A semigroup approach to the time dependent parabolic initial-boundary value problem

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Abstract. In this paper a general initial-boundary value problem for a higher order linear parabolic equation with time dependent coefficients and non homogeneous data is studied by the abstract results obtained from a generalization of the classical theory of analytic semigroups of linear operators.

1 Introduction and notations

This paper gives a proof of existence, uniqueness, and optimal Hölder regularity of the solutions of the linear parabolic initial-boundary value problem of higher order:

\[
\begin{align*}
D_t u(t, x) &= \sum_{|\gamma| \leq 2m} a_\gamma(t, x) D_\gamma^2 u(t, x) + f(t, x), \\
(t, x) &\in Q = [0, T] \times \Omega \\
u(0, x) &= u_0(x), \quad x \in \Omega \\
\sum_{|\beta| \leq m_j} b_{j\beta}(t, x) D_\beta^2 u(t, x) &= g_j(t, x), \\
(t, x) &\in S = [0, T] \times \Gamma, \quad j = 1, \ldots, m.
\end{align*}
\]

Here $\Omega$ is a bounded open set in $\mathbb{R}^n$ with smooth boundary $\Gamma$. For every multi-index $\beta = (\beta_1, \cdots, \beta_n) \in \mathbb{N}^n$ we set $|\beta| = \beta_1 + \cdots + \beta_n$ and $D_\beta^2 u = D_{\beta_1}^2 \cdots D_{\beta_n}^2 u$. We assume that, for every $t \in [0, T]$, the operator

\[
A(t) = \sum_{|\gamma| \leq 2m} a_\gamma(t, \cdot) D_\gamma^2
\]

is elliptic and the boundary operators

\[
B_j(t) = \sum_{|\beta| \leq m_j} b_{j\beta}(t, \cdot) D_\beta^2, \quad j = 1, \ldots, m
\]

of order $m_j < 2m$ satisfy the usual requirements which guarantee the existence of the resolvent in a sufficiently large set of the complex plane (see section 2 for precise conditions). In [S] V. A. Solonnikov considered a larger
class of parabolic problems using the methods of potential theory. He showed in particular, the unique solvability of (1.1) in the Hölder classes \( C^{2m, \alpha}(Q) \) for any nonintegral \( \alpha > 0 \) if the \( A(t) \) are uniformly strongly elliptic and if the \( B_j \) fulfill conditions being very similar to ours.

If the leading coefficients of \( A(t) \) are real valued the conditions in [S] coincide with ours in the case of a single equation, which is considered here. The space \( C^{2m, \alpha}(Q) \) is the set of the \( \alpha \)-Hölder continuous functions with respect to the parabolic distance \( d((t, x), (s, y)) = |t - s|^{\frac{1}{2m}} + |x - y|, \ t, s \in [0, T]; x, y \in \Omega \). In section 1 we will show that a function \( u : [0, T] \times \bar{\Omega} \to \mathbb{C} \) belongs to \( C^{2m, \alpha}(Q) \) if and only if \( t \to u(t, \cdot) \) belongs to \( C^{2m}([0, T]; C^0(\bar{\Omega})) \) and \( t \to D_t^k u(t, \cdot) \) is bounded with values in \( C^{\alpha - 2mk}(\bar{\Omega}) \) for \( k = 0, \ldots, \lfloor \frac{\alpha}{2m} \rfloor \) (\( \lfloor r \rfloor \) denotes the integral part of the real number \( r \)). This result is important for the abstract approach to problem (1.1) that we will develop. In fact we reduce (1.1) to a problem for an ordinary differential equation in the Banach space \( X \) of continuous functions on \( \bar{\Omega} \), namely

\[
\begin{align*}
\begin{cases}
  u'(t) = A(t)u(t) + f(t), & t \in [0, T] \\
  u(0) = u_0 \\
  B_j(t)u(t) = g_j(t), & t \in [0, T], \ j = 1, \ldots, m
\end{cases}
\end{align*}
\]

(1.4)

in the unknown \( u : [0, T] \to X \), where we have set \( u(t) = u(t, \cdot), f(t) = f(t, \cdot) \) and \( g_j(t) = g_j(t, \cdot) \). For each \( t_0 \in [0, T] \) the linear operator

\[
\begin{align*}
\begin{cases}
  \Lambda : D(\Lambda) \subseteq X \to X \\
  D(\Lambda) = \{ \varphi \in \bigcap_{p \geq 1} W^{2m, p}(\Omega); A(t_0)\varphi \in X, \ B_j(t_0)\varphi = 0, & j = 1, \ldots, m \} \\
  \Lambda \varphi = A(t_0)\varphi, \ \varphi \in D(\Lambda)
\end{cases}
\end{align*}
\]

(1.5)

generates an analytic semigroup in \( X \) (not necessary strongly continuous at \( t = 0 \)) but the classical theory of semigroups cannot be directly used to solve (1.4) when \( g_j \neq 0 \) for some \( j = 1, \ldots, m \) or when \( B_j \) has order \( m_j = 0 \) for some \( j = 1, \ldots, m \): in this latter case \( D(\Lambda) \neq X \), so that we must employ
the results of the theory of analytic semigroups with non densely defined
generators (see [Si]). We study problem (1.4) in two steps: first we consider
the autonomous case in an arbitrary interval \([t_0, t_1]\) contained in \([0, T]\):

\[
\begin{align*}
  v'(t) &= A(t_0)v(t) + f(t), \quad t \in [t_0, t_1] \\
  v(t_0) &= v_0 \\
  B_j(t_0)v(t) &= g_j(t), \quad t \in [t_0, t_1], \quad j = 1, \ldots, m.
\end{align*}
\]

Then we use a perturbation method to solve (1.4) in a suitable interval \([0, \tau]\)
and by means of a sharp estimate of the solution we repeat the same pro-
cedure in \([\tau, 2\tau]\) so that we can extend our solution on the whole of \([0, T]\).
To overcome the above mentioned problem caused by the non homogeneous
boundary data \(g_j\) we adapt to our situation a device introduced by A. Bal-
akraishnan ([B]): we construct a function \(n : [0, T] \to X\) such that

\[
B_j(t_0)(n(t)) = g_j(t), \quad t \in [t_0, t_1], \quad j = 1, \ldots, m.
\]

If \(v\) is a solution of 1.6 and \(n\) is sufficiently regular then, \(w(t) = v(t) - n(t)\)
satisfies

\[
\begin{align*}
  w'(t) &= A(t_0)w(t) + A(t_0)n(t) - n'(t) + f(t), \quad t \in [t_0, t_1] \\
  w(t_0) &= v_0 - n(t_0) \\
  B_j(t_0)w(t) &= 0, \quad t \in [t_0, t_1]
\end{align*}
\]

so that \(w\) is given by the variation of constants formula

\[
\begin{align*}
  w(t) &= e^{(t-t_0)\Lambda} \left[ v_0 - n(t_0) \right] + \int_{t_0}^{t} e^{(t-s)\Lambda} \left[ A(t_0)n(s) - n'(s) + f(s) \right] ds,
\end{align*}
\]

\(t \in [t_0, t_1]\)

where \(e^{\cdot\Lambda}\) is the semigroup generated by \(\Lambda\) in \(X\). Integrating by parts we get
a representation formula for the solution of (1.6):

\[
\begin{align*}
  v(t) &= e^{(t-t_0)\Lambda} v_0 + \int_{t_0}^{t} e^{(t-s)\Lambda} [A(t_0)n(s) + f(s)] ds -
\end{align*}
\]
\[-\Lambda \int_{t_0}^{t} e^{(t-s)\Lambda} n(s) ds, \quad t \in [t_0, t_1]\]

which makes sense even if \( n \) is merely Hölder continuous. In fact we will choose \( n \) in such a way that \( v \) defined by (1.10) is the unique solution of (1.6) in the mentioned Hölder classes. In [L] for each \( t \in [t_0, t_1] \), \( n(t) = n(t, \cdot) \) is defined as the solution of the elliptic problem

\[
(1.11) \begin{cases}
    A(t_0) n(t, \cdot) = \omega n(t, \cdot) \\
    B_j(t_0) n(t, \cdot) = g_j(t), \quad j = 1, \ldots, m
\end{cases}
\]

where \( \omega \in \rho(\Lambda) \), the resolvent set of \( \Lambda \). But this choice is not useful in our case and thus we are led to construct explicitly a linear operator \( N \) such that

\[
(1.12) \begin{cases}
    \varphi_j \in C^{2m-m_j}(\Gamma), \quad j = 1, \ldots, m \Rightarrow N(\varphi_1, \ldots, \varphi_m) \in C^{2m}(\Omega) \\
    \varphi_j \in C^{2m+\alpha-m_j}(\Gamma), \quad j = 1, \ldots, m \Rightarrow N(\varphi_1, \ldots, \varphi_m) \in C^{2m+\alpha}(\Omega) \\
    B_j(t_0) N(\varphi_1, \ldots, \varphi_m) = \varphi_j, \quad j = 1, \ldots, m
\end{cases}
\]

and then set \( n(t) = N(\varphi_1(t), \ldots, \varphi_m(t)), \quad t \in [t_0, t_1] \).

In this paper we use the following notations: If \( X \) is a Banach space, \( k \in \mathbb{N}, \sigma \in [0, 1] \) then \( C^k([t_0, t_1]; X) \) is the space of the \( k \) times continuously differentiable functions from \([t_0, t_1]\) to \( X \) and \( C^{k+\sigma}([t_0, t_1]; X) \) is the space of functions whose \( k \)-th derivative is \( \sigma \)-Hölder continuous. These spaces are endowed with their usual norms. \( B([t_0, t_1]; X) \) denotes the space of all bounded functions from \([t_0, t_1]\) to \( X \) with the sup-norm. If \( \Omega \) is a bounded open set of \( \mathbb{R}^n \) and \( \alpha > 0, \alpha \notin \mathbb{N} \) then \( C^{\alpha}(\Omega) \) is the space of the \( \alpha \)-Hölder continuous functions from \( \Omega \) to \( C \) with the usual norm. If \( \Gamma \) is the boundary of \( \Omega \) we say that \( \Gamma \) is of class \( \alpha \) if there exist a finite number of closed balls \( B_1, \ldots, B_N \) such that \( \Gamma \subseteq \bigcup_{i=1}^{N} \hat{B}_i \) and a \( C^{\alpha} \)-diffeomorphism \( \varphi_i \) from \( B_i \) into the closed unit ball \( B \) of \( \mathbb{R}^n \) such that \( \varphi_i(B_i \cap \Omega) = B_+ = \{(y_1, \ldots, y_n) \in B; y_n > 0\} \) and \( \varphi_i(B_i \cap \Gamma) = \Sigma = \{(y_1, \ldots, y_n) \in B; y_n = 0\} \). The space \( C^{\beta}(\Gamma) \) with \( \beta \in [0, \alpha] \) is the set of all \( g : \Gamma \to C \) such that
\( g \circ \varphi^{-1}_i \in C^\beta(\Sigma) \) for \( i = 1, \cdots, N \). If \( \alpha \in ]0,1[ \) and \( u : \bar{\Omega} \to \mathbb{C} \) we set
\[
[u]_{C^\alpha(\bar{\Omega})} = \sup \{ |u(x) - u(y)| / |x - y|^\alpha ; \ x, y \in \bar{\Omega}, \ x \neq y \};
\]
an analogous definition is given for \([u]_{C^\alpha([t_0,t_1])}\).

The structure of the paper is the following: in section 2 we study the spaces \( C^{\alpha_m,\alpha}_2([0,T] \times \Omega) \) and in section 3 we prove some new maximal regularity results for abstract parabolic equations: they are the basic tools of our proofs. Sections 4, 5, 7, 8, concern the autonomous problem (1.8): they can be considered as the Hölder counterpart of [L] where several \( L^2 \) and \( H^\alpha \) regularity results are given for the function \( \nu \) defined by (1.10). In section 6 we construct the extension operator \( N \) from \( \Gamma \) to \( \bar{\Omega} \) satisfying (1.12). Finally in section 9 we treat the non autonomous problem: in this case the regularity theorem is obtained from the autonomous case by perturbation techniques and by sharp estimates. The homogeneous Dirichlet problem was treated by Sinestrari and von Wahl (see [W1]).

2 Hölder spaces in cylindrical domains

Let \( \Omega \) be a bounded set of \( \mathbb{R}^n \) with boundary \( \Gamma \) of class \( 2m+\alpha \), where \( m \in \mathbb{N} \) and \( \alpha > 0 \) is not integer. Setting \( Q = [t_0,t_1] \times \Omega \), we recall the following (see [S]).

**Definition 2.1** \( C^{\alpha_m,\alpha}_2(Q) \) is the Banach space of the functions \( u : Q \to \mathbb{C} \) such that:

\[
(2.1) \quad \text{There exist continuous } D_t^k D_x^\beta u \text{ in } Q \text{ if } 2mk + |\beta| \leq [\alpha].
\]

\[
(2.2) \quad I_1 = \sum_{\alpha-2m < 2mk + |\beta| \leq [\alpha]} \sup_{t \in [t_0,t_1]} [D_t^k D_x^\beta u(t,\cdot)]_{C^{\alpha-|\beta|}(\bar{\Omega})}
\]
\[
= \sum_{k=0} \sum_{|\beta|=|\alpha|-2mk} \sup_{t \in [t_0,t_1]} [D_t^k D_x^\beta u(t,\cdot)]_{C^{\alpha-|\beta|}(\bar{\Omega})} < \infty
\]

\[
(2.3) \quad I_2 = \sum_{\alpha-2m < 2mk + |\beta| \leq [\alpha]} \sup_{x \in \Omega} [D_t^k D_x^\beta u(\cdot, x)]_{C^{\alpha-|\beta|-2mk}(\bar{\Omega})}
\]
\[
= \sum_{0 \leq |\beta| \leq [\alpha]} \sup_{x \in \Omega} [D_t^k D_x^\beta u(\cdot, x)]_{C^{\alpha-|\beta|-|\beta|}(\bar{\Omega})} < \infty
\]
with the norm

\[(2.4) \quad \|u\|_{C^{\frac{\alpha}{2m}, \alpha}(Q)} = I_0 + I_1 + I_2\]

where

\[(2.5) \quad I_0 = \sum_{2mk + |\beta| \leq \lfloor \alpha \rfloor} \sup_{(t,x) \in Q} |D_t^k D_x^\beta u(t,x)|\]

\[= \sum_{k=0}^{[\frac{\alpha}{2m}]} \sum_{|\beta| \leq \lfloor \alpha \rfloor - 2mk} \sup_{(t,x) \in Q} |D_t^k D_x^\beta u(t,x)|\]

We prove now a result which gives a characterization of the space $C^{\frac{\alpha}{2m}, \alpha}(Q)$.

**Theorem 2.2** $u \in C^{\frac{\alpha}{2m}, \alpha}(Q)$ if and only if, setting $u(t, \cdot) = u(t)$ for $t \in [t_0, t_1]$, we have

\[(2.6) \quad u \in C^{\frac{\alpha}{2m}}([t_0, t_1]; C^0(\bar{\Omega}))\]

\[(2.7) \quad u^{(k)} \in B([t_0, t_1]; C^{\alpha - 2mk}(\bar{\Omega})), \quad k = 0, \ldots, [\frac{\alpha}{2m}]\]

Moreover defining

\[(2.8) \quad \|\|u\||_{C^{\frac{\alpha}{2m}, \alpha}(Q)} = J_1 + J_2\]

with

\[(2.9) \quad J_1 = \|u\|_{C^{\frac{\alpha}{2m}}([t_0, t_1]; C(\bar{\Omega}))}\]

\[(2.10) \quad J_2 = \sum_{k=0}^{[\frac{\alpha}{2m}]} \|u^{(k)}\|_{B([t_0, t_1]; C^{\alpha - 2mk}(\bar{\Omega}))}\]

there exists $c_1 = c_1(\alpha, m, \Omega)$ such that

\[(2.11) \quad c_1 \|u\|_{C^{\frac{\alpha}{2m}, \alpha}(Q)} \leq \|\|u\||_{C^{\frac{\alpha}{2m}, \alpha}(Q)} \leq 2\|u\|_{C^{\frac{\alpha}{2m}, \alpha}(Q)}\]
Finally if \( u \in C^{\frac{\alpha}{2m},\alpha}(Q) \) then

\[
(2.12) \quad u \in C^{\frac{\alpha-h}{2m},h}([t_0, t_1]; C^h(\Omega)), \quad h = 0, \cdots, [\alpha]
\]

and there exists \( c_2 = c_2(\alpha, m, \Omega) \) such that

\[
(2.13) \quad \sum_{h=0}^{[\alpha]} \| u \|_{C^{\frac{\alpha-h}{2m},h}([t_0, t_1]; C^h(\Omega))} \leq c_2 \| u \|_{C^{\frac{\alpha}{2m},\alpha}(Q)}.
\]

**Proof.** Conditions (2.1) and (2.2) are equivalent to

\[
(2.14) \quad u \in C^k([t_0, t_1]; C^{[\alpha]-2mk}(\Omega)), \quad k = 0, \cdots, \left[ \frac{\alpha}{2m} \right]
\]

\[
(2.15) \quad u^{(k)} \in B([t_0, t_1]; C^{\alpha-2mk}(\Omega)), \quad k = 0, \cdots, \left[ \frac{\alpha}{2m} \right].
\]

Moreover there exists \( c_{11} = c_{11}(\alpha, m) \) such that

\[
(2.16) \quad J_2 \leq I_0 + I_1 \leq c_{11}J_2.
\]

In addition conditions (2.1) and (2.3) imply that

\[
(2.17) \quad u \in C^{\frac{\alpha}{2m},0}([t_0, t_1]; C^0(\Omega))
\]

and we have

\[
(2.18) \quad J_1 \leq I_0 + I_2.
\]

In conclusion, if \( u \in C^{\frac{\alpha}{2m},\alpha}(Q) \) then (2.6) and (2.7) are true and

\[
(2.19) \quad \| u \|_{C^{\frac{\alpha}{2m},\alpha}(Q)} \leq 2\| u \|_{C^{\frac{\alpha}{2m},\alpha}(Q)}.
\]

Before proving the converse let us show that (2.6) and (2.7) imply (2.12): we can suppose \([\alpha] > 0\). From (2.7) we first deduce for \( h = 1, \cdots, [\alpha]\)
\[(2.20) \quad u(r) \in B([t_0, t_1]; C^h(\bar{\Omega})), \quad r = 0, \ldots, \lfloor \frac{\alpha - h}{2m} \rfloor \]

then setting

\[(2.21) \quad k = \lfloor \frac{\alpha - h}{2m} \rfloor, \quad \theta = \frac{\alpha - h}{2m} - \lfloor \frac{\alpha - h}{2m} \rfloor \]

we have

\[(2.22) \quad u(k) \in C^\theta([t_0, t_1]; C^h(\bar{\Omega})).\]

Actually, by using interpolatory estimates (see e.g. [ADN], p. 657, [W2], p. 255) we obtain when \(k = \lfloor \frac{\alpha}{2m} \rfloor\)

\[
(2.23) \quad \begin{cases} 
||u^{(k)}(t'') - u^{(k)}(t')||_{C^h(\bar{\Omega})} \\
\leq c_{12}||u^{(k)}(t'') - u^{(k)}(t')||_{C^{\frac{h}{\alpha} - \frac{2m}{2m} - k} (\bar{\Omega})}||u^{(k)}(t'') - u^{(k)}(t')||_{C^\theta(\bar{\Omega})}^{1 - \frac{h}{\alpha - \frac{2m}{2m}}} \\
\leq 2c_{12}(J_1 + J_2)||t'' - t'||^{\frac{h - \alpha}{2m} - k}; \quad t', t'' \in [t_0, t_1]
\end{cases}
\]

and when \(k < \lfloor \frac{\alpha}{2m} \rfloor\)

\[
(2.24) \quad \begin{cases} 
||u^{(k)}(t'') - u^{(k)}(t')||_{C^h(\bar{\Omega})} \\
\leq c_{13}||u^{(k)}(t'') - u^{(k)}(t')||_{C^{\frac{h - \alpha + k + 1}{2m} - \frac{2m}{2m}} (\bar{\Omega})} \\
\leq 2c_{13}J_1||t'' - t'||^{\frac{h - \alpha}{2m} - k}; \quad t', t'' \in [t_0, t_1]
\end{cases}
\]

where \(c_{12}\) and \(c_{13}\) depend on \(\alpha, m\) and \(\Omega\). Now (2.20) - (2.24) prove (2.12) and (2.13). But (2.12) implies (2.14) and (2.7) is (2.15); hence from what we have seen at the beginning of the proof we can say that (2.6) and (2.7) imply (2.1) and (2.2); but also (2.3) is verified because from (2.23) and (2.24) we deduce (setting \(\gamma = \frac{h - \alpha}{2m}\))

\[
(2.25) \quad I_2 \leq \sum_{h=0}^{[y]} \sup_{t', t'' \in [t_0, t_1]} \frac{||u^{[y]}(t'') - u^{[y]}(t')||_{C^h(\bar{\Omega})}}{||t'' - t'||^{\gamma - \theta}} \leq 2(c_{12} + c_{13})(J_1 + J_2) + J_2
\]
which (together with (2.16)) gives also the first of (2.11).

\[ \square \]

Remark: From definition 1.1 it follows that if \( u \in C_{2m}^{\alpha}(Q) \) and \( \gamma \in \mathbb{N}^n \) with \( 1 \leq |\gamma| \leq |\alpha| \) then \( D_\gamma^\alpha u \in C_{2m}^{\alpha'}(Q) \) with \( \alpha' = \alpha - |\gamma| \) and if \( h \in \mathbb{N} \) with \( 1 \leq h \leq \lfloor \frac{\alpha}{2m} \rfloor \) then \( D^h u \in C_{2m}^{\alpha''}(Q) \) with \( \alpha'' = \alpha - 2mh \).

In addition, from theorem 1.2 it follows that \( C_{2m}^{\alpha''}(Q) \hookrightarrow C_{2m}^{\alpha'}(Q) \) for \( \alpha' < \alpha'' \).

3 A maximal regularity result for abstract parabolic equations

Let \( X \) be a Banach space with norm \( || \cdot || \) and let \( \Lambda : D(\Lambda) \subseteq X \to X \) be a linear operator satisfying the following condition:

\[
\begin{cases}
\text{There exist } \omega \in \mathbb{R}, \eta \in ]\frac{\pi}{2}, \pi[ \text{ and } M > 0 \text{ such that if } \lambda \in \mathbb{C}, \\
\lambda \neq \omega \text{ and } \left| \arg(\lambda - \omega) \right| < \eta \text{ then } \lambda \text{ is in the resolvent} \\
\text{set of } \Lambda \text{ and } ||(\lambda - \omega)(\lambda - \Lambda)^{-1}||_{L(X)} \leq M.
\end{cases}
\]

(3.1)

In this case \( \Lambda \) generates an analytic semigroup \( e^{t\Lambda} \) which is not strongly continuous at \( t = 0 \) if \( D(\Lambda) \neq X \) (for analytic semigroups with non dense domain we refer to ([Si]).

A family of intermediate spaces between \( D(\Lambda) \) and \( X \) can be defined by

\[
D_\Lambda(\beta, \infty) = \left\{ x \in X; \left[ x \right]_\beta = \sup_{0 < t \leq 1} \left\| t^{1-\beta} \Lambda e^{t\Lambda} x \right\| < \infty \right\}, \ 0 < \beta < 1.
\]

(3.2)

They are Banach spaces under the norm

\[
\| x \|_{D_\Lambda(\beta, \infty)} = \| x \| + [x]_\beta
\]

(3.3)

and given \( k \in \mathbb{N} \) and \( \beta \in ]0, 1[ \) there exists a continuous and increasing function \( M_{k, \beta} : \mathbb{R}_+ \to \mathbb{R}_+ \) such that

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\[ ||t^{k-\beta} A^k e^{t\Lambda}||_{L^1(D_\Lambda(\beta,\infty),X)} \leq M_{k,\theta}(T), \quad 0 < t \leq T. \]

The spaces \( D_\Lambda(\beta, \infty) \) are necessary to study the maximal H"older regularity in time up to \( t = t_0 \) of the initial value problem

\[
\begin{cases}
  v'(t) = \Lambda v(t) + \varphi(t), & t \in [t_0, t_1] \\
  v(t_0) = v_0.
\end{cases}
\]

Here we recall first a sharp regularity result proved in [Si], Theorem 4.5 in the case \( \omega = 0 \): the case \( \omega \neq 0 \) is obtained by changing the unknown \( v(t) \) with \( e^{\omega t}v(t) \).

**Theorem 3.1** Let (3.1) hold and let \( \varphi \in C^\theta([t_0, t_1]; X) \), \( v_0 \in D(\Lambda) \), \( \Lambda v_0 + \varphi(t_0) \in D_\Lambda(\theta, \infty) \), \( 0 < \theta < 1 \). Then the unique solution of problem (3.5) is given by

\[
v(t) = e^{(t-t_0)\Lambda}v_0 + \int_{t_0}^{t} e^{(t-s)\Lambda}\varphi(s)ds, \quad t \in [t_0, t_1].
\]

Moreover \( v', \Lambda v \in C^\theta([t_0, t_1]; X) \), \( v' \in B([t_0, t_1]; D_\Lambda(\theta, \infty)) \) and there exists a continuous and increasing function \( c_3 : \mathbb{R}_+ \to \mathbb{R}_+ \) such that

\[
\begin{cases}
  ||v||_{C^{1+\theta}([t_0, t_1]; X)} + ||v'||_{B([t_0, t_1]; D_\Lambda(\theta, \infty))} \\
  \leq c_3(t_1 - t_0) \left\{ ||\varphi||_{C^\theta([t_0, t_1]; X)} + ||v_0|| + ||\Lambda v_0 + \varphi(t_0)||_{D_\Lambda(\theta, \infty)} \right\}.
\end{cases}
\]

The function \( c_3 \) depends also on \( M, \omega, \eta \) and \( \theta \) (see (3.1)).

By using this theorem we can prove a further result of maximal regularity under the assumption that \( \varphi \) has values in some intermediate space: this will be used in the application to the parabolic problem (1.1).

**Theorem 3.2** Let (3.1) hold and let \( \varphi \in C^\theta([t_0, t_1]; D_\Lambda(\beta, \infty)) \) with \( \theta, \beta \in [0,1] \). If \( \varphi(t_0) = v_0 = 0 \) then the solution \( v \) of (3.5) is such that

\[
v', \Lambda v \in C^\theta([t_0, t_1]; D_\Lambda(\beta, \infty)).
\]
If in addition $\theta + \beta \neq 1$ then we have also

\begin{equation}
(3.9) \quad \Lambda v \in C^{\theta+\beta}([t_0, t_1]; X).
\end{equation}

Moreover when $0 < \theta + \beta < 1$ we have

\begin{equation}
(3.10) \quad v' \in B([t_0, t_1]; D_\Lambda(\theta + \beta, \infty))
\end{equation}

and there exists a continuous and increasing function $c_4 : \mathbb{R}_+ \to \mathbb{R}_+$ (depending also on $M, \eta, \omega, \theta$ and $\beta$) such that

\begin{equation}
(3.11) \quad \begin{cases}
\|v\|_{C^{1+\theta}([t_0, t_1]; D_\Lambda(\beta, \infty))} + \|\Lambda v\|_{C^{\theta+\beta}([t_0, t_1]; X)} + \|v'\|_{B([t_0, t_1]; D_\Lambda(\theta+\beta, \infty))} \\
\leq c_4(t_1 - t_0)[\varphi]_{C^\theta([t_0, t_1]; D_\Lambda(\beta, \infty))}.
\end{cases}
\end{equation}

In the case $1 < \theta + \beta < 2$ there exists $(\Lambda v)'(t) = \Lambda v'(t)$ for $t \in [t_0, t_1]$ and

\begin{equation}
(3.12) \quad (\Lambda v)' \in B([t_0, t_1]; D_\Lambda(\theta + \beta - 1, \infty))
\end{equation}

and there exists a continuous and increasing function $c_5 : \mathbb{R}_+ \to \mathbb{R}_+$ (depending also on $M, \eta, \omega, \theta$ and $\beta$) such that:

\begin{equation}
(3.13) \quad \begin{cases}
\|v\|_{C^{1+\theta}([t_0, t_1]; D_\Lambda(\beta, \infty))} + \|\Lambda v\|_{C^{\theta+\beta}([t_0, t_1]; X)} \\
+\|(\Lambda v)'\|_{B([t_0, t_1]; D_\Lambda(\theta+\beta-1, \infty))} \\
\leq c_5(t_1 - t_0)[\varphi]_{C^\theta([t_0, t_1]; D_\Lambda(\beta, \infty))}.
\end{cases}
\end{equation}

Finally if $\theta + \beta \neq 1$ we have $v \in C^{\theta+\beta}([t_0, t_1]; D(\Lambda))$ (with $D(\Lambda)$ endowed with the graph norm).

Proof. For brevity we will write $C^\theta$ instead of $C^\theta([t_0, t_1]; D_\Lambda(\beta, \infty))$ and we will denote by $c$ a generic function with the same properities as $c_4$ and $c_5$ in the statement above. Setting $D(A) = \{x \in D(\Lambda); \Lambda x \in D_\Lambda(\beta, \infty)\}$ and $\Lambda x = \Delta x$ for $x \in D(A)$, the operator $A : D(A) \subseteq D_\Lambda(\beta, \infty) \to D_\Lambda(\beta, \infty)$ satisfies (3.1) (see proposition 1.10 of [Sil]). Applying the previous theorem we find that (3.5) with $v_0 = 0$ has a unique solution $v$ given by (3.6) (with $v_0 = 0$) and (3.8) holds. Moreover from (3.7) we obtain
\begin{align}
(3.14) \quad \|v\|_{C^{1+\theta}([t_0,t_1]; D_A(\beta,\infty))} & \leq c[\varphi]_{C^\theta}.
\end{align}

Suppose now $0 < \theta + \beta < 1$. From (4.1) of [Si] we get

\begin{align}
(3.15) \quad \Lambda v(t) &= \Lambda \int_{t_0}^{t} e^{(t-s)A}[\varphi(s) - \varphi(t)] ds + [e^{(t-t_0)A} - I]\varphi(t), \quad t \in [t_0, t_1]
\end{align}

and so when $t_0 \leq r \leq t \leq t_1$

\begin{align}
\begin{cases}
\Lambda v(t) - \Lambda v(r) \\
= \Lambda \int_{t_0}^{r} e^{(t-s)A} - e^{(r-s)A}[\varphi(s) - \varphi(r)] ds + \Lambda \int_{r}^{t} e^{(r-s)A}[\varphi(s) - \varphi(t)] ds + [e^{(t-t_0)A} - e^{(r-t_0)A}] \varphi(r) + [I - e^{(t-r)A}] [\varphi(r) - \varphi(t)].
\end{cases}
\end{align}

By using (3.4) we have for $t_0 \leq s \leq r \leq t \leq t_1$:

\begin{align}
(3.17) \quad \begin{cases}
\|\Lambda [e^{(t-s)A} - e^{(r-s)A}][\varphi(s) - \varphi(r)]\| \\
= \|\Lambda \int_{r}^{t} \Lambda e^{\xi A} [\varphi(s) - \varphi(r)] d\xi\| \\
\leq c[\varphi]_{C^\theta} \int_{r}^{t} (r - s)^{\theta} \xi^{\beta - 2} d\xi.
\end{cases}
\end{align}

Moreover, as $\varphi(t_0) = 0$:

\begin{align}
(3.18) \quad \begin{cases}
\|[e^{(t-t_0)A} - e^{(r-t_0)A}] \varphi(r)\| = \|\int_{r_0}^{t_0} \Lambda e^{\xi A} [\varphi(r) - \varphi(t_0)] d\xi\| \\
\leq c[\varphi]_{C^\theta} \int_{r_0}^{t_0} (r - t_0)^{\theta} \xi^{\beta - 1} d\xi
\end{cases}
\end{align}

and

\begin{align}
(3.19) \quad \begin{cases}
\|[I - e^{(t-r)A}] [\varphi(r) - \varphi(t)]\| = \|\int_{0}^{r} \Lambda e^{\xi A} [\varphi(r) - \varphi(t)] d\xi\| \\
\leq c[\varphi]_{C^\theta} \int_{0}^{r} \xi^{\beta - 1} d\xi.
\end{cases}
\end{align}
Now we can deduce from (3.16) for $t_0 \leq r \leq t \leq t_1$:

\[
(3.20) \begin{cases}
||\Lambda v(t) - \Lambda v(r)|| \\
\leq c[\varphi]c^o \{ \int_{t_0}^{r} (r-s)^\theta ds \int_{s}^{t} \xi^{\beta-2} d\xi + \int_{t}^{r} (t-s)^{\theta+\beta-1} ds + \\
(r-t_0)^{\theta} \int_{r-t_0}^{t} \xi^{\beta-1} d\xi + (t-r)^{\theta} \int_{0}^{t} \xi^{\beta-1} d\xi \}
\end{cases}
\]

As $\beta + \theta < 1$ we get

\[
(3.21) \begin{cases}
\int_{t_0}^{r} (r-s)^\theta ds \int_{s}^{t} \xi^{\beta-2} d\xi \\
\leq \int_{t_0}^{r} ds \int_{s}^{t} \xi^{\beta+\theta-2} d\xi \\
= (1 - \beta - \theta)^{-1}(\beta + \theta)^{-1}[(t-r)^{\beta+\theta} - (t-t_0)^{\beta+\theta} + (r-t_0)^{\beta+\theta}] \\
\leq (1 - \beta - \theta)^{-1}(\beta + \theta)^{-1}(t-r)^{\beta+\theta}
\end{cases}
\]

and also

\[
(3.22) \begin{cases}
(r-t_0)^{\theta} \int_{r-t_0}^{t} \xi^{\beta-1} d\xi \\
\leq \int_{r-t_0}^{t} \xi^{\beta+\beta-1} d\xi = (\theta + \beta)^{-1}[(t-t_0)^{\beta+\theta} - (r-t_0)^{\beta+\theta}] \\
\leq (\theta + \beta)^{-1}(t-r)^{\beta+\theta}
\end{cases}
\]

and so from (3.20) we get

\[
||\Lambda v(t) - \Lambda v(r)|| \leq c[\varphi]c^o(t-r)^{\theta+\beta}, \ t_0 \leq r \leq t \leq t_1
\]

hence $\Lambda v \in C^{\theta+\beta}([t_0, t_1]; X)$, i.e. (3.9) holds. Moreover

\[
(3.23) \ [\Lambda v]_{C^{\theta+\beta}([t_0, t_1]; X)} \leq c[\varphi]c^o.
\]

From the previous theorem we deduce
(3.24) \[ \|v'\|_{B([t_0,t_1]; D_A(\vartheta, \infty))} \leq c[\varphi]_{C^\theta} \]

and by using (3.2) and (3.4) we get for \( t \in [t_0, t_1] \)

\[
\begin{aligned}
\|v'(t)\|_{D_A(\vartheta + \beta, \infty)} &= \|v'(t)\| + \sup_{0 < s \leq 1} \|s^{1-\beta} \Lambda e^{sA} v'(t)\|
\leq \|v'(t)\| + c \sup_{0 < s \leq 1} \|s^{1-\theta} \Lambda e^{sA} v'(t)\|_{D_A(\beta, \infty)}
\leq c\|v'(t)\|_{D_A(\vartheta, \infty)}.
\end{aligned}
\]

(3.25)

Hence from (3.23) we obtain (3.10) and

(3.26) \[ \|v'\|_{B([t_0,t_1]; D_A(\vartheta + \beta, \infty))} \leq c[\varphi]_{C^\theta}. \]

To prove (3.11) we will use (3.15) and (3.4) to get a first estimate which holds also when \( \theta + \beta > 1 \):

\[
\begin{aligned}
\|\Lambda v(t)\| &\leq \int_{t_0}^{t} \|\Lambda e^{(t-s)A}\|_{L(D_A(\beta, \infty), X)} \|\varphi(s) - \varphi(t)\|_{D_A(\beta, \infty)} ds + \\| \int_{t_0}^{t} \Lambda e^{sA} [\varphi(t) - \varphi(t_0)] d\xi \| \\
&\leq c \int_{t_0}^{t}(t-s)^{\theta+\beta-1} ds + \int_{t_0}^{t} (t-t_0)^{\theta} \xi^{\beta-1} d\xi [\varphi]_{C^\theta} \\
&\leq c[(\theta + \beta)^{-1} + \beta^{-1}](t_1 - t_0)^{\theta+\beta}[\varphi]_{C^\theta}.
\end{aligned}
\]

(3.27)

Hence by using also (3.23) we obtain

(3.28) \[ \|\Lambda v\|_{C^{\theta+\beta}[t_0, t_1]; X} \leq c[\varphi]_{C^\theta}. \]

Now (3.11) is a consequence of (3.14), (3.26), and (3.28).

Let us examine the case \( 1 < \theta + \beta < 2 \), setting \( \gamma = \theta + \beta \). From (4.2) of [Si] we have

\[
(3.29) \quad v'(t) = \int_{t_0}^{t} \Lambda e^{(t-s)A} [\varphi(s) - \varphi(t)] ds + e^{(t-t_0)A} \varphi(t), \quad t \in [t_0, t_1]
\]

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hence by using (3.4) we obtain for \( t \in [t_0, t_1] \):

\[
\|\Lambda v'(t)\| = \left\| \int_{t_0}^{t} \Lambda^2 e^{(t-s)\Lambda} [\varphi(s) - \varphi(t)] ds \right. \\
+ \Lambda e^{(t-t_0)\Lambda} [\varphi(t) - \varphi(t_0)] \right\| \\
\leq \int_{t_0}^{t} \|\Lambda^2 e^{(t-s)\Lambda}\|_{L(D_\Lambda(\beta,\infty),\mathcal{X})} \|\varphi(s) - \varphi(t)\|_{D_\Lambda(\beta,\infty)} ds \\
+ \|\Lambda e^{(t-t_0)\Lambda}\|_{L(D_\Lambda(\beta,\infty),\mathcal{X})} \|\varphi(t) - \varphi(t_0)\|_{D_\Lambda(\beta,\infty)} \\
\leq c[\varphi] c_\sigma \left[ \int_{t_0}^{t} (t-s)^{\gamma-2} ds + (t-t_0)^{\gamma-1} \right] \leq c[\varphi] c_\sigma
\]

whereas for \( t_0 \leq s \leq t \leq t_1 \) we have

\[
\Lambda v'(t) - \Lambda v'(s) = \left\| \int_{t_0}^{s} \Lambda^2 [e^{(t-s)\Lambda} - e^{(s-\sigma)\Lambda}] [\varphi(\sigma) - \varphi(s)] d\sigma \right. \\
+ \left. \int_{s}^{t} \Lambda^2 e^{(t-\sigma)\Lambda} [\varphi(\sigma) - \varphi(t)] d\sigma \right. \\
+ \left. \Lambda [e^{(t-t_0)\Lambda} - e^{(s-t_0)\Lambda}] \varphi(s) \right. \\
+ \left. \Lambda e^{(t-s)\Lambda} [\varphi(t) - \varphi(s)] \right\|. 
\]

Now from (3.4) we get

\[
\left\| \int_{t_0}^{s} \Lambda^2 [e^{(t-s)\Lambda} - e^{(s-\sigma)\Lambda}] [\varphi(\sigma) - \varphi(s)] d\sigma \right\| \\
= \left\| \int_{t_0}^{s} \Lambda \int_{t_0}^{t} e^{\tau\Lambda} [\varphi(\sigma) - \varphi(s)] d\tau d\sigma \right\| \\
\leq \int_{t_0}^{s} d\sigma \int_{t_0}^{t} \|\Lambda^3 e^{\tau\Lambda}\|_{L(D_\Lambda(\beta,\infty),\mathcal{X})} \|\varphi(\sigma) - \varphi(s)\|_{D_\Lambda(\beta,\infty)} d\tau \\
\leq c[\varphi] c_\sigma \int_{t_0}^{s} (s-\sigma)^{\theta} d\sigma \int_{t_0}^{t} \tau^{\beta-3} d\tau \\
\leq c[\varphi] c_\sigma \int_{t_0}^{s} d\sigma \int_{s}^{t} \tau^{\gamma-3} d\tau \\
= c[\varphi] c_\sigma (2-\gamma)^{-1} (\gamma-1)^{-1} [((t-s)^{\gamma-1} - (t-t_0)^{\gamma-1} + (s-t_0)^{\gamma-1})] \\
\leq c[\varphi] c_\sigma (2-\gamma)^{-1} (\gamma-1)^{-1} (t-s)^{\gamma-1}
\]
and also

\[
\left\| \int_s^t \Lambda^2 e^{(t-s)\Lambda} \varphi(\sigma) - \varphi(t) \right\| d\sigma \leq c[\varphi]_{C^\delta} \int_s^t (t-\sigma)^{\gamma-2} d\sigma \\
= c(2-\gamma)^{-1} [\varphi]_{C^\delta} (t-s)^{\gamma-1}
\]

\[
\left\| \Lambda e^{(t-t_0)\Lambda} - e^{(s-t_0)\Lambda} \varphi(s) \right\| \\
\leq \left\| \int_s^{t-t_0} \Lambda^2 e^{\sigma\Lambda} (\varphi(s) - \varphi(t_0)) d\sigma \right\| \\
\leq c[\varphi]_{C^\delta} \int_s^{t-t_0} \sigma^{\delta-2} (s-t_0)^{\delta} d\sigma \\
\leq c[\varphi]_{C^\delta} \int_s^{t-t_0} \sigma^{\gamma-2} d\sigma \\
\leq c[\varphi]_{C^\delta} (\gamma-1)^{-1} (t-s)^{\gamma-1}
\]

\[
\left\| \Lambda e^{(t-t)\Lambda} (\varphi(t) - \varphi(s)) \right\| \leq \left\| \Lambda e^{(t-s)\Lambda} \right\|_{L(D_{\Lambda}(\beta,\infty);X)} [\varphi]_{C^\delta} (t-s)^{\delta} \\
\leq c[\varphi]_{C^\delta} (t-s)^{\gamma-1}.
\]

From (3.31), by virtue of (3.32) - (3.35), we obtain

\[(3.36) \quad \Lambda v' \in C^{\gamma-1}([t_0,t_1];X).\]

But for \( t, t+h \in [t_0,t_1], \; h \neq 0 \) we have:

\[
(3.37) \quad \left\| \frac{\Lambda v(t+h) - \Lambda v(t)}{h} - \Lambda v'(t) \right\| = \frac{1}{h} \int_t^{t+h} (\Lambda v'(s) - \Lambda v'(t)) ds
\]

and so the continuity of \( \Lambda v' \) implies that there exists

\[(3.38) \quad (\Lambda v)'(t) = \Lambda v'(t), \; t \in [t_0,t_1].\]

Then from (3.36) we deduce that \( \Lambda v \in C^\gamma([t_0,t_1];X) \), i.e. (3.9) holds; by using (3.27), (3.30), (3.31) and (3.32) - (3.35) we obtain also
\[(3.39) \quad \|\Lambda v\|_{C^{\theta+\beta}([t_0,t_1];X)} \leq c[\varphi]_{C^\theta}.
\]

Let us prove (3.12). For \(t \in [t_0, t_1]\) and \(\xi \in ]0, 1[\) we get from (3.29) and (3.4)

\[
\begin{aligned}
&\|\xi^{2-\gamma} \Lambda e^{\xi^\Lambda} \Lambda v'(t)\|
\leq c[\varphi]_{C^\theta} [\xi^{2-\gamma} \int_{t_0}^t (t-s)^{\theta}(t-s+\xi)^{\beta-3} ds \\
&\quad + \xi^{2-\gamma}(t-t_0 + \xi)^{\beta-2}(t-t_0)^{\theta}]
\leq c[\varphi]_{C^\theta} [(2-\gamma)^{-1} [1 - (\frac{t-t_0 + \xi}{t-t_0 + \xi})^{2-\gamma} ] + (\frac{t-t_0 + \xi}{t-t_0 + \xi})^{2-\gamma}]
\leq c[\varphi]_{C^\theta} [(2-\gamma)^{-1} + 1].
\end{aligned}
\]

This proves that \(\Lambda v' \in B([t_0, t_1];D_\Lambda(\gamma-1, \infty)),\) i.e. (3.12) holds. Moreover from (3.30) and (3.40) we obtain

\[
(3.41) \quad \|\Lambda v'\|_{B([t_0, t_1];D_\Lambda(\gamma-1, \infty))} = \sup_{t \in [t_0, t_1]} [\|\Lambda v'(t)\| + \sup_{0 < \xi \leq 1} \|\xi^{2-\gamma} \Lambda e^{\xi^\Lambda} \Lambda v'(t)\|]
\leq c[\varphi]_{C^\theta}
\]

and also estimate (3.13) as a consequence of (3.14), (3.39), and (3.41). The last assertion of the theorem follows from the fact that (3.8) implies \(v \in C^{\theta+\beta}([t_0, t_1];X)\) and \(\Lambda v \in C^{\theta+\beta}([t_0, t_1];X)\) (see (3.9)) when \(\theta + \beta \neq 1.\) \(\square\)

4 The case of time independent coefficients

Let \(\Omega\) be a bounded open set in \(\mathbb{R}^n\) with \(C^{2m+\alpha}\) boundary \(\Gamma\) where \(m, \alpha > 0\) with \(m \in \mathbb{N}\) and \(\alpha \notin \mathbb{N}\). For each \(x \in \Gamma,\) we denote by \(\nu(x) =\)

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$(\nu_1(x), \cdots, \nu_n(x))$ the unit exterior normal vector to $\Gamma$ at the point $x$. Given $[t_0, t_1] \subseteq \mathbb{R}$ we set again $Q = [t_0, t_1] \times \bar{\Omega}$ and $S = [t_0, t_1] \times \Gamma$.

We consider the linear parabolic initial-boundary value problem (1.1) when the coefficients are independent on time, i.e.:

$$
\begin{equation}
\begin{cases}
D_t u(t, x) = \sum_{|\gamma| \leq 2m} a_\gamma(x) D_\gamma^2 u(t, x) + f(t, x), \quad (t, x) \in Q \\
u(t_0, x) = u_0(x), \quad x \in \bar{\Omega} \\
\sum_{|\beta| \leq m} b_{j\beta}(x) D_\beta^2 u(t, x) = g_j(t, x), \quad j = 1, \cdots, m, \quad (t, x) \in S.
\end{cases}
\end{equation}
$$

(4.1)

The coefficients $a_\gamma$ and $b_{j\beta}$ are subject to the following assumptions:

$$
\begin{equation}
\begin{cases}
(a) \text{ (regularity) } a_\gamma \in C^\alpha(\bar{\Omega}) \text{ for each } \gamma \in \mathbb{N}^n \text{ such that } |\gamma| \leq 2m. \\
(b) \text{ (ellipticity) There exist } \mu > 0, \eta_0 \in \left[\frac{\pi}{2}, \pi\right[ \text{ such that for each } \ x \in \bar{\Omega}, \eta \in [-\eta_0, \eta_0], \xi \in \mathbb{R}^n, r \in \mathbb{R} \text{ with } |\xi|^2 + r^2 \neq 0 \text{ we have } \\
| \sum_{|\gamma|=2m} a_\gamma(x)\xi^\gamma + (-1)^m r^{2m} e^{i\eta}| \geq \mu(|\xi|^{2m} + r^{2m}) \\
(c) \text{ (roots condition) For each } x \in \Gamma, \eta \in [-\eta_0, \eta_0], \xi \in \mathbb{R}^n, r \in \mathbb{R} \\
\text{with } |\xi|^2 + r^2 \neq 0 \text{ and } \langle \xi, \nu(x) \rangle = 0, \text{ the polynomial } \\
p(z) = \sum_{|\gamma|=2m} a_\gamma(x)\xi^\gamma + z\nu(x) \mid_{\xi}^\eta - (-1)^m r^{2m} e^{i\eta} \\
\text{has exactly } m \text{ roots } z_j^+(x, \xi, r, \eta), j = 1, \cdots, m \text{ with positive imaginary part}.
\end{cases}
\end{equation}
$$

(4.2)
\[
\begin{align*}
(\text{a}) \quad \text{(regularity) } b_{j\beta} \in C^{2m+\alpha-m_j}(\Gamma) \text{ for each } j = 1, \ldots, m \\
\quad \text{and for each } \beta \in \mathbb{N}^n \text{ such that } |\beta| \leq m_j \\
(\text{b}) \quad \text{(normality) } 0 \leq m_1 < m_2 < \cdots < m_m \leq 2m - 1 \text{ and for each } j = 1, \ldots, m, \sum_{|\beta|=m_j} b_{j\beta}(x)(\nu(x))^{\beta} \neq 0 \text{ for } x \in \Gamma. \\
(\text{c}) \quad \text{(complementing condition) For each } x \in \Gamma, \xi \in \mathbb{R}^n, r \in \mathbb{R} \\
\quad \text{with } |\xi|^2 + r^2 \neq 0 \text{ and } \langle \xi, \nu(x) \rangle = 0, \text{ the polynomials } \\
p_j(z) = \sum_{|\beta|=m_j} b_{j\beta}(x)(\xi + rz\nu(x))^{\beta}, j = 1, \ldots, m \text{ are linearly} \\
\quad \text{independent modulo the polynomial } \\
q(z) = \prod_{j=1}^m (z - z_j^+(x, \xi, r, \eta)) \text{ where } z_j^+ \text{ are defined in 4.2(c).}
\end{align*}
\]

Let us observe that the last condition of (b) is a consequence of (c).

Concerning the data \(f, u_0\) and \(g_j \ (j = 1, \ldots, m)\) we assume that

\[
\begin{align*}
(4.4) \quad \begin{cases} 
 f \in C^{2m,\alpha}(Q), & u_0 \in C^{2m+\alpha}(\bar{\Omega}), & g_j \in C^{\frac{2m+\alpha-m_j}{2m},2m+\alpha-m_j}(S), \\
j = 1, \ldots, m.
\end{cases}
\end{align*}
\]

We want to prove the existence of a solution \(u\) of (4.1) such that

\[
(4.5) \quad u \in C^{\frac{2m+\alpha}{2m},2m+\alpha}(Q).
\]

Then it is known (see [S]) that the following compatibility conditions must hold
\[
\begin{aligned}
\sum_{|\beta| \leq m_j} b_{j\beta}(x) D_2^\beta u^k(x) = D_t^k g_j(t_0, x), \quad x \in \Gamma, \ j = 1, \ldots, m; \\
\quad k = 0, \ldots, \left[\frac{2m + \alpha - m_j}{2m}\right]
\end{aligned}
\]

(4.6)

where we have set for each \( x \in \tilde{\Omega} \):

\[
\begin{align*}
&u^{(0)}(x) = u_0(x) \\
&u^{(k)}(x) = \sum_{|\gamma| \leq 2m} a_{\gamma}(x) D_2^\gamma u^{(k-1)}(x) + D_t^{k-1} f(t_0, x), \\
&\quad k = 1, \ldots, \left[\frac{2m + \alpha}{2m}\right],
\end{align*}
\]

and we have also

(4.7) \[ u^{(k)}(x) = D_t^k u(t_0, x), \quad x \in \tilde{\Omega}, \ k = 1, \ldots, \left[\frac{2m + \alpha}{2m}\right]. \]

We will prove in sections 6 and 7 that the conditions (4.6) are also sufficient to obtain a solution of (4.1) verifying (4.5). More precisely the following result holds true:

**Theorem 4.1** Let assumptions (4.2)-(4.3) hold. Given \( f, u_0 \) and \( g_j (j = 1, \ldots, m) \) satisfying (4.4) and the compatibility conditions (4.6), there exists a unique solution \( u \in C^{\frac{2m + \alpha}{2m}, \frac{2m + \alpha}{2m}}(Q) \) of problem (4.1). Moreover there exists a continuous and increasing function \( c : \mathbb{R}_+ \to \mathbb{R}_+ \) (depending also on bounds of the norms of \( a_{\gamma}, b_{j\beta} \) and on \( \Omega, \eta_0 \)) such that

\[
\begin{aligned}
&\| u \|_{C^{\frac{2m + \alpha}{2m}, \frac{2m + \alpha}{2m}}(Q)} \\
\leq & \quad c(t_1 - t_0) \{ \| f \|_{C^{\frac{2m + \alpha - m_j}{2m}, \frac{2m + \alpha - m_j}{2m}}(Q)} + \| u_0 \|_{C^{2m + \alpha} (\Omega)} \\
&\quad + \sum_{j=1}^m \| g_j \|_{C^{\frac{2m + \alpha - m_j}{2m}, \frac{2m + \alpha - m_j}{2m}}(S)} \}.
\end{aligned}
\]

(4.8)

Before giving our proof of this theorem we will give an abstract version of problem (4.1) in order to apply the results of section 2.
5 The abstract setting

We shall use the following notation

\begin{align}
A &= \sum_{|\alpha| \leq 2m} a_{\alpha}(\cdot)D_{x}^{\alpha}, \quad B_{j} = \sum_{|\beta| \leq m_{j}} b_{j\beta}(\cdot)D_{x}^{\beta}, \quad j = 1, \ldots, m \\
f(t) &= f(t, \cdot), \quad g_{j}(t) = g_{j}(t, \cdot), \quad t \in [t_{0}, t_{1}], \quad j = 1, \ldots, m \\
u^{(0)} &= u_{0}, \quad u^{(k)} = Au^{(k-1)} + f^{(k-1)}(t_{0}), \quad k = 1, \ldots, \left[\frac{2m + \alpha}{2m}\right] 
\end{align}

where \( f^{(k)}(t) = D_{t}^{k}f(t, \cdot) \), so that problem (4.1) can be written as

\begin{align}
\begin{cases}
\quad u'(t) = Au(t) + f(t), \quad t \in [t_{0}, t_{1}] \\
\quad u(t_{0}) = u_{0} \\
B_{j}u(t) = g_{j}(t), \quad j = 1, \ldots, m; \quad t \in [t_{0}, t_{1}]
\end{cases}
\end{align}

and the compatibility conditions as

\begin{align}
B_{j}u^{(k)} = g_{j}^{(k)}(t_{0}), \quad j = 1, \ldots, m; \quad k = 0, \ldots, \left[\frac{2m + \alpha - m_{j}}{2m}\right].
\end{align}

By virtue of theorem 1.2, theorem 3.1 is equivalent to

**Theorem 5.1** Let assumptions (4.2)-(4.3) hold. Assume that

\begin{align}
\begin{cases}
\quad f \in C_{2m}^{2m}([t_{0}, t_{1}]; C(\bar{\Omega})) \text{ is such that} \\
\quad f^{(k)} \in B([t_{0}, t_{1}]; C^{\alpha-2mk}(\bar{\Omega})), \quad k = 0, \ldots, \left[\frac{\alpha}{2m}\right] \\
\quad u_{0} \in C^{2m+\alpha}(\bar{\Omega})
\end{cases}
\end{align}

and assume that for each \( j = 1, \ldots, m \)

\begin{align}
\begin{cases}
\quad g_{j} \in C_{2m+\alpha-m_{j}}^{2m+\alpha-m_{j}}([t_{0}, t_{1}]; C(\Gamma)) \text{ is such that} \\
\quad g_{j}^{(k)} \in B([t_{0}, t_{1}]; C^{2m+\alpha-m_{j}-2mk}(\Gamma)), \quad k = 0, \ldots, \left[\frac{2m+\alpha-m_{j}}{2m}\right].
\end{cases}
\end{align}
Moreover suppose that, defining $u^{(k)}$ by (5.1), conditions (5.3) hold. Then problem (5.2) has a unique solution

\begin{equation}
\left\{ \begin{array}{l}
u \in C^{2m+\alpha}_{2m}([t_0, t_1]; C(\bar{\Omega})) \text{ such that} \\
u^{(k)} \in B([t_0, t_1]; C^{2m+\alpha-2mk}(\bar{\Omega})), \quad k = 0, \ldots, [\frac{2m+\alpha}{2m}] \\
\end{array} \right.
\end{equation}

and there exists a continuous and increasing function $c : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ (depending also on bounds of the norms of $a_\gamma, b_{j\beta}$ and on $\Omega, \eta_0$) such that

\begin{equation}
\left\| \left\| u \right\| \right\|_{C^{\frac{2m+\alpha}{2m}, 2m+\alpha}(Q)} \leq c(t_1 - t_0) \{ \left\| \int F \right\|_{C^{\alpha}_{2m, \alpha}(Q)} + \left\| u_0 \right\|_{C^{2m+\alpha}(\bar{\Omega})}
+ \sum_{j=1}^{m} \left\| g_j \right\|_{C^{\frac{2m+\alpha - m_j}{2m}, \frac{2m+\alpha - m_j}{2m}}(S)} \}.
\end{equation}

In the proof of this theorem we shall use also known results about generators of analytic semigroups, characterization of interpolation spaces and Hölder regularity for elliptic equations which we collect in the following theorem.

**Theorem 5.2** Let (4.2)-(4.3) hold and let $A$ and $B_j$ ($j = 1, \ldots, m$) be defined by (5.1).

(i) If $X = C(\bar{\Omega})$ is given the sup-norm and

\begin{equation}
\left\{ \begin{array}{l}
D(\Lambda) = \{ \varphi \in \bigcap_{p>1} W^{2m,p}(\Omega); \ A\varphi \in C(\bar{\Omega}); \ B_j\varphi = 0, \\
\quad \quad \quad \quad \quad j = 1, \cdots, m \} \\
\Lambda \varphi = A\varphi
\end{array} \right.
\end{equation}

then $\Lambda : D(\Lambda) \subseteq X \rightarrow X$ satisfies (3.1), where $\omega, \eta, M$ depend on bounds of the norms of $a_\gamma, b_{j\beta}$, and on $\Omega, \eta_0$.

(ii) For each $\theta \in [0, 1[$ such that $2m\theta \notin \mathbb{N}$ we have

\begin{equation}
D(\theta, \infty) = \{ \varphi \in C^{2m\theta}(\bar{\Omega}); \ B_j\varphi = 0 \text{ if } m_j \leq [2m\theta] \}
\end{equation}
and the $C^{2m\theta}$-norm is equivalent to the $D_{\Lambda}(\theta, \infty)$-norm.

(iii) For each $k = 1, \cdots, 2m - 1$ we have

$$C_{B}^{k}(\Omega) = \{ \varphi \in C^{k}(\Omega); \quad B_{j}\varphi = 0 \text{ if } m_{j} < k \} \hookrightarrow D_{\Lambda}(\frac{k}{2m}, \infty)$$

where $C_{B}^{k}(\Omega)$ is given the norm of $C^{k}(\Omega)$.

(iv) $D = \{ \varphi \in \bigcap_{\rho > 1} W^{2m, \rho}(\Omega); \quad A\varphi \in C^{\alpha}(\Omega); \quad B_{j}\varphi \in C^{2m+\alpha-m_{j}}(\Gamma); \quad j = 1, \cdots, m \} \subset C^{2m+\alpha}(\Omega)$ and there exists $c_{0} > 0$ such that

$$\| \varphi \|_{C^{2m+\alpha}(\Omega)} \leq c_{0}\{ \| A\varphi \|_{C^{\alpha}(\Omega)} + \| \varphi \|_{C(\Omega)} + \sum_{j=1}^{m} \| B_{j}\varphi \|_{C^{2m+\alpha-m_{j}}(\Gamma)} \}.$$ 

Proof. (i) is a consequence of the results on the resolvent of $\Lambda$ proved in [GG] and on the estimate of $\| (\lambda - \Lambda)^{-1} \|$ demonstrated in [St] and [AT]. (ii) is a result of [AT] and (iii) follows from the characterization of $D_{\Lambda}(\frac{k}{2m}, \infty)$ proved in [A]. Concerning (iv) one must observe that the results of [ADN] prove estimate (5.12) whereas the inclusion of the set $D$ in $C^{2m+\alpha}(\Omega)$ is a consequence of the existence theorems proved in [GG].

6 An extension operator

As mentioned in the introduction the theory of semigroups of linear operators cannot be applied to problem (1.1) when $g_{j} \neq 0$ for some $j = 1, \cdots, m$. For this reason we need an operator $N$ satisfying (1.12): its existence will be proved in three steps. We can suppose that $n > 1$ because when $n = 1$ the operator $N$ can be constructed by using suitable polynomials.

The first step is essentially lemma 10 of [Se]:

Theorem 6.1 Given $\varphi \in C_{0}^{\infty}(\mathbb{R}^{n-1})$ such that $\int_{\mathbb{R}^{n-1}} \varphi(t_{1}, \cdots, t_{n-1})dt_{1} \cdots dt_{n-1} = 1$, set for each $k \in \mathbb{N}$ and $f \in C^{\alpha}(\mathbb{R}^{n-1})$:
\[
F_k(y_1, \ldots, y_n) = \left\{
\begin{aligned}
&\frac{y_i^k}{k!} \int_{\mathbb{R}^{n-1}} \varphi(t_1, \ldots, t_{n-1}) dt_1 \cdots dt_{n-1}.
&f(y_1 + t_1 y_n, \ldots, y_{n-1} + t_{n-1} y_n) dt_1 \cdots dt_{n-1}.
\end{aligned}
\right.
\]

Then we have:

(6.2) \quad F_k \in C^k(\mathbb{R}^n)

and

(6.3) \quad D^l_y F_k(y_1, \ldots, y_{n-1}, 0) = \left\{
\begin{aligned}
&0 \text{ if } |l| \leq k \text{ and } l \neq (0, \ldots, 0, k)
&f(y_1, \ldots, y_{n-1}) \text{ if } l = (0, \ldots, 0, k)
&\text{for } (y_1, \ldots, y_{n-1}) \in \mathbb{R}^{n-1}.
\end{aligned}
\right.

Moreover for each \( m \in \mathbb{N} \) and \( \theta \in [0, 1[: 

(6.4) \quad f \in C^{m+\theta}(\mathbb{R}^{n-1}) \Rightarrow F_k \in C^{k+m+\theta}(\mathbb{R}^n).

As second step we prove the following result.

**Theorem 6.2** For each \( j = 1, \ldots, m \) there exists \( N_j \in L(C^0(\Gamma); C^{m_j}(\Omega)) \) such that setting \( v_j = N_j \psi \) for \( \psi \in C^0(\Gamma) \), we have

(6.5) \quad D^l_x v_j(x) = 0, \quad x \in \Gamma, \quad l \in \mathbb{N}^n, \quad |l| < m_j

(6.6) \quad (B_j v_j)(x) = \psi(x), \quad x \in \Gamma.

Moreover

(6.7) \quad N_j \in L(C^{r}(\Gamma); C^{r+m_j}(\Omega)), \quad \forall r \in [0, 2m + \alpha - m_j].
Proof. If \( \bigcup_{i=1}^{N} \hat{B}_i \) is an open covering of \( \Gamma \) (see the notation of section 0) let \( \varphi_i = \{ \varphi_{i1}, \ldots, \varphi_{i\infty} \} (i = 1, \ldots, N) \) be the \( C^{2m+\alpha} \)-diffeomorphism of \( B_i \) onto \( B \). Then setting for \( i = 1, \ldots, N \) and \( j = 1, \ldots, m \)

\[
\gamma_{ij}(x) = \sum_{\beta=(\beta_1, \ldots, \beta_n), \|\beta\|=m_j} b_{ij}(x) \left( \frac{\partial \varphi_{in}(x)}{\partial x_1} \right)^{\beta_1} \cdots \left( \frac{\partial \varphi_{in}(x)}{\partial x_n} \right)^{\beta_n}, \quad x \in \Gamma \cap B_i
\]

we have \( \gamma_{ij} \in \mathcal{C}^{2m+\alpha-m_j}(\Gamma \cap B_i) \). Given \( \psi \in \mathcal{C}^r(\Gamma) \) with \( r \in [0, 2m+\alpha-m_j] \) we set

\[
f_{ij}(y_1, \ldots, y_{n-1}, 0) = \frac{\psi[\varphi_i^{-1}(y_1, \ldots, y_{n-1}, 0)]}{\gamma_{ij}[\varphi_i^{-1}(y_1, \ldots, y_{n-1}, 0)]}, \quad (y_1, \ldots, y_{n-1}, 0) \in \Sigma.
\]

As \( f_{ij} \in \mathcal{C}^r(\Sigma) \), we can define (by virtue of theorem 5.1) \( u_{ij}^* \in \mathcal{C}^{r+m_j}(\mathbb{R}^n) \) such that for each \( (y_1, \ldots, y_{n-1}, 0) \in \Sigma \) we have:

\[
\begin{aligned}
D_y^l u_{ij}^*(y_1, \ldots, y_{n-1}, 0) \\
= 0 & \text{ if } |l| \leq m_j \text{ and } l \neq (0, \ldots, 0, m_j) \\
f_{ij}(y_1, \ldots, y_{n-1}, 0) & \text{ if } l = (0, \ldots, 0, m_j).
\end{aligned}
\]

Let \( \varphi_i^* \) and \( \xi_i^* \) be the extensions of \( \varphi_i \) and \( \xi_i \) obtained by setting \( \varphi_i^*(x) = (0, \cdots, 0) \) and \( \xi_i^*(x) = 0 \) for \( x \in \Omega \setminus B_i \). We will prove that

\[
v_j(x) = \sum_{i=1}^{N} \xi_i^*(x) u_{ij}^*(\varphi_i^*(x)), \quad x \in \bar{\Omega}
\]

satisfies (6.5)-(6.7). Let us fix \( x_0 \in \Gamma \) and let \( N_0 = \{ i \in \{1, \cdots, N\}; x_0 \in \hat{B}_i \} \): there exists \( r > 0 \) such that \( B(x_0, r) = \{ x \in \mathbb{R}^n, |x - x_0| < r \} \subseteq \bigcap_{i \in N_0} \hat{B}_i \),

\[
v_j(x) = \sum_{i \in N_0} \xi_i(x) u_{ij}^*(\varphi_i(x)), \quad x \in B(x_0, r) \cap \bar{\Omega}.
\]

Hence for \( l \in \mathbb{N}^n \) such that \( |l| < m_j \) we deduce (6.5) from the first of (6.10), while for \( |l| = m_j \) we obtain
\begin{equation}
\begin{aligned}
(B_j v_j)(x_0) &= \sum_{|\beta| = m_j} b_{j_\beta}(x_0) \sum_{i \in \mathbb{N}_0} D_x^\beta [\xi_i(x) u_{ij}^*(\varphi_i(x))]_{x=x_0} \\
&= \sum_{i \in \mathbb{N}_0} \xi_i(x_0) \sum_{|\beta| = m_j} b_{j_\beta}(x_0) \left( \frac{\partial^{m_j} u_{ij}^*(y)}{\partial y_m} \right)_{y=\varphi_i(x_0)} \\
&\left( \frac{\partial \varphi_i(x)}{\partial x_l} \right)_{l=1}^{l_1} \cdots \left( \frac{\partial \varphi_i(x)}{\partial x_{m_n}} \right)_{l=1}^{l_m} \bigg|_{x=x_0}.
\end{aligned}
\end{equation}

Hence (6.6) follows from (6.9) and (6.10). (6.7) is a consequence of the definition of $u_{ij}^*$ and the regularity assumptions on $\Gamma$. \hfill \Box

We can prove now the existence of the extension operator $N$ verifying (1.12).

\textbf{Theorem 6.3} Given $s = 1, \ldots, m$ there exists

\begin{equation}
M_s \in L(\prod_{j=1}^s C^{\theta-m_j}(\Gamma), C^{\theta-m_s}(\Gamma)),
\end{equation}

\forall \theta \in [m_s, 2m + \alpha]

and such that setting

\begin{equation}
N(\psi_1, \ldots, \psi_m) = \sum_{s=1}^m N_s M_s (\psi_1, \ldots, \psi_s)
\end{equation}

we have

\begin{equation}
N \in L(\prod_{j=1}^m C^{2m+\theta'-m_j}(\Gamma), C^{2m+\theta'}(\hat{\Omega})),
\end{equation}

\forall \theta' \in [0, \alpha]

and

\begin{equation}
B_j(N(\psi_1, \ldots, \psi_m))(x) = \psi_j(x), \ x \in \Gamma, \ j = 1, \ldots, m.
\end{equation}

\textbf{Proof.} Let us first observe that by virtue of (4.3) (a) we have for each $j = 1, \ldots, m$: 

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(6.18) \( B_j \in L(C^\theta(\Omega); C^{\theta-m_j}(\Gamma)), \ \theta \in [m_j, 2m + \alpha] \).

For each \((r, s) \in \mathbb{N}^2\) with \(1 \leq r \leq s \leq m\) we define the operator \(H_{s,r}\) by induction:

(6.19) \( H_{r,r} = I_{\Gamma}, \ 1 \leq r \leq m \)

(where \(I_{\Gamma}\) is the identity in the set of functions defined in \(\Gamma\)) and

(6.20) \( H_{r+h,r} = -B_{r+h} \sum_{j=0}^{h-1} N_{r+j} H_{r+j,r} \).

From (6.7) and (6.18) we deduce, for \(1 \leq r \leq s \leq m\):

(6.21) \( H_{s,r} \in L(C^{\theta-m_r}(\Gamma); C^{\theta-m_s}(\Gamma)) \) if \(\theta \in [m_s, 2m + \alpha]\).

Let us define now the following operators

\[
\begin{align*}
M_1(\psi_1) & = H_{1,1}(\psi_1) \\
M_2(\psi_1, \psi_2) & = H_{2,1}(\psi_1) + H_{2,2}(\psi_2) \\
& \vdots \\
M_m(\psi_1, \ldots, \psi_m) & = H_{m,1}(\psi_1) + \cdots + H_{m,m}(\psi_m).
\end{align*}
\]

(6.22)

From (6.21) we obtain (6.14) which implies (6.16). To prove (6.17) let us fix

(6.23) \( \psi_j \in C^{2m+\theta'-m_j}(\Gamma), \ j = 1, \ldots, m, \ \theta' \in [0, \alpha] \)

and set

(6.24) \( u_k = \sum_{s=1}^{k} N_s M_s(\psi_1, \ldots, \psi_s), \ k = 1, \ldots, m. \)

By using properties (6.5)-(6.6) one proves by induction that

(6.25) \( B_j u_k = \psi_j \) for \(1 \leq j \leq k\)

which, for \(k = m\), coincides with (6.17). \(\square\)
7 Proof of theorem 4.1 for $\alpha < 2m$

From (5.6) and (2.12) we deduce

$$
\begin{align*}
(7.1) \quad \left\{ \begin{array}{l}
g_j \in C^{2m+\alpha-m_j-h}_{2m-2m}([t_0, t_1]; C^h(\Gamma)), \\
h \in \mathbb{N}, \ 0 \leq h \leq [\alpha] + 2m - m_j
\end{array} \right. 
\end{align*}
$$

so that from (6.7), (6.14), and (7.1) we obtain (for $s = 1, \ldots, m$):

$$
(7.2) \quad N_s M_s(g_1(\cdot), \ldots, g_s(\cdot)) \in B([t_0, t_1]; C^{2m+\alpha}(\Omega)) \cap C^{2m}(\Omega) \cap C^{2m-m_s}_{2m}([t_0, t_1]; C^{m_s}(\Omega)).
$$

Hence if

$$
(7.3) \quad m_1 < \cdots < m_s < \alpha
$$

there exists, for $t \in [t_0, t_1]$

$$
(7.4) \quad \frac{d}{dt} N_s M_s(g_1(t), \ldots, g_s(t)) = N_s M_s (g'_1(t), \ldots, g'_s(t))
$$

and

$$
(7.5) \quad \left\{ \begin{array}{l}
N_1 M_1(g'_1(\cdot)) \in C^{0}_{2m}([t_0, t_1]; C^0(\Omega)) \text{ if } m_1 = 0 \\
N_s M_s(g'_1(\cdot), \ldots, g'_s(\cdot)) \in C^{a-m_s}_{2m}([t_0, t_1]; D_\lambda(m_s, \infty)) \\
\text{if } 1 \leq m_s < \alpha.
\end{array} \right.
$$

Actually, from (5.11) we get that the subspace of $C^{m_s}(\Omega)$ defined as

$$
(7.6) \quad \{ \varphi \in C^{m_s}(\Omega); B_j \varphi = 0 \text{ if } m_j < m_s \}
$$

is continuously embedded in $D_\lambda(m_s, \infty)$ if $m_s \geq 1$. Moreover from (6.5) we deduce that for each $u \in C^0(\Gamma)$:

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(7.7) \( (B_j N_s u)(x) \equiv 0 \) on \( \Omega \) for \( j < s \)

and from (5.10) we obtain

(7.8) \( D\lambda_{\frac{m_s}{2m}, \infty} = \{ u \in C^m(\Omega); B_j u = 0 \text{ if } m_j \leq m_s \} \) for \( 0 < m_s < 2m \)

which implies (7.5)(2); while (7.5)(1) is a consequence of (7.2) with \( s = 1 \).

Let us define for each \( t \in [t_0, t_1] \):

(7.9) \[ n(t) = N(g_1(t), \cdots, g_m(t)) = \sum_{s=1}^{m} N_s M_s(g_1(t), \cdots, g_s(t)) \]

(7.10) \[
\begin{cases}
0 \text{ if } \alpha < m_1 \\
\sum_{s=1}^{p} N_s M_s(g_1(t), \cdots, g_s(t)) \text{ if } m_1 < \alpha \\
n_2(t) = n(t) - n_1(t),
\end{cases}
\]

where \( p \) is the greatest integer \( j \leq m \) such that \( m_j < \alpha \). From (7.2) and (7.4) it follows that

(7.11) \[
\begin{align*}
n &\in B([t_0, t_1]; C^{2m+\alpha}(\Omega)) \cap C^m_{\alpha}([t_0, t_1]; C^{2m}(\Omega)) \\
A n &\in B([t_0, t_1]; C^{\alpha}(\Omega)) \cap C^{2\alpha}_{\alpha}([t_0, t_1]; C^0(\Omega)),
\end{align*}
\]

(7.12) \[
\begin{align*}
n_1 &\in B([t_0, t_1]; C^{2m+\alpha}(\Omega)) \\
A n_1 &\in B([t_0, t_1]; C^{\alpha}(\Omega))
\end{align*}
\]

(7.13) \[
\begin{align*}
n_1 &\in C^1([t_0, t_1]; C^0(\Omega)) \\
n'_1 &\in B([t_0, t_1]; C^{\alpha}(\Omega)) \\
n'_1(t) &= \sum_{s=1}^{p} N_s M_s(g'_1(t), \cdots, g'_s(t)), \ t \in [t_0, t_1], \text{ if } m_1 < \alpha.
\end{align*}
\]
As we mentioned in the introduction, we will prove that a solution of problem (5.2) can be obtained by the representation formula (1.10) with some adaptations due to the compatibility conditions that must be satisfied to get solutions having the required regularity. More precisely we will solve the three problems

\[
\begin{align*}
  v'(t) &= Av(t) + f(t) + An(t) - n'_1(t_0), \quad t \in [t_0, t_1] \\
  v(t_0) &= u_0 - n(t_0) \\
  B_j v(t) &= 0, \quad j = 1, \cdots, m, \quad t \in [t_0, t_1]
\end{align*}
\]

(7.14)

\[
\begin{align*}
  w'(t) &= Aw(t) - An_1(t) + n'_1(t_0), \quad t \in [t_0, t_1] \\
  w(t_0) &= n_1(t_0) \\
  B_j w(t) &= B_j n_1(t), \quad j = 1, \cdots, m, \quad t \in [t_0, t_1]
\end{align*}
\]

(7.15)

\[
\begin{align*}
  z'(t) &= Az(t) - An_2(t), \quad t \in [t_0, t_1] \\
  z(t_0) &= n_2(t_0) \\
  B_j z(t) &= B_j n_2(t), \quad j = 1, \cdots, m, \quad t \in [t_0, t_1]
\end{align*}
\]

(7.16)

and will prove that

\[
(7.17) \quad u(t) = v(t) + w(t) + z(t), \quad t \in [t_0, t_1]
\]

is the solution of problem (5.2). Uniqueness of the solution is a consequence of theorem 3.1.

To solve the three problems we will apply the abstract results of section 3 with \( X \) and \( \Lambda \) defined in (i) of theorem 5.2.

Let us first consider problem (7.14). To apply theorem 3.1 with \( \theta = \frac{\alpha}{2m} \) we must show that

\[
(7.18) \quad f(\cdot) + An(\cdot) - n'_1(t_0) \in C^{\frac{\alpha}{2m}}([t_0, t_1]; C(\Omega))
\]

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(7.19) \( u_0 - n(t_0) \in D(\Lambda) \)

and

(7.20) \( \Lambda(u_0 - n(t_0)) + f(t_0) + An(t_0) - n'_1(t_0) \in D_{\Lambda}(\frac{\alpha}{2m}, \infty) \)

where

(7.21) \( D_{\Lambda}(\frac{\alpha}{2m}, \infty) = \{ \varphi \in C^\alpha(\bar{\Omega}); B_j \varphi = 0 \text{ if } m_j < \alpha \} \).

Now (7.18) follows from (7.11) and (7.13); (7.20) is a consequence of (7.12), (6.17) and (5.3). To prove (7.20) we can use (7.11)-(7.13) and the fact that due to (6.25) and (5.3) we have, for \( m_j < \alpha \)

(7.22) \[
    \begin{cases}
    B_j(\Lambda u_0 + f(t_0) - n'_1(t_0)) \\
    = B_j(\Lambda u_0 + f(t_0)) - B_j \sum_{s=1}^{p} N_s M_s(g'_1(t_0), \ldots, g'_s(t_0)) \\
    = B_j(\Lambda u_0 + f(t_0)) - g'_j(t_0) = 0.
    \end{cases}
\]

Theorem 3.1 gives now a solution \( v \) of (7.14) such that

(7.23) \[
    \begin{cases}
    v \in C^{1+\frac{\alpha}{2m}}([t_0, t_1]; C(\bar{\Omega})), \ v' \in B([t_0, t_1]; C^\alpha(\bar{\Omega})) \\
    v(t) \in \bigcap_{p>1} W^{2m,p}(\bar{\Omega}), \ Av(t) \in C^\alpha(\bar{\Omega}), \ t \in [t_0, t_1]
    \end{cases}
\]

where we have used the fact that \( D_{\Lambda}(\frac{\alpha}{2m}, \infty) \simeq C^\alpha(\bar{\Omega}) \) (see theorem 5.2). To solve problem (7.15) we write

(7.24) \( n'_1(t) - n'_1(t_0) = \sum_{s=1}^{p} \varphi_s(t), \ t \in [t_0, t_1] \)

with

(7.25) \( \varphi_s(t) = N_s M_s(g'_1(t) - g'_1(t_0), \ldots, g'_s(t) - g'_s(t_0)), \ t \in [t_0, t_1] \)

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and for each \( s = 1, \cdots, p \), we consider the problem

\[
\begin{aligned}
{w}'(t) &= Aw_s(t) - \varphi_s(t), \quad t \in [t_0, t_1] \\
{w}_s(t_0) &= 0 \\
B_jw_s(t) &= 0, \quad j = 1, \cdots, m, \quad t \in [t_0, t_1].
\end{aligned}
\] (7.26)

If \( m_1 = 0 \) we have \( \varphi_1 \in C^\alpha_{\infty}(t_0, t_1; C^0(\Omega)) \) by virtue of (7.5): hence from theorem 2.1 we deduce the existence of a solution \( w_1 \) to (7.26)(1) such that

\[
\begin{aligned}
w_1 &\in C^1([t_0, t_1]; X), \\
\Lambda w_1 &\in C^\alpha_{\infty}([t_0, t_1]; X), \\
{w}_1' &\in B([t_0, t_1]; D_\Lambda(\frac{\alpha}{2m}, \infty)).
\end{aligned}
\] (7.27)

If \( m_1 > 0 \) we get from (7.5) that \( \varphi_s \in C^\alpha_{\infty}(t_0, t_1; D_\Lambda(\frac{m_1}{2m}, \infty)) \) for each \( s = 1, \cdots, p \). Hence from theorem 3.2 we obtain

\[
\begin{aligned}
w_s &\in C^1([t_0, t_1]; X), \\
\Lambda w_s &\in C^\alpha_{\infty}([t_0, t_1]; X), \\
{w}_s' &\in B([t_0, t_1]; D_\Lambda(\frac{\alpha}{2m}, \infty))
\end{aligned}
\] (7.28)

where \( w_s \) is the solution of (7.26). Setting

\[
w(t) = \sum_{s=1}^{p} w_s(t) + n_1(t), \quad t \in [t_0, t_1]
\] (7.29)

we obtain a solution of (7.15) verifying the same properties as \( v \) (see (7.23)).

We can consider problem (7.16) when \( n_2(t) \neq 0 \) i.e. \( p < m \). In this case we set for each \( s = p + 1, \cdots, m \):

\[
\psi_s(t) = N_sM_s(g_1(t) - g_1(t_0), \cdots, g_s(t) - g_s(t_0)), \quad t \in [t_0, t_1]
\] (7.30)
and first we solve problem

\begin{equation}
\begin{align*}
(7.31) \quad \begin{cases}
  y'_s(t) &= \Lambda y_s(t) + \psi_s(t), \quad t \in [t_0, t_1] \\
  y_s(t_0) &= 0.
\end{cases}
\end{align*}
\end{equation}

We have

\begin{equation}
(7.32) \quad \psi_s \in C^{2m+\alpha-\frac{m}{2m}}([t_0, t_1]; D_\Lambda(\frac{m_s}{2m}, \infty))
\end{equation}

by virtue of (5.11), (7.2) and the fact that \( B_j N_s = 0 \) for \( j < s \). Hence theorem 5.2 gives a solution \( y_s \) of (7.31) such that

\begin{equation}
(7.33) \quad \begin{cases}
  \Lambda y_s &\in C^{1+\frac{m}{2m}}([t_0, t_1]; X), \\
  \Lambda y'_s &\in B([t_0, t_1]; D_\Lambda(\frac{\alpha}{2m}, \infty)).
\end{cases}
\end{equation}

Then, setting

\begin{equation}
(7.34) \quad z(t) = -\Lambda(\sum_{s=p+1}^{m} y_s(t)) + n_2(t_0), \quad t \in [t_0, t_1]
\end{equation}

we obtain a solution of (7.16) verifying the same properties of \( v \) (see (7.23)). We conclude that \( u = v + w + z \) is a solution of (5.2) because \( B_j n(t) = g_j(t), \ j = 1, \cdots, m, \ t \in [t_0, t_1] \). Moreover we have:

\begin{equation}
(7.35) \quad \begin{cases}
  u \in C^{1+\frac{m}{2m}}([t_0, t_1]; C(\hat{\Omega})), \quad u' \in B([t_0, t_1]; C^\alpha(\hat{\Omega})) \\
  u(t) \in \bigcap_{p>1} W^{2m,p}(\Omega), \quad Au(t) \in C^\alpha(\hat{\Omega}), \quad t \in [t_0, t_1].
\end{cases}
\end{equation}

As \( \alpha < 2m \), to prove (5.7) we only need to show that:

\begin{equation}
(7.36) \quad u \in B([t_0, t_1]; C^{2m+\alpha}(\hat{\Omega}))
\end{equation}
and this can be done by using (iv) of theorem 5.2, by virtue of (7.35) and the fact that $B_j u(t) = g_j(t) \in C^{2m + \alpha - m_j}(\Gamma)$, $t \in [t_0, t_1]$. Estimate (5.8) is a consequence of estimates (2.11), (2.13), (3.7), (3.13), and (5.12).

Remark: From (3.5) it follows that the following representation formulas hold for each $t \in [t_0, t_1]$:

\begin{align}
(7.37) \quad v(t) &= e^{(t-t_0) \Lambda} (u_0 - n(t_0)) + \int_{t_0}^{t} e^{(t-s) \Lambda} (f(s) + An(s) - n_1'(t_0)) ds \\
(7.38) \quad w(t) &= n_1(t) - \int_{t_0}^{t} e^{(t-s) \Lambda} (n_1'(s) - n_1'(t_0)) ds \\
(7.39) \quad z(t) &= n_2(t_0) - \Lambda \int_{t_0}^{t} e^{(t-s) \Lambda} (n_2(s) - n_2(t_0)) ds
\end{align}

and so, since $n_1 \in C^1([t_0, t_1]; X)$, we deduce that (1.10) holds, i.e.

\begin{equation}
(7.40) \quad \begin{cases}
u(t) = e^{(t-t_0) \Lambda} u_0 + \int_{t_0}^{t} e^{(t-s) \Lambda} (f(s) + An(s)) ds \\
&- \Lambda \int_{t_0}^{t} e^{(t-s) \Lambda} n(s) ds, \ t \in [t_0, t_1].
\end{cases}
\end{equation}

8 Proof of theorem 4.1 for $\alpha > 2m$

We use the notations of section 6 and we set

\begin{equation}
(8.1) \quad \alpha = 2mk + \sigma, \ k \in \mathbb{N}, \ 0 < \sigma < 2m.
\end{equation}

The result will be proved by recurrence on $k$. Suppose $k = 1$ and consider the problem obtained differentiating formally (5.2):
\[
\begin{align*}
&y'(t) = Ay(t) + f'(t), \ t \in [t_0, t_1] \\
&y(t_0) = Au_0 + f(t_0) \\
&B_j y(t) = g_j(t), \ j = 1, \ldots, m, \ t \in [t_0, t_1]
\end{align*}
\]

Since the conditions of the case \( \alpha < 2m \) are verified, we deduce the existence of a unique solution \( y \) such that
\[
\begin{align*}
&y \in C^{1+\frac{\sigma}{2m}}([t_0, t_1]; C(\bar{\Omega})) \cap B([t_0, t_1]; C^{2m+\sigma}(\bar{\Omega})), \\
y' \in B([t_0, t_1]; C^\sigma(\bar{\Omega}))
\end{align*}
\]
and the estimate
\[
\begin{align*}
&\|y\|_{C^{1+\frac{\sigma}{2m}}([t_0, t_1]; C(\bar{\Omega}))} + \|y\|_{B([t_0, t_1]; C^{2m+\sigma}(\bar{\Omega}))} \\
&\quad + \|y'\|_{B([t_0, t_1]; C^\sigma(\bar{\Omega}))} \\
\leq c_\sigma (t_1 - t_0) \left( \|Au_0 + f(t_0)\|_{C^{2m+\sigma}(\bar{\Omega})} + \|f'(t)\|_{C^{\frac{\sigma}{2m}}(Q)} \\
&\quad + \sum_{j=1}^m \|g_j\|_{C^{\frac{2m+\sigma-m_j}{2m}}(\bar{\Omega})} \right)
\end{align*}
\]
holds, where \( c_\sigma \) has the same properties as \( c \) in (5.8).

We want to show that \( y = u' \), where \( u \) is the solution of (5.2). To this end we fix \( \bar{t} \in ]t_0, t_1[ \), we define for each \( h \in ]0, t_1 - \bar{t}[ \)
\[
(8.5) \quad u_h(t) = h^{-1}(u(t + h) - u(t)), \ t \in [t_0, \bar{t}]
\]
and we define similarly \( f_h \) and \( g_{jh} \). Setting \( u_h(t_0) = u_{oh} \) we have for each \( h \):\[
\begin{align*}
u'_h(t) &= Au_h(t) + f_h(t), \ t \in [t_0, \bar{t}] \\
u_h(t_0) &= u_{oh} \\
B_j u_h(t) &= g_{jh}(t), \ j = 1, \ldots, m; \ t \in [t_0, \bar{t}].
\end{align*}
\]
Taking into account the representation formula (7.40) we obtain:
\[
\begin{aligned}
\mathcal{L}h(t) - y(t) &= e^{(t-t_0)\Lambda}(u_{0h} - u'(t_0)) \\
&\quad + \int_{t_0}^{t} e^{(t-s)\Lambda}[f_h(s) - f'(s) + A(n_h(s) - n'(s))]ds \\
&\quad - \Lambda \int_{t_0}^{t} e^{(t-s)\Lambda}(n_h(s) - n'(s))ds
\end{aligned}
\]

where

\[
\begin{aligned}
n_h(t) = N(g_{1h}(t), \cdots, g_{mh}(t)) = h^{-1}(n(t + h) - n(t)), \quad t \in [t_0, \bar{t}].
\end{aligned}
\]

As \(u, f\) and \(An\) are in \(C^1([t_0, t_1]; X)\) (see (7.11)) we have \(\lim_{h \to 0} u_{0h} = u'(t_0)\) and \(\lim_{h \to 0} f_h(t) = f(t)\) uniformly in \([t_0, \bar{t}]\). Moreover from (7.11) we deduce \(n \in C^{1+\frac{\alpha}{2m}}([t_0, t_1]; X)\). Then for \(\varepsilon \in [0, \sigma]\) we have

\[
\lim_{h \to 0} \|n_h - n'\|_{C^{1+\frac{\alpha}{2m}}([t_0, \bar{t}]}; X).
\]

Now for \(t \in [t_0, \bar{t}]\) we have

\[
\begin{aligned}
&\quad \mathcal{L} \int_{t_0}^{t} e^{(t-s)\Lambda}(n_h(s) - n'(s))ds \\
= &\quad \int_{t_0}^{t} \Lambda e^{(t-s)\Lambda}(n_h(s) - n'(s) - n_h(t) + n'(t))ds \\
&\quad + (e^{(t-t_0)\Lambda} - 1)(n_h(t) - n'(t))
\end{aligned}
\]

and so from (8.7) we obtain

\[
\begin{aligned}
\|u_h(t) - y(t)\|_X &\leq M\{\|u_{0h} - u'(t_0)\|_X \\
&\quad + (\bar{t} - t_0)\|f_h - f'\|_{B([t_0, \bar{t}]}; X) \\
&\quad + \|A(n_h - n')\|_{B([t_0, \bar{t}]}; X)\} \\
&\quad + 2m\varepsilon^{-1}M_1[n_h - n']_{C^{1+\frac{\alpha}{2m}}([t_0, \bar{t}]}; X) \\
&\quad \cdot (\bar{t} - t_0)^{\frac{\alpha}{2m}} + (M + 1)\|n_h - n'\|_{B([t_0, \bar{t}]}; X)
\end{aligned}
\]

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where $M = \sup_{t_0 \leq t \leq t_1} \|e^{tA}\|_{L(X)}$ and $M_1 = \sup_{t_0 \leq t \leq t_1} \|tAe^{tA}\|_{L(X)}$.

This shows that $\lim_{h \to 0} u_h(t) = y(t)$ uniformly in $[t_0, \bar{t}]$: as $\bar{t}$ is arbitrary in $[t_0, t_1]$ we have $u'(t) = y(t)$ for $t_0 \leq t < t_1$ and, by continuity, also for $t = t_1$. From (8.3) it follows that

$$
\begin{align*}
\begin{cases}
    u &\in C^{2 + \frac{\sigma}{2m}}([t_0, t_1]; C(\bar{\Omega})), \\
    u' &\in B([t_0, t_1]; C^{2m+\sigma}(\bar{\Omega})), \\
    u'' &\in B([t_0, t_1]; C^\sigma(\bar{\Omega})).
\end{cases}
\end{align*}
$$

(8.12)

Moreover, since $Au = u' - f = y - f$, from (8.3) we deduce also

$$
(8.13) \quad Au \in B([t_0, t_1]; C^{2m+\sigma}(\bar{\Omega}))
$$

and $B_j u = g_j \in B([t_0, t_1]; C^{4m+\sigma-m_j}(\Gamma))$: now from (iv) of theorem 5.2 we get $u \in B([t_0, t_1]; C^{4m+\sigma}(\bar{\Omega}))$; this is sufficient to conclude that $u \in C^{2m+\sigma}(Q)$ by virtue of theorem 2.2. Estimate (5.8) follows from (5.12) and (8.4), and the statement is proved for $k = 1$.

Assume now that the statement holds for $k = 0, 1, \cdots, \bar{k} - 1$ with $\bar{k} \geq 2$: we will prove that it holds also for $k = \bar{k}$. We know that (8.2) is satisfied by $y = u'$ and (also by virtue of theorem 2.2) that

$$
\begin{align*}
\begin{cases}
    f' &\in C^{k-1 + \frac{\sigma}{2m}}([t_0, t_1]; C(\bar{\Omega})), \\
    \frac{d^k f'}{dt^k} &\in B([t_0, t_1]; C^{2m(k-1-k)+\sigma}(\bar{\Omega})), \quad k = 1, \cdots, \bar{k} - 1 \\
    Au_0 + f(t_0) &\in C^{2mk+\sigma}(\bar{\Omega}) \\
    g_j' &\in C^{k + \frac{\sigma-m_j}{2m}}([t_0, t_1]; C(\Gamma)), \\
    \frac{d^k g_j'}{dt^k} &\in B([t_0, t_1]; C^{2m(k-k)+\sigma-m_j}(\Gamma)), \quad k = 1, \cdots, \bar{k} + \frac{\sigma-m_j}{2m}.
\end{cases}
\end{align*}
$$

(8.14)

Moreover, due to (5.3), the compatibility conditions are satisfied by $f'$, $Au_0 + f(t_0)$ and $g_j'$. By assumption the statement is true when $k = \bar{k} - 1$ and so $u' \in C^{k+\frac{\sigma}{2m}}([t_0, t_1]; C(\bar{\Omega}))$ and $\frac{d^k u'}{dt^k} \in B([t_0, t_1]; C^{2m(k-k)+\sigma}(\bar{\Omega}))$ for $k = 1, \cdots, \bar{k}$,
so that \( u \in C^{k+1+k\sigma}(l_0, t_1]; C(\Omega)) \) and \( u^{(k)} \in B([l_0, t_1]; C^{2m(k+1-k)+\sigma}(\Omega)) \) for \( k = 1, \cdots, \bar{k}+1 \). As in the case \( k = 1 \) one proves that \( u \in B([l_0, t_1]; C^{2m(k+1)+\sigma}(\Omega)) \).

Estimate (5.8) with \( \alpha = 2m\bar{k} + \sigma \) follows now easily, and the statement is proved for \( k = \bar{k} \).

\[ \square \]

9 The time dependent coefficient case

In what follows \( \Omega \) is again a bounded open set of \( \mathbb{R}^n \) with \( C^{2m+\alpha} \) boundary \( \Gamma \) with \( m \in \mathbb{N} \) and \( \alpha > 0, \alpha \notin \mathbb{N} \). Given \( T > 0 \) we set \( Q_0 = [0, T] \times \Omega \) and \( S_0 = [0, T] \times \Gamma \) and we consider the problem:

\[
\begin{align*}
D_t u(t, x) &= \sum_{|\gamma| \leq 2m} a_{\gamma}(t, x) D_\gamma u(t, x) + f(t, x), \ (t, x) \in Q_0 \\
u(0, x) &= u_0(x), \ x \in \Omega \\
\sum_{|\beta| \leq m_j} b_{j\beta}(t, x) D_\beta u(t, x) &= g_j(t, x), \ j = 1, \cdots, m, \ (t, x) \in S_0.
\end{align*}
\]

(9.1)

We assume that

\[
\begin{align*}
f, a_{\gamma} &\in C^{2m, \alpha}_-(Q_0) \text{ for each } \gamma \in \mathbb{N}^n \text{ with } |\gamma| \leq 2m, \\
g_j, b_{j\beta} &\in C^{2m+\alpha-m_j, 2m+\alpha-m_j}(S_0) \\
\text{for each } \beta \in \mathbb{N}^n \text{ with } |\beta| \leq m_j, \ j = 1, \cdots, m, \\
u_0 &\in C^{2m+\alpha}(\Omega)
\end{align*}
\]

(9.2)

and moreover that

\[
\begin{align*}
f(t, \cdot) &\text{ satisfy } (4.2)(b)(c) \\
\text{for each } t \in [0, T], \ a_{\gamma}(t, \cdot) &\text{ satisfy } (4.2)(b)(c) \text{ with } \eta_0 \text{ independent on } t.
\end{align*}
\]

(9.3)

Also in this case we look for solutions \( u \in C^{2m+\alpha, 2m+\alpha}_-(Q_0) \): this implies (see [S]) that the following compatibility conditions hold:
\[
\begin{align*}
&\sum_{r=0}^{k} \binom{k}{r} \sum_{|\beta| \leq m_j} \Delta_t^{k-r} b_{j\beta}(0, x) \Delta_x^\beta u^{(r)}(x) = D_t^{k} g_j(0, x), \\
x \in \Gamma; \ k = 0, \ldots, \left[\frac{2m+\alpha-m_j}{2m}\right]; \ j = 1, \ldots, m \\
\text{where we have set for each } x \in \bar{\Omega} \\
u^{(0)}(x) = u_0(x) \\
u^{(k)}(x) = \sum_{s=0}^{k-1} \binom{k-1}{s} \sum_{|\gamma| \leq 2m} \Delta_t^{k-1-s} a_{\gamma}(0, x) \Delta_x^{\gamma} u^{(s)}(x) + D_t^{k-1} f(0, x), \\
k = 1, \ldots, \left[\frac{2m+\alpha}{2m}\right].
\end{align*}
\]

We have also

\[
u^{(k)}(x) = D_t^{k} u(0, x), \ k = 1, \ldots, \left[\frac{2m+\alpha}{2m}\right], \ x \in \bar{\Omega}.
\]

In this section we will prove the following result:

**Theorem 9.1** Let assumptions (9.2)-(9.4) hold. Then problem (9.1) has a unique solution \( u \in C^{2m+\alpha, 2m+\alpha}(Q_0) \) and there exists a continuous and increasing function \( \tilde{c} : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) (depending also on bounds of the norms of \( a_{\gamma}, b_{j\beta} \) and on \( \Omega, \eta_0 \)) such that

\[
\begin{align*}
\left\| u \right\|_{C^{2m+\alpha, 2m+\alpha}(Q_0)} & \\
& \leq \tilde{c}(T) \left\{ \left\| f \right\|_{C^{2m+\alpha, \alpha}(Q_0)} + \left\| u_0 \right\|_{C^{2m+\alpha}(\bar{\Omega})} \\
& \quad + \sum_{j=1}^{m} \left\| g_j \right\|_{C^{\frac{2m+\alpha-m_j}{2m}, \frac{2m+\alpha-m_j}{2m}}(S_0)} \right\}.
\end{align*}
\]

**Proof.** We will prove the existence of \( \delta > 0 \) such that if \( u_0 \in C^{2m+\alpha}(\Omega), \ [t_0, t_1] \subseteq [0, T] \) and \( t_1 - t_0 \leq \delta \), then problem

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\[
D_t v(t, x) = \sum_{|\gamma| \leq 2m} a_\gamma(t, x) D_\gamma^2 v(t, x) + f(t, x),
\]
\[(t, x) \in Q = [t_0, t_1] \times \tilde{\Omega}\]
\[
v(t_0, x) = v_0(x), \quad x \in \tilde{\Omega}
\]
\[
\sum_{|\beta| \leq m} b_{j\beta}(t, x) D_\beta^2 v(t, x) = g_j(t, x), \quad j = 1, \ldots, m,
\]
\[(t, x) \in S = [t_0, t_1] \times \Gamma
\]

has a unique solution \(v \in C^{2m+\alpha}_{2m+\alpha}(Q)\), depending on \(f, v_0\) and \(g_j\) in the sense of estimate (9.6), provided the following compatibility conditions hold

\[
\sum_{r=0}^{k} \binom{k}{r} \sum_{|\beta| \leq m_j} D_t^{k-r} b_{j\beta}(t_0, x) D_\beta^2 v_r(x) = D_t^k g_j(t_0, x);
\]
\[x \in \Gamma; \quad k = 0, \ldots, \left\lfloor \frac{2m+\alpha-2m_j}{2m} \right\rfloor; \quad j = 1, \ldots, m\]
\[(9.8)\]

where we have set for each \(x \in \tilde{\Omega}\)

\[
v_k(x) = \sum_{s=0}^{k-1} \binom{k-1}{s} \sum_{|\gamma| \leq 2m} D_t^{k-1-s} a_\gamma(t_0, x) D_\gamma^2 v_s(x) + D_t^{k-1} f(t_0, x);
\]
\[k = 1, \ldots, \left\lfloor \frac{2m+\alpha}{2m} \right\rfloor.
\]

This will imply the statement because we can first choose \(v_0 = u_0\) and \(t_0 = 0\) and obtain a solution in \([0, \delta] \times \tilde{\Omega}\): if \(\delta < T\), taking \(v_0 = u(\delta)\) and \(t_0 = \delta\) conditions (9.8) are satisfied so that we can extend the solution to \((0, 2\delta) \cap (0, T]) \times \tilde{\Omega}\): after a finite number of steps we obtain the conclusion. As in section 5 we write (9.7) in an abstract form

\[
\begin{aligned}
v'(t) &= A(t)v(t) + f(t), \quad t \in [t_0, t_1] \\
v(t_0) &= v_0 \\
B_j v(t) &= g_j(t), \quad j = 1, \ldots, m; \quad t \in [t_0, t_1]
\end{aligned}
\]

(9.9)

where for each \(t \in [t_0, t_1]\) and \(j = 1, \ldots, m\) we have set:
\[
\begin{aligned}
A(t) &= \sum_{|\gamma| \leq 2m} a_\gamma(t, \cdot) D_\xi^\gamma, \\
B_j(t) &= \sum_{|\delta| \leq m_j} b_{j\delta}(t, \cdot) D_\xi^\delta, \\
f(t) &= f(t, \cdot), g_j(t) = g_j(t, \cdot), v(t) = v(t, \cdot).
\end{aligned}
\]

We will solve (9.9) by a perturbation method by means of the existence results and the sharp estimates of the corresponding time independent case. Setting

\[(9.11) \quad \alpha = 2m\bar{k} + \sigma, \quad \bar{k} \in \mathbb{N}, \quad \sigma \in [0, 2m[\]

we list some consequences of assumptions (9.2).

There exist \(\hat{a} > 0\) depending on \(\Omega, T\) and the norms of \(a_\gamma\) such that:

\[(9.12) \quad \left\{ \begin{array}{l}
(i) \quad A(\cdot) \in C_2^{2m}([0, T]; L(C^{2m}(\bar{\Omega}), C^0(\bar{\Omega}))) \text{ and }
\|A\|_{C_2^{2m}([0, T]; L(C^{2m}(\bar{\Omega}), C^0(\bar{\Omega}))} \leq \hat{a} \\
(ii) \quad \text{for } k = 0, \ldots, \bar{k} \text{ and } t \in [0, T], \\
A^{(k)}(t) \in L(C^{\alpha+2m-2mk}(\bar{\Omega}); C^{\alpha-2mk}(\bar{\Omega})) \text{ and }
\|A^{(k)}(t) - A^{(k)}(r)\|_{C^{\alpha-2mk}(\bar{\Omega})} \\
\leq \hat{a} \{\|\varphi\|_{C^{2m(k-k+1)+\epsilon}(\bar{\Omega})} + (t-r)^{\frac{\epsilon}{2m}}\} \|\varphi\|_{C^{\alpha+2m-2mk}(\bar{\Omega})}
\text{ for each } \varphi \in C^{\alpha+2m-2mk}(\bar{\Omega}) \text{ and } 0 \leq r < t \leq T \\
(iii) \quad \text{for } 0 \leq k \leq \bar{k} \leq h \leq \bar{k} \text{ and } t \in [0, T], \\
A^{(k)}(t) \in L(C^{\alpha+2m-2mh}(\bar{\Omega}); C^{\alpha-2mh}(\bar{\Omega})) \text{ and }
\|A^{(k)}(t)\|_{L(C^{\alpha+2m-2mh}(\bar{\Omega}); C^{\alpha-2mh}(\bar{\Omega}))} \leq \hat{a}
\end{array} \right\}
\]

and for each \(j = 1, \ldots, m\) there is \(\hat{b}_j > 0\), depending on \(\Omega, T\) and the norms of \(b_{j\delta}\) such that
Let us set

\[
Y = \{ v \in C^{2m+\alpha}([t_0, t_1]; C(\bar{\Omega})), \quad v^{(k)} \in B([t_0, t_1]; C^{2m+\alpha - 2mk}(\bar{\Omega})), \\
\quad v^{(k)}(t_0) = v_k, \quad k = 0, \ldots, \bar{k} + 1 \}
\]

where \( v_k \) is defined in (9.8). For each \( v \in Y \) let us consider the perturbed problem

\[
\begin{align*}
& u'(t) = A(t_0)u(t) + [A(t) - A(t_0)]v(t) + f(t), \quad t \in [t_0, t_1] \\
& u(t_0) = v_0 \\
& B_j(t_0)u(t) = [B_j(t_0) - B_j(t)]v(t) + g_j(t), \quad t \in [t_0, t_1].
\end{align*}
\]

Setting for \( t \in [t_0, t_1] \):

\[
\begin{align*}
\varphi(t) &= [A(t) - A(t_0)]v(t) + f(t) \\
\psi_j(t) &= [B_j(t_0) - B_j(t)]v(t) + g_j(t), \quad j = 1, \ldots, m
\end{align*}
\]
and recalling that for \( j = 1, \ldots, m \) we have

\[
(9.17) \quad Y \subseteq C^{2m}(\{t_0, t_1\}; C^{2m}(\tilde{\Omega}))) \cap C^{\frac{2m + \alpha - m_j}{2}}(\{t_0, t_1\}; C^{m_j}(\Omega))
\]

we get

\[
(9.18) \quad \begin{cases}
\varphi \in C^{2m}(\{t_0, t_1\}; C^0(\tilde{\Omega})), \\
\varphi^{(k)} \in B(\{t_0, t_1\}; C^{\alpha - 2mk}(\tilde{\Omega})) \text{ for } k = 0, \ldots, \bar{k}
\end{cases}
\]

\[
(9.19) \quad \begin{cases}
\psi_j \in C^{\frac{2m + \alpha - m_j}{2m}}(\{t_0, t_1\}; C^0(\Gamma)), \\
\psi_j^{(k)} \in B(\{t_0, t_1\}; C^{\alpha - m_j - 2m(k-1)}(\Gamma)) \\
\quad \text{ for } k = 0, \ldots, \left[\frac{1 + \alpha - m_j}{2m}\right] \text{ and } j = 1, \ldots, m.
\end{cases}
\]

Therefore the assumptions of theorem 4.1 are satisfied by the data \( \varphi, \psi, v_0 \), because also the compatibility conditions (5.3) are verified: in fact we have

\[
(9.20) \quad \begin{cases}
\varphi^{(k)}(t_0) = \sum_{r=1}^{k} \binom{k}{r} A^{(k-r)}(t_0)v_r + f^{(k)}(t_0), \quad k = 1, \ldots, \left[\frac{\alpha}{2m}\right] \\
\psi_j^{(k)}(t_0) = -\sum_{r=1}^{k} \binom{k}{r} B^{(k-r)}(t_0)v_r + g_j^{(k)}(t_0), \quad j = 1, \ldots, m; \\
\quad \text{ for } k = 1, \ldots, \left[\frac{2m + \alpha - m_j}{2m}\right]
\end{cases}
\]

so that setting

\[
(9.21) \quad \begin{cases}
u^{(0)} = v_0 \\
u^{(k)} = A(t_0)u^{(k-1)} + \varphi^{(k-1)}(t_0), \quad k = 1, \ldots, \left[\frac{2m + \alpha}{2m}\right]
\end{cases}
\]

we get from (9.20) \( u^{(k)} = v_k, k = 0, \ldots, \left[\frac{2m + \alpha}{2m}\right] \), and so from (9.8) and (9.20) we deduce, for \( j = 1, \ldots, m \) and \( k = 0, \ldots, \left[\frac{2m + \alpha - m_j}{2m}\right] \):

\[
B_j(t_0)u^{(k)} - \psi_j^{(k)}(t_0) = B_j(t_0)v_k + \sum_{r=1}^{k} \binom{k}{r} B_j^{k-r}(t_0)v_r - g_j^{(k)}(t_0) = 0
\]
so that (5.3) holds. We are now able to apply theorem 5.1 and deduce the existence of a solution \( u \in C^{\frac{2m+\alpha}{2m},2m+\alpha}(Q) \) of problem (9.15). Let us define

\[
(9.22) \begin{cases}
    \mathcal{T} : Y \to Y \\
    \mathcal{T}v = u, \text{ where } u \text{ is the solution of (9.15)}.
\end{cases}
\]

We will prove that \( \mathcal{T} \) is a contraction on \( Y \) in the norm \( ||| \cdot |||_{C^{\frac{2m+\alpha}{2m},2m+\alpha}(Q)} \) provided \( t_1 - t_0 \) is sufficiently small. Let us set \( ||| \cdot |||_{C^{\frac{2m+\alpha}{2m},2m+\alpha}(Q)} = ||| \cdot |||_\alpha \) and \( ||| \cdot |||_{C^{\frac{2m+\alpha}{2m},2m+\alpha}(S)} = ||| \cdot |||_\alpha \).

First from (2.13) we get

\[
(9.23) \sum_{h=0}^{[2m+\alpha]} \|v\|_{C^{\frac{2m+\alpha-h}{2m},(t_0,t_1);C^h(\Omega)}} \leq c_2 \|v\|.
\]

Now let \( w_1, w_2 \in Y \), and set for \( i = 1, 2, j = 1, \ldots, m \) and \( t \in [t_0, t_1] \):

\[
(9.24) \begin{cases}
    \varphi_i(t) = [A(t) - A(t_0)]w_i(t) \\
    \psi_{ij}(t) = [B_j(t_0 - B_j(t)]w_i(t).
\end{cases}
\]

From estimate (5.8) we obtain

\[
(9.25) \begin{cases}
    |||\mathcal{T}w_1 - \mathcal{T}w_2|||_{2m+\alpha} \\
    \leq c(T)\{|||\varphi_1 - \varphi_2|||_\alpha + \sum_{j=1}^{m} |||\psi_{1j} - \psi_{2j}|||_{2m+\alpha-m_j}\}.
\end{cases}
\]

Recalling that

\[
(9.26) \begin{cases}
    |||\varphi|||_\alpha = \sum_{k=0}^{k} \|\varphi^{(k)}\|_{B([t_0,t_1];C^{\alpha-2mk}(\Omega))} \\
    + [\varphi^{(k)}]_{C^{\frac{2m+\alpha}{2m},\alpha}(S)}([t_0,t_1];C^\alpha(\Omega)).
\end{cases}
\]
we consider first the case $k = 0$ i.e. $\alpha < 2m$. From now on we denote by $c$
a generic function independent on $[t_0, t_1] \subseteq [0, T]$ and on $w_1, w_2 \in Y$, and by
$\rho : \mathbb{R}_+ \to \mathbb{R}_+$ a continuous function with $\rho(0) = 0$.

From (9.12)(i)(ii), (9.13)(i)(ii), and (9.14)(i)(ii) we get

\begin{align}
(9.27) \quad |||\varphi_1 - \varphi_2|||_\alpha & \leq c \cdot \rho(t_1 - t_0)|||w_1 - w_2|||_{2m+\alpha} \\
(9.28) \quad |||\psi_{1j} - \psi_{2j}|||_{2m+\alpha - m_j} & \leq c \cdot \rho(t_1 - t_0)|||w_1 - w_2|||_{2m+\alpha}, \; j = 1, \ldots, m
\end{align}

and so from (9.25) we deduce that for $t_1 - t_0$ small $T$ is a contraction; hence
it has a unique fixed point $v \in Y$ which is the unique solution $v$ of (9.9) such
that $v \in C^{\frac{2m+\alpha}{2m}}([t_0, t_1]; C(\Omega))$ and $v^{(k)} \in B([t_0, t_1]; C^{2m+\alpha - 2mk}(\Omega))$ for $k = 0, \ldots, 1 + \lfloor \frac{\alpha}{2m} \rfloor$. By using again (iv) of theorem 5.2 we get $u \in C^{\frac{2m+\alpha}{2m} - 2m+\alpha}(Q)$,
and the conclusion follows. \hfill \square

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