We characterize lines of reversibility for the centre-problem by using their slope as parameter. As a result of our method we formulate in a rather concise way the conditions for reversibility for cubic systems with nonvanishing quadratic terms.

1 Introduction

Consider a system of differential equations of the form

\[
\begin{align*}
\dot{x} &= y + q(x, y) \\
\dot{y} &= -x - p(x, y)
\end{align*}
\]

where \(p, q\) are polynomials whose terms of lowest order are of degree at least two. A well-known sufficient condition, due to Poincaré, for the origin to be a centre is that the system be reversible with respect to a line \(L\), which passes through the origin, i.e. that the system be invariant under a reflection in the line \(L\), and under a simultaneous reversal of the independent variable \(t\). Thus system (1.1) is reversible with respect to the line \(x = 0\) if and only if it is invariant under the transformation \((x, y, t) \rightarrow (-x, y, -t)\), i.e. if and only if \(q(-x, y) = q(x, y)\) and \(p(-x, y) = -p(x, y)\). Thus \(q\) contains only even powers of \(x\) and \(p\) only odd ones. Reversibility with respect to \(y = 0\) is thus equivalent to \(q(x, -y) = -q(x, y), p(x, -y) = p(x, y)\), i.e. \(q\) contains only odd powers of \(y\) and \(p\) only even ones. As for a general line \(L\) one can apply a rotation which transforms \(L\) into the line \(x = 0\) or \(y = 0\) and a criterion for reversibility may then readily be attained [1]. Collins, by using tensor-calculus, derives in [2] a necessary and sufficient condition for the existence of such a line without involving its unknown equation.

Here we discuss another method which neither uses purely orthogonal transformations nor tensor calculus. By a suitable change of variables we reduce the problem of finding a line of reversibility \(y = \frac{1}{m}x\) with \(m \in \mathbb{R} - \{0\}\) to the question whether the system is reversible to \(y = 0\). This access therefore leaves out the coordinate-axes as possible lines of reversibility. As mentioned above it is however easy to decide if one of the coordinate axes is a line of reversibility. If we write

\[
\begin{align*}
p &= p_2 + p_3 + \ldots + p_N \\
q &= q_2 + q_3 + \ldots + q_N
\end{align*}
\]

with homogeneous polynomials \(p_i, q_i\) of degree \(i\) it turns out that possible lines of reversibility are already determined by the quadratic case \(\dot{x} = y + q_2, \dot{y} = -x - p_2\) provided \((p_2, q_2) \neq (0, 0)\). The values of \(m\) found there have to be inserted into polynomial equations corresponding to \(p_2, q_2, \ldots, p_N, q_N\). These equations then provide the necessary and sufficient conditions for (1.1) to have a line of reversibility. The coefficients of the polynomial equations linearly depend on the coefficients of \(p_3, q_3, \ldots, p_N, q_N\) respectively. We use this method to discuss the case

\[
\begin{align*}
p &= p_2 + p_3 \\
q &= q_2 + q_3
\end{align*}
\]

for \((p_2, q_2) \neq (0, 0)\)

and to bring the conditions for the existence of a line of reversibility into a manageable form.
2 Reversibility in Polynomial Systems

In what follows we frequently use instead of (1.1) the single equation
\[
y' = -\frac{x + p(x, y)}{y + q(x, y)}
\]  
(2.1)
as done in [3] or [4]. For \( m \neq 0 \) we employ the linear transformation of variables.
\[
\begin{align*}
\xi &= y - mx, \\
\eta &= y + \frac{1}{m}x 
\end{align*}
\]  
(2.2)
or
\[
\begin{align*}
x &= \frac{m}{m^2 + 1} (\eta - \xi) = \varphi(\xi, \eta), \\
y &= \frac{m^2}{m^2 + 1} \eta + \frac{1}{m^2 + 1} \xi = \psi(\xi, \eta).
\end{align*}
\]  
(2.3)
We set \( \Phi(\xi, \eta) = (\varphi(\xi, \eta), \psi(\xi, \eta))^T \) with \( T \) for transposition. \( \Phi^{-1} \) consists of a rotation and a stretching of the \( x, y \)-coordinates. So does \( \Phi \) but in opposite order. Thus reversibility with respect to a line is a property which is invariant under \( \Phi^{-1} \) and \( \Phi \). Then (2.1) becomes
\[
\eta' = -\frac{\xi + (q - mp) \circ \Phi}{m^2 \eta + m(mp + p) \circ \Phi}.
\]  
(2.4)
Now we arrive at

**Theorem 2.1:** (2.1) has a centre at \((0, 0)\) with line of reversibility \( y = -\frac{1}{m}x(m \neq 0) \) if and only if each
\[
\begin{align*}
(q_i - mp_i) \circ \Phi \text{ contains only even powers of } \eta \text{ and each} \\
(mq_i + p_i) \circ \Phi \text{ contains only odd powers of } \eta, \quad 2 \leq i \leq N.
\end{align*}
\]  
(2.5)

**Proof:**
If (2.4) satisfies (2.5), then (2.4) has a centre at \((0, 0)\) with line of reversibility \( \eta = 0 \). Thus (2.1) has a centre at \((0, 0)\) with line of reversibility \( y = -\frac{1}{m}x \). If conversely (2.1) has a centre at \((0, 0)\) with line of reversibility \( y = -\frac{1}{m}x \), then (2.4) has so with line of reversibility \( \eta = 0 \). Consequently (2.5) is satisfied. \( \square \)

(2.5) can be transformed into a more explicit form.

**Theorem 2.2:** (2.5) is equivalent to \( N - 1 \) matrix equations
\[
\begin{align*}
\mathcal{L}_{i+1}(p_i, q_i) \left( \begin{array}{c}
m \\
m^2 \\
\vdots \\
m^{i+1}
\end{array} \right) &= \left( \begin{array}{c}
b_1(p_i, q_i) \\
b_2(p_i, q_i) \\
\vdots \\
b_i+1(p_i, q_i)
\end{array} \right), \quad 2 \leq i \leq N,
\end{align*}
\]  
(2.6)
where \( \mathcal{L}_{i+1}(p_i, q_i) \), \((b_1(p_i, q_i), \ldots, b_{m+1}(p_i, q_i))^T\) are \((i+1) \times (i+1), \) \((i+1) \times 1\) matrices respectively whose coefficients linearly depend on the coefficients of \( p_i, q_i \).

**Proof:**
We have to evaluate \((q_i - mp_i) \circ \Phi, (mq_i + p_i) \circ \Phi\). These expressions are of the form
\[
\frac{1}{(m^2 + 1)^i} \sum_{k,l,k+l=i} (q_{ikl} - mp_{ikl}) \sum_{j=0}^{k} \binom{k}{j} m^j \eta^{k-j} (-\xi)^j \cdot \sum_{q=0}^{l} \binom{l}{q} m^{2(l-q)} \eta^{l-q} \xi^q =
\]
\[
= \frac{m^i}{(m^2 + 1)^i} \sum_{\lambda=0}^{i} \eta^{i-\lambda} \sum_{k,l,k+l=i} (q_{ikl} - mp_{ikl}) m^l \cdot \sum_{j,q,j+q=\lambda \atop j \leq \min(k,l)} \sum_{q=\min(j,l)}^{\lambda} m^{-2q} (-\xi)^j \xi^q \binom{k}{j} \binom{l}{q}
\]
if we set \( \lambda = j + q \) and if \( p_{ikl}, q_{ikl} \) denote the coefficients of \( p_i, q_i \) respectively. If \( i - \lambda \) is odd, the coefficient of \( \eta^{i-\lambda} \) has to vanish. If \( i \) is odd the values
furnish the powers in question. For \( \lambda = 0 \) the largest occurring power of \( m \) is \( 2i + 1 \), the smallest one \( i \). For \( \lambda = 2 \) we obtain \( 2i - 1 \) as largest one and \( i - 2 \) as smallest one and so on. Dividing by \( m, m^{-2}, \ldots \) and multiplying by \((m^2 + 1)^i\) we end up with \((i + 1)/2\) polynomials in \( m \) of degree \( i + 1 \) which have to vanish. If \( i \) is even the values
\[
\lambda = i - 1, i - 3, \ldots, 0
\]
furnish the powers in question. For \( \lambda = 1 \) the largest occurring power of \( m \) is \( 2i \), the smallest one is \( i - 1 \). Observe that these values are assumed for \( l = i - 1, q = 0 \) and \( l = 1, q = 1 \). For \( \lambda = 3 \) we obtain \( 2i - 2 \) as largest one and \( i - 3 \) as smallest one and so on. Dividing by \( m^{i-1}, m^{i-3}, \ldots \) and multiplying by \((m^2 + 1)^i\) we arrive at \( i/2 \) polynomials in \( m \) of degree \( i + 1 \) which have to vanish. As for \( (mq_i + p_i) \circ \Phi \), the coefficients of even powers of \( \eta \) have to vanish. The calculations are very similar to the preceding ones. If \( i \) is odd we again obtain \((i + 1)/2\) polynomials in \( m \) of degree \( i + 1 \) which have to vanish; if \( i \) is even we arrive at \( i/2 + 1 \) polynomials in \( m \) which have to vanish. \( \square \)

The systems (2.6) have to be considered as necessary and sufficient conditions on the coefficients of \( p_i, q_i \) for the existence of a line of reversibility different from the coordinate-axes. This can be seen as follows. If \((p_j, q_j)\) is the first pair where \( p_j, q_j \) do not vanish identically we can find the possible values of \( m \) from (2.6) for \( i = j \) in terms of the coefficients of \( p_j, q_j \). These then have to be inserted into (2.6) for \( i = j, \ldots, N \). For instance let us assume that in
\[
\mathcal{L}_{j+1}(p_j, q_j) = (l_{ik})_{i=1, \ldots, j+1}
\]
the matrix
\[
(l_{ik})_{i=2, \ldots, j+1} \text{ has rank } j
\]
then we can possibly obtain the value of \( m \) from the first row of (2.6, \( i = j \)). At least this is so if \( \mathcal{L}_{j+1}(p_j, q_j) \) has rank \( j + 1 \). This value of \( m \) if \( \neq 0 \) then has to be inserted into the remaining equations in (2.6). It is an expression in the coefficients of \( p_j, q_j \), Thus we obtain the necessary and sufficient conditions for the existence of a line of reversibility different from the coordinate-axes. In the example to follow in the next section we will see that in more detail.

### 3 Cubic Systems with Nonvanishing Quadratic Parts

Let us consider
\[
y' = -\frac{x + p_2 + p_3}{y + q_2 + q_3}
\]
with
\[
\begin{align*}
p_2 &= \tilde{a}x^2 + (2\tilde{b} + \alpha)xy + \tilde{c}y^2, \\
q_2 &= \tilde{b}x^2 + (2\tilde{c} + \beta)xy + \tilde{d}y^2, \\
p_3 &= ax^3 + bx^2y + cxy^2 + dy^3, \\
q_3 &= Ax^3 + Bx^2y + Cxy^2 + Dy^3.
\end{align*}
\]
Here we adopted the usual notation for the quadratic parts \( p_2, q_2 \) (cf. [3, 4, 5]). The conditions (2.6, \( i = 2, 3 \)) read as follows.
\[
m^3(2\tilde{b} + \alpha) + m^2(-4\tilde{c} + \beta + 2\tilde{a}) + m(-4\tilde{b} + \alpha + 2\tilde{d}) = -(2\tilde{c} + \beta),
\]
\[
m^3\tilde{b} + m^2(-2\tilde{c} + \beta + \tilde{a}) + m(-2\tilde{b} + \alpha + \tilde{d}) = -\tilde{c},
\]
\[
m^3\tilde{d} + m^2((2\tilde{c} + \beta) + \tilde{c}) + m((2\tilde{b} + \alpha) + \tilde{b}) = -\tilde{a},
\]
\[
-m^4d + m^3(D - c) + m^2(C - b) + m(B - a) = -A
\]
\[-m^4 b + m^3 (B - (3a - 2c)) + m^2 (3A - 2C - (3d - 2b)) + m(3D - 2B - c) = -C, \]  
\[-m^4 C + m^3 (3D - 2B - c) + m^2 (2C - 3A - (2b - 3d)) + m(B - (3a - 2c)) = -b, \]  
\[-m^4 A + m^3 (B - a) + m^2 (-C + b) + m(D - c) = -d. \]  
(3.5, 3.6, 3.7) stem from \(i = 2, (3.4, 3.5, 3.6, 3.7) \) from \(i = 3. \) We start with \(i = 2. \) Then (3.1, 3.2, 3.3) are equivalent to

\[
m^3 \alpha + m^2 \beta + m \alpha + \beta = 0
\]  
\[
m^2 (\hat{b} + \hat{d}) + m^2 (\hat{a} + \hat{c}) + m(\hat{b} + \hat{d}) + \hat{a} + \hat{c} = 0
\]  
\[
m^3 \hat{d} + m^2 ((2\hat{c} + \beta) + \hat{c}) + m((2\hat{b} + \alpha) + \hat{\beta}) + \hat{a} = 0
\]  
(3.8, 3.9, 3.10)

For further treatment we introduce the vector

\[
a = (\hat{a} + \hat{c}, \hat{b} + \hat{d}, \alpha, \beta) \in \mathbb{R}^4
\]

If \(a \) has only nonvanishing components (3.8, 3.9) admit within \(R - \{0\} \) only the solutions \(-\frac{\beta}{\alpha}, -\frac{\hat{a} + \hat{c}}{\beta + \hat{d}}\) respectively. Thus we obtain as necessary and sufficient conditions for the solvability of (3.1, 3.2, 3.3) the relations

\[
\beta(\hat{b} + \hat{d}) = \alpha(\hat{a} + \hat{c}), \tag{3.11}
\]

\[
-\beta^2 \hat{d} + \alpha \beta^2 (3\hat{c} + \beta) - \alpha^2 \beta (3\hat{b} + \alpha) + \alpha^2 \hat{a} = 0. \tag{3.12}
\]

(3.11, 3.12) coincide with condition II in [5, p. 13].

Inserting \(m = -\frac{\hat{d}}{\hat{d}}\) into (3.4, ..., 3.7) we obtain together with (3.11, 3.12) the necessary and sufficient conditions for the existence of a line of reversibility, different from the coordinate-axes.

We briefly discuss the other possibilities for \(a \). If \(a \neq 0 \) there are only two cases where we may have a line of reversibility different from the coordinate-axes, namely

\[
\hat{a} + \hat{c} \neq 0, \quad \hat{b} + \hat{d} \neq 0, \quad \alpha = 0, \quad \beta = 0,
\]
\[
\hat{a} + \hat{c} = 0, \quad \hat{b} + \hat{d} = 0, \quad \alpha \neq 0, \quad \beta \neq 0;
\]
then \(m = -\frac{\hat{a} + \hat{c}}{\hat{b} + \hat{d}}\) in the first case and then necessary and sufficient conditions for the existence of a line of reversibility as above are

\[
-(\hat{a} + \hat{c})^2 \hat{d} + 3(\hat{b} + \hat{d})(\hat{a} + \hat{c})^2 \hat{c} - 3(\hat{b} + \hat{d})^2 (\hat{a} + \hat{c})\hat{b} + (\hat{b} + \hat{d})^3 \hat{a} = 0,
\]

\[(3.4, \ldots, 3.7) \text{ with } m = -\frac{\hat{a} + \hat{c}}{\hat{b} + \hat{d}}. \]

In the second case we have \(m = -\frac{\beta}{\alpha} \) and an analogous result. It remains to deal with \(a = 0 \). In this case we are left with

\[
m^3 \hat{d} + 3m^2 \hat{c} - 3m\hat{d} - \hat{c} = 0.
\]

If \(\hat{d} \neq 0 \) we obtain three distinct real solutions \(m_1, m_2, m_3 \) since the discriminant is \(< 0 \). If \(\hat{c} \neq 0 \) these solutions do not vanish and \(y = -\frac{1}{m} x \) is a line of reversibility if and only if \((3.4, \ldots, 3.7) \) are satisfied with \(m = m_1 \). Since \(m_1, m_2, m_3 \) can be computed by means of Cardano’s formula we arrive thus at the necessary and sufficient conditions for the existence of a line of reversibility different from the coordinate axes. If \(\hat{d} \neq 0, \hat{c} = 0 \) one of \(m_i \) vanishes, say \(m_3 \). For \(m_1 = \sqrt{3}, m_2 = -\sqrt{3} \) the conclusion before holds. The case \(\hat{d} = 0, \hat{c} \neq 0 \) furnishes two roots, namely \(m_1 = \frac{1}{\sqrt{3}}, m_2 = -\frac{1}{\sqrt{3}} \) and we can proceed as before. The case \(\hat{d} = \hat{c} = 0 \) implies \((p_2, q_2) = (0, 0) \) since \(a = 0 \). It therefore contradicts our assumption.
References


