \mathcal{Z}_{\Lambda+2n-\kappa}(\varphi)(\kappa + 2n) \ldots \) in (3.2) has the following properties: Let \( \kappa = 0, \ldots, \lambda - 1 \). If

\[ \hat{f}_{2n+\kappa}(\varphi + \pi) + \hat{f}_{2n+\kappa}(\varphi) = 0, \kappa \text{ odd}, \]  
(4.10)

\[ \hat{f}_{2n+\kappa} \text{ is } \pi\text{-periodic, } \kappa \text{ even,} \]  
(4.11)

then for \( \lambda \) odd we have \( \mathcal{R}(\varphi + \pi) = \mathcal{R}(\varphi) = 0 \). Moreover there is one and only one constant \( \hat{d}_{\Lambda+4n-1} = c_{\Lambda+4n-1} \) such that every solution of (3.2) is \( 2\pi \)-periodic. This is in fact equivalent to (3.2) having one \( 2\pi \)-periodic solution. Cf. (3.7). \( \hat{d}_{\Lambda+4n-1} \) vanishes if and only if there is an \( \hat{f}_{2n+\lambda} \) with \( \hat{f}_{2n+\lambda}(\varphi + \pi) + \hat{f}_{2n+\lambda}(\varphi) = 0 \) and this particular one is uniquely determined. Now let \( \lambda \) be even. Then (4.10, 4.11) imply \( \mathcal{R}(\varphi + \pi) - \mathcal{R}(\varphi) = 0 \) and there is a uniquely determined constant \( \hat{d}_{\Lambda+4n-1} = c_{\Lambda+4n-1} \) such that every solution (equivalent: one solution) of (3.2) is \( 2\pi \)-periodic. As it is evident, any solution \( \hat{f}_{2n} \) of (3.3) is \( \pi \)-periodic and \( \hat{d}_{4n-1} = 0 \). Now let us consider (3.5). We have \( \mathcal{R}(\varphi + \pi) = \mathcal{R}(\varphi) = 0 \). Thus

\[ \int_0^{2\pi} e^{i\int_0^\varphi \frac{(1+2n)\mathcal{Z}_{2n-1}(\psi)}{N_{2n-1}(\psi)} d\psi} \mathcal{R}(\varphi) d\varphi = \int_0^\pi e^{i\int_0^\varphi \frac{(1+2n)\mathcal{Z}_{2n-1}(\psi)}{N_{2n-1}(\psi)} d\psi} \mathcal{R}(\varphi) d\varphi + \int_0^{2\pi} e^{i\int_0^\varphi \frac{(1+2n)\mathcal{Z}_{2n-1}(\psi)}{N_{2n-1}(\psi)} d\psi} \mathcal{R}(\varphi) d\varphi, \]

\[ = \int_0^\pi e^{i\int_0^\varphi \frac{(1+2n)\mathcal{Z}_{2n-1}(\psi)}{N_{2n-1}(\psi)} d\psi} \mathcal{R}(\varphi) d\varphi + \mathcal{R}(\varphi + \pi) d\varphi, \]

(4.12)

\[ = 0, \]  
(4.13)

Thus the first \( d_j \) which does not vanish has the form \( \hat{d}_{4n-\Lambda+1} \) with \( \Lambda \) even.

As in the beginning of the present section one can show that if there is a first \( \hat{d}_{4n+\Lambda-1} \neq 0 \) then the origin is a focus for \( y' = \frac{x^2 + ay}{y^2 + bx^2} \). Now we consider \( y' = \frac{x^2 + ay}{y^2 + bx^2} \). By some lengthy calculations we again end up with \( \hat{d}_{4n+\Lambda-1} = d_0 = 0 \). (0, 0) is however likely a focus, at least for appropriate values of \( a, b \). This can be seen from the computer-graphics to follow. They show the integral curves in the \( x, y \)-space for initial values \( (0, 2, 0), (0, 1, 0) \) and \( (0, 1, 0, 1) \).
Figure 1: \( y' = -\frac{x^3 + y^3}{y^3 + x^3} \) in \((0.2, 0)\)
Figure 2: $y' = -\frac{x^3 + y^3}{y^2 + x^2}$ in $(0.1, 0)$
Figure 3: $y' = -\frac{x^3 + y^4}{y^2 + x^2}$, in (0.1, 0.1)
References
