Existence and Stability of Static Shells for the Vlasov-Poisson System

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Abstract
We prove the existence and stability of stationary solutions to the Vlasov-Poisson System with spherical symmetry, which describe static shells, i.e., the support of their densities is bounded away from the origin. We use a variational approach which was established by Y. Guo and G. Rein.

1 Introduction
In stellar dynamics, the evolution of a large ensemble of particles (e.g. stars) which interact only by their self-consistent, self-generated gravitational field, is described by the Vlasov-Poisson system

$$\frac{\partial}{\partial t}f + v \cdot \nabla_x f - \nabla_x U \cdot \nabla_v f = 0,$$

$$\Delta U = 4\pi \rho, \quad \lim_{|x| \to \infty} U(t,x) = 0,$$

$$\rho(t,x) = \int f(t,x,v)dv.$$ (1.1) (1.2) (1.3)

Here $f = f(t,x,v) \geq 0$ is the phase-space density, where $t \in \mathbb{R}$ denotes time, and $x,v \in \mathbb{R}^3$ denote position and velocity. $U = U(t,x)$ is the gravitational potential of the ensemble, and $\rho = \rho(t,x)$ is its spatial density.

In this paper we examine existence and stability of steady states of this system. We are interested in stationary solutions of the form

$$f(x,v) = (E_0 - E)^k (L - L_0)^l,$$ (1.4)

with $-1 < l$ and $0 < k < l + \frac{3}{2}$. Here $(\cdot)_+$ denotes the positive part, $E_0, L_0$ are constants,

$$E = \frac{1}{2} |v|^2 + U(x).$$
is the particle energy which is conserved along characteristics of the Vlasov-equation (1.1) if \( U \) is time-independent and

\[
L = |x \times v|^2 = |x|^2|v|^2 - (x \cdot v)^2
\]

(1.5)
denotes the modulus of angular momentum squared which is conserved, if \( U \) is spherically symmetric.

The existence of steady states of form (1.4) was established in [4], where the ansatz (1.4) was plugged into the Poisson-equation (1.2) and the solvability of the resulting semilinear elliptic equation was shown. These stationary solutions are spherically symmetric, i.e.,

\[
f(x,v) = f(Ax,Av) \quad \forall A \in O(3),
\]

(1.6)
where \( O(3) \) is the group of orthogonal \( 3 \times 3 \) matrices. For \( L_0 > 0 \) the support of the induced spatial density \( \rho(x) = \rho(|x|) \) is contained in some interval \([R_1, R_2]\) with \( R_1 > 0 \) and the steady state describes a shell.

The ansatz (1.4) also leads to steady states and shells of the Vlasov-Einstein system, the general relativistic pendant of the Newtonian Vlasov-Poisson system, and they provide an access to study stability and critical phenomena numerically, cf. [1].

We examine the shells in the Newtonian framework and we use a similar approach as in [2], where existence and stability of the above steady states was shown in the case \( L_0 = 0 \). We briefly sketch the main ideas:

The Vlasov-Poisson system is conservative, i.e., the total energy

\[
H(f) := E_{\text{kin}}(f) + E_{\text{pot}}(f) := \frac{1}{2} \int |v|^2 f(x,v) dv dx - \frac{1}{8\pi} \int |\nabla U_f(x)|^2 dx
\]

(1.7)
of a state \( f \) is conserved along solutions and hence is a natural candidate for a Lyapunov function in a stability analysis; \( U_f \) denotes the potential induced by \( f \). However, the energy does not have critical points, but for any reasonable function \( \Phi \) the so-called Casimir functional

\[
\mathcal{C}(f) := \iint \Phi(f(x,v)) dv dx
\]

is conserved as well. Now one tries to minimize the energy-Casimir functional

\[
\mathcal{H}_C := H + \mathcal{C}
\]

in the class of allowed perturbations \( \mathcal{F}_M \), which consists of positive \( L^1(\mathbb{R}^6) \)-functions with prescribed mass \( M \), i.e. \( \iint f = M \) and an integrability condition.

The aim is to prove that a minimizer \( f_0 \) is a stationary solution of (1.1) – (1.3) and to deduce its stability. One of the main difficulties is to show that the mass remains concentrated along a minimizing sequence, a fact which is essential in order to prove that a function, constructed as the weak limit of the sequence, indeed is a minimizer.
We are only able to show stability against spherically symmetric perturbations, because our approach requires an $L$-dependence in the Casimir functional, more precisely, we define

$$C(f) := \int_{\mathbb{R}^3} \Phi((L-L_0)^{-1} f(x,v)) (L-L_0)^l_+ dv,$$  \hspace{1cm} (1.8)

with $-1 < l$ as in (1.4), $\Phi$ convex, satisfying certain growth conditions, and this will be a conserved quantity for spherically symmetric $f$ only. To simplify our presentation, we focus on the case

$$\Phi(f) = f^{1+1/k}$$

which will lead to stationary solutions of the form (1.4). The Casimir functional then reads

$$C(f) := \int_{\mathbb{R}^2} f^{1+1/k}(x,v)(L-L_0)^{-l/k} dv.$$  \hspace{1cm} (1.9)

At one point we need a scaling argument, which gets complicated in the case of a translation in $L$ in the Casimir-functional. Here we exploit the spherical symmetry and use coordinates adapted to it: If $f = f(x,v)$ is spherically symmetric, we have

$$f(x,v) = \tilde{f}(r,w,L),$$

with $r = |x|, w = \frac{x}{|x|}$ and $L$ as in (1.5), see Section 3. Altogether, we want to minimize the energy-Casimir functional

$$\mathcal{H}_C(f) = E_{\text{kin}}(f) + E_{\text{pot}}(f) + C(f),$$

with $E_{\text{kin}}, E_{\text{pot}}$ from (1.7) and $C(f)$ as in (1.9) over the set

$$\mathcal{F}_M := \left\{ f \in L^1(\mathbb{R}^3) \mid f \geq 0, f \text{ is spherically symmetric, } \int\int f = M, E_{\text{kin}}(f) + C(f) < \infty, f(x,v) = 0 \text{ a.e. for } 0 \leq L < L_0 \right\},$$  \hspace{1cm} (1.10)

see (1.6) for the definition of spherical symmetry.

Our paper is organized as follows: In the next Section, we show that $\mathcal{H}_C$ is bounded from below. Then we prove a scaling property and that the mass remains concentrated along a minimizing sequence in Section 3. In Section 4 and 5 we show the existence of a minimizer and analyse its properties; it is a stationary solution, and it is nonlinearly stable against spherically symmetric perturbations.

## 2 A lower bound on $\mathcal{H}_C$

We want to establish a lower bound on $\mathcal{H}_C$ and we will need several estimates for $\rho_f$ and $U_f$ induced by an element $f \in \mathcal{F}_M$. We will show that $E_{\text{pot}}(f)$ makes sense, that is, $\nabla U_f \in L^2(\mathbb{R}^3)$. 

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Lemma 2.1 Let $n := k + l + \frac{3}{2}$. Then there exists $C > 0$, such that

$$\int \rho_f^{1 + \frac{1}{n}}(x)|x|^{-2l/n}dx \leq C(f + E_{\text{kin}}(f)), \quad f \in \mathcal{F}_M.$$ 

Proof. For any $R > 0$, we have

$$\rho_f(x) = \int f(x,v)dv$$

$$= \int_{|v| \leq R} f(x,v)dv + \int_{|v| \geq R} f(x,v)dv$$

$$\leq \int_{|v| \leq R} (L - L_0)^{1/k} f(x,v)(L - L_0)^{-1/k} dv + \frac{1}{R^2} \int |v|^2 f(x,v)dv$$

$$\leq C \left( \int_{|v| \leq R} (L - L_0)^{1/k} dv \right)^{1/k} \left( \int f^{1 + \frac{1}{n}}(x,v)(L - L_0)^{-1/k} dv \right)^{1/n}$$

$$+ \frac{1}{R^2} \int |v|^2 f(x,v)dv$$

$$\leq C |x|^{\frac{2n}{n+1}} R^{\frac{2n}{n+2}} \left( \int f^{1 + \frac{1}{n}}(x,v)(L - L_0)^{-1/k} dv \right)^{\frac{n}{n+1}}$$

$$+ \frac{1}{R^2} \int |v|^2 f(x,v)dv.$$ 

Optimization in $R$ yields

$$\rho_f(x) \leq C|x|^{\frac{2n}{n+1} + \frac{3}{2}} \left( \int |v|^2 f(x,v)dv + \int f^{1 + \frac{1}{n}}(x,v)(L - L_0)^{-1/k} dv \right)^{\frac{n}{n+1}}.$$ 

Taking both sides of the inequality to the power $1 + \frac{1}{n}$, dividing by $r^{\frac{3}{n}}$ and integrating with respect to $x$ proves the assertion. \qed

From Lemma 2.1 we see that a function $f$ lying in $\mathcal{F}_M$ and its induced density $\rho_f$ automatically are elements of certain Banach spaces which we now define:

$$L^{k,l}(\mathbb{R}^6) := \left\{ f: \mathbb{R}^6 \to \mathbb{R} \text{ measurable, spherically symmetric and} \right.$$ \n
$$\left. \int \int f^{1 + \frac{1}{k}}(L - L_0)^{-1/k} dx dv < \infty \right\}$$

equipped with the norm

$$\|f\|_{k,l} := \left( \int \int f^{1 + \frac{1}{k}}(L - L_0)^{-1/k} dx dv \right)^{\frac{1}{n+1}}$$

and

$$L^{n,l}(\mathbb{R}^3) := \left\{ \rho: \mathbb{R}^3 \to \mathbb{R} \text{ measurable, spherically symmetric and} \right.$$ \n
$$\left. \int \rho^{1 + \frac{1}{n}} |x|^{-2l/n} dx < \infty \right\}$$

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with norm
\[ \|\rho\|_{n,l} := \left( \int \rho^{1+\frac{1}{n}} |x|^{-2l/n} \, dx \right)^{\frac{n}{n+1}}. \]

Both spaces are reflexive Banach spaces. More precisely, \( f \) and \( \mu_f \) are contained in the subsets \( L^{k,l}_+(\mathbb{R}^n) \) and \( L^{n,l}_+(\mathbb{R}^3) \), respectively, which consist of the a.e.-nonnegative functions of these spaces.

**Lemma 2.2** For \( \rho \in L^{n,l}(\mathbb{R}^3) \) define
\[
m_\rho(r) := \int_{|x|\leq r} \rho(x) \, dx = 4\pi \int_0^r s^2 \rho(s) \, ds. \tag{2.1}
\]

Then the following holds:

(a) There exist constants \( C > 0 \) and \( q > 0 \) such that for \( \rho \in L^{n,l}(\mathbb{R}^3) \) with \( \int \rho(x) \, dx = M \) we have
\[
-E_{pot}(\rho) := \frac{1}{8\pi} \int |\nabla U_\rho|^2 \, dx \\
\leq \frac{1}{2} \int_0^R \frac{m_\rho^2(r)}{r^2} \, dr + \frac{M^2}{2R} \\
\leq CR^q(1 + \|\rho\|_{n,l}^{1+\frac{1}{n}}) + \frac{M^2}{2R}, \quad R > 0
\]

where \( U_\rho \) denotes the potential induced by \( \rho \).

(b) For every \( R > 0 \) the mapping
\[ T : L^{n,l}(\mathbb{R}^3) \ni \rho \mapsto \frac{m_\rho}{r}|_{[0,R]} \in L^2([0,R]) \]

is compact.

(c) For \( \rho_1, \rho_2 \in L^{n,l}(\mathbb{R}^3) \cap L^1(\mathbb{R}^3) \) we have
\[ \int \nabla U_{\rho_1} \cdot \nabla U_{\rho_2} \, dx = -4\pi \int U_{\rho_1} \rho_2 \, dx. \]

**Proof.** From spherical symmetry, we have \( U_{\rho}(x) = U_{\rho}(|x|) \) and
\[ \nabla U_{\rho}(x) = U'_{\rho}(r) \cdot \frac{x}{r} = \frac{m_\rho(r)}{r^2} \cdot \frac{x}{r}, \]
where \( r = |x| \) and \( ' \) denotes the derivative with respect to \( r \). Together with \( m_\rho(r) \leq M \), this shows the first estimate of (a). Now for \( \rho \in L^{n,k}(\mathbb{R}^3) \), Hölder’s inequality implies
\[ |m_\rho(r)| \leq Cr^{(2l+3)/(n+1)}\|\rho\|_{n,l}, \quad r \geq 0, \]

and thus
\[ \int_0^R \frac{m^2(r)}{r^2} dr \leq C \| \rho \|_{n,l}^2 R^{(4l+5-n)/(n+1)}. \]
Then the estimate in (a) follows for \( n \leq 1 \). For \( n > 1 \) we have
\[ \int_0^R \frac{m^2(r)}{r^2} dr \leq M_1^{-\frac{1}{2}} \int_0^R \frac{m^{1+\frac{1}{n}}(r)}{r^2} dr \leq C \| \rho \|_{n,l} R^2 R^{(2l+3-n)/n}. \]
Since both powers of \( R \) are positive, this completes the proof of (a). As to (b), the compactness of \( T \) can be shown using the Fréchet-Kolmogorov criterion, cf. [2], and (c) follows by an integration by parts. □

**Lemma 2.3** There exists a constant \( C > 0 \), such that
\[ \mathcal{H}_C(f) \geq \frac{1}{2} (E_{\text{kin}}(f) + \mathcal{C}(f)) - C, \quad f \in \mathcal{F}_M \]
in particular,
\[ h_M := \inf \{ \mathcal{H}_C(f) \mid f \in \mathcal{F}_M \} > -\infty. \tag{2.2} \]
**Proof.** Using the previous two lemmas we have
\begin{align*}
\mathcal{H}_C(f) & \geq E_{\text{kin}}(f) + \mathcal{C}(f) - CR^n (1 + \| \rho f \|_{n,l}^{1+\frac{1}{n}}) - \frac{M^2}{2R} \\
& \geq (E_{\text{kin}}(f) + \mathcal{C}(f))(1 - CR^n) - CR^n - \frac{M^2}{2R},
\end{align*}
where \( C > 0 \) is some constant which does not depend on \( R > 0 \). The assertion follows by a suitable choice of \( R \). □

### 3 Scaling and Splitting

In this section we first show that \( h_M \) is negative. We also examine the behaviour of \( h_M \), if \( M \) varies.

**Lemma 3.1** Define \( h_M \) as in (2.2).

(a) Let \( M > 0 \). Then \( -\infty < h_M < 0 \).

(b) There exists \( \alpha > 0 \) such that for \( 0 < M_1 \leq M_2 \)
\[ h_{M_1} \geq \left( \frac{M_1}{M_2} \right)^{1+\alpha} h_{M_2}. \]

**Proof.** As already mentioned in the introduction, we will use coordinates adapted to spherical symmetry. If \( f(x,v) = f(Ax, Av) \forall A \in O(3) \), we have
\[ f(x,v) = f(r,w,L), \]
where \( r := |x|, \ w := \frac{2v}{r}, \ L := |x \times v|^2 \) and we will write again \( f \) instead of \( f \). For \( f \in \mathcal{F}_M \), we define
\[
m_f(r) := 4\pi \int_0^r s^2 \rho_f(s) \, ds,
\]
where \( \rho_f(x) = \rho_f(|x|) \) is the induced density of \( f \).

It is easy to check that, in the new coordinates, the energies and the Casimir functional read
\[
E_{\text{kin}}(f) = 2\pi^2 \int_{\mathbb{R}^+} \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} (w^2 + \frac{L}{r^2}) f(r,w,L) \, dl \, dw \, dr,
\]
\[
E_{\text{pot}}(f) = -\frac{1}{2} \int_{\mathbb{R}^+} \frac{m_f^2(r)}{r^2} \, dr,
\]
\[
\mathcal{C}(f) = 4\pi^2 \int_{\mathbb{R}^+} \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} f^{1+1/k}(r,w,L)(L - L_0)^{-1/k} \, dl \, dw \, dr,
\]
with \( \mathbb{R}^+ := [0, \infty[ \).

Given any function \( f \in \mathcal{F}_M \), we define a rescaled and translated function
\[
\bar{f}(r,w,L) = af(b^{-1}r,c^{-1}w, b^2c^2L - (b^2c^2 - 1)L_0),
\]
where \( a,b,c > 0 \).

Then \( \bar{f}(r,w,L) = 0 \) a.e. if \( L < L_0 \),
\[
\int\int\int \bar{f}(r,w,L) \, dl \, dw \, dr = a(bc)^{-3} \int\int\int f(r,w,L) \, dl \, dw \, dr
\]
and if \( f \in \mathcal{F}_M \), we have \( \bar{f} \in \mathcal{F}_M \) with \( M = a(bc)^{-3}M \). Furthermore,
\[
E_{\text{kin}}(\bar{f}) = 2\pi^2 ab^{-3}c^{-5} \int\int\int \left( w^2 + \frac{L + (b^2c^2 - 1)L_0}{r^2} \right) f(r,w,L) \, dr \, dw \, dl,
\]
\[
\mathcal{C}(\bar{f}) = a^{1+\frac{1}{k}}b^{-3+\frac{2}{k}}c^{-3+\frac{2}{k}} \mathcal{C}(f),
\]
\[
E_{\text{pot}}(\bar{f}) = -\frac{1}{2} \int_{\mathbb{R}^+} a^2b^{-6}c^{-6} \frac{m_f^2(br)}{r^2} \, dr = a^2b^{-6}c^{-6}E_{\text{pot}}(f),
\]
and the mapping \( \mathcal{F}_M \rightarrow \mathcal{F}_M, \ f \mapsto \bar{f} \) is bijective and its inverse is
\[
f(r,w,L) \mapsto f \left( b^{-1}r,c^{-1}w, \frac{L + (b^2c^2 - 1)L_0}{b^2c^2} \right).
\]

To prove (a) we consider the case \( bc < 1 \). Here we have
\[
E_{\text{kin}}(f) \leq ab^{-3}c^{-5}E_{\text{kin}}(f).
\]

Now we fix some \( f \in \mathcal{F}_1 \) with compact support and let
\[
a = M(bc)^3.
\]
Consequently,

\[ \mathcal{H}_C(f) \leq a^{1+\frac{5}{4}}b^{-3+\frac{2l}{k}}c^{-3+\frac{2l}{k}}C(f) + ab^{-3}c^{-5}E_{\text{kin}}(f) + a^2b^{-5}c^{-6}E_{\text{pot}}(f) \]

\[ \leq C_1a^{\frac{1}{4}}(bc)^{\frac{3}{4}} + C_2c^{-2} - C_3b, \]

where \( C_1, C_2, C_3 > 0 \) depend on \( f \). Since we want the last term to dominate as \( b \to 0 \), we let \( c = b^{-n/2} \), so that \( bc = b^{1 - \frac{n}{2}} \) for some \( \eta \in ]1, 2[ \). For \( b \) small enough we have \( bc < 1 \) and

\[ \mathcal{H}_C(\bar{f}) \leq C_1b^{(1 - \frac{n}{2})(2l + 3)/k} + C_2b^{\eta} - C_3b. \]

Now fix \( \eta \in ]1, 2[ \) such that \( (1 - \frac{n}{2})(2l + 3)/k > 1 \); such an \( \eta \) exists by the assumptions on \( k \) and \( l \). For \( b > 0 \) sufficiently small, the sum of the last three terms will be negative and the assertion (a) follows.

As to (b), we distinguish the cases \( l \geq 0 \) and \(-1 < l < 0\). If \( l \geq 0 \), we take \( \bar{f} \) defined by (3.2), but now with \( a = c = 1 \) and \( b > 1 \). From (3.3), we have

\[ E_{\text{kin}}(\bar{f}) \leq b^{-1}E_{\text{kin}}(f) \]

and

\[ E_{\text{kin}}(\bar{f}) \geq b^{-3}E_{\text{kin}}(f). \]

We take \( b = (M_1/M_2)^{-1/3} > 1 \). This implies

\[ \mathcal{H}_C(\bar{f}) \geq b^{-3}E_{\text{kin}}(f) + b^{-5}E_{\text{pot}}(f) + b^{-3+\frac{2l}{k}}C(f) \]

\[ \geq b^{-5}E_{\text{kin}}(f) + b^{-5}E_{\text{pot}}(f) + b^{-5}C(f) \]

\[ = b^{-5}\mathcal{H}_C(f). \]

This implies \( a = \frac{2}{3} \), notice that \( b^\omega < 1 \) for \( \omega < 0 \).

If \(-1 < l < 0\), we first consider the case \( 0 < k \leq 1/2 \) and we assume \( bc \geq 1 \) in (3.2). Together with (3.3),

\[ E_{\text{kin}}(\bar{f}) \leq ab^{-1}c^{-3}E_{\text{kin}}(f), \]

and

\[ E_{\text{kin}}(\bar{f}) \geq ab^{-3}c^{-5}E_{\text{kin}}(f). \]

Now we choose \( f \in \mathcal{F}_{M_2} \) and \( \bar{f} \in \mathcal{F}_{M_1} \), so that

\[ ab^{-3}c^{-3} = \frac{M_1}{M_2} =: m \leq 1. \]

Then

\[ \mathcal{H}_C(\bar{f}) \geq ab^{-3}c^{-5}E_{\text{kin}}(f) + a^2b^{-5}c^{-6}E_{\text{pot}}(f) + a^{1+\frac{5}{4}}(bc)^{-3+\frac{2l}{k}}C(f) \]

\[ = mc^{-2}E_{\text{kin}}(f) + m^2bE_{\text{pot}}(f) + ma^{\frac{1}{4}}(bc)^{\frac{3}{4}}C(f). \]
We require
\[
ma^{\frac{1}{c}}(bc)^{\frac{2}{c}} = m^2b = mc^{-2}.
\]
Hence we choose
\[
c = m^{\frac{2l+2}{2l-3}}, \quad b = m^{\frac{2l+2k+4}{2l-2k}}, \quad a = (bc)^{3}m
\]
and this implies \(bc = m^{(2l+2)/(3+2l-2k)-1} = m^{(2l-1)/(3+2l-2k)} \geq 1\). Then
\[
\mathcal{H}_C(f) \geq m^{1+\alpha}(E_{\text{kin}}(f) + E_{\text{pot}}(f) + C(f)) = m^{1+\alpha}\mathcal{H}_C(f),
\]
where \(\alpha := (4l+4)/(2l+3-2k) > 0\).

For \(-1 < l < 0\) and \(k > \frac{2}{4}\) we assume \(bc < 1\) in (3.2) and it is easy to check that
\[
E_{\text{kin}}(\bar{f}) \geq ab^{-1}c^{-3}E_{\text{kin}}(f).
\]
We take \(f, \bar{f}\) and \(m\) as in (3.4) above. Finally,
\[
\mathcal{H}_C(\bar{f}) \geq ab^{-1}c^{-3}E_{\text{kin}}(f) + a^2b^{-5}c^{-6}E_{\text{pot}}(f) + a^{1+\frac{4}{c}}(bc)^{-3+\frac{2}{c}}C(f)
\]
\[
= mb^2E_{\text{kin}}(f) + m^2bE_{\text{pot}}(f) + ma^{\frac{1}{c}}(bc)^{\frac{2}{c}}C(f).
\]
Defining
\[
b := m, \quad c := m^{(2k-4-2l)/(3+2l)},
\]
so that \(bc = m^{(2k-1)/(3+2l)} < 1\), the assertion follows with \(\alpha = 2\).

The scaling estimate above can be used to show that, along a minimizing sequence, the mass has to remain concentrated.

**Lemma 3.2** Let \(M > 0\). Then there exists a radius \(R_M > 0\) such that if \((f_j) \subset F_M\) is a minimizing sequence of \(\mathcal{H}_C\),
\[
\lim_{j \to \infty} \int_{|x| > R} f_j \, dx = 0, \quad R > R_M.
\]

**Proof.** We define the ball \(B_R := \{x \in \mathbb{R}^3 : |x| < R\}\). Let \(\chi_{B_R \times \mathbb{R}^3}\) be the characteristic function of \(B_R \times \mathbb{R}^3\). For \(f \in F_M\) we split
\[
f_1 := \chi_{B_R \times \mathbb{R}^3} f, \quad f_2 = f - f_1
\]
and define \(m_i(r) := m_{f_i}(r), \quad i = 1, 2,\) with \(m_{f_i}\) as in (3.1). We abbreviate \(\lambda = M - m_f(R)\). Then
\[
\mathcal{H}_C(f) = \mathcal{H}_C(f_1) + \mathcal{H}_C(f_2) - \int_0^\infty \frac{m_1(r)m_2(r)}{r^2} \, dr \\
\geq h_{M-\lambda} + h_\lambda - \int_0^\infty \frac{m_1(r)m_2(r)}{r^2} \, dr,
\]
since \(f_1 \in F_{M-\lambda}\) and \(f_2 \in F_\lambda\). Next,
\[
\int_0^\infty \frac{m_1(r)m_2(r)}{r^2} \, dr \leq \lambda(M-\lambda) \int_R^\infty \frac{1}{r^2} \, dr = \frac{\lambda(M-\lambda)}{R}.
\]
Using Lemma 3.1 (b), we find that

\[ \mathcal{H}_C(f) \geq \left[ \left( 1 - \frac{\lambda}{M} \right)^{1+\alpha} + \left( \frac{\lambda}{M} \right)^{1+\alpha} \right] h_M - \frac{\lambda(M-\lambda)}{R}. \]

Since the function \( q \) defined by

\[ q(x) := x^{\alpha+1} + (1-x)^{\alpha+1} + C_\alpha x (1-x) \]

is convex in \([0,1]\) for suitable \( C_\alpha > 0 \), we have the inequality

\[ (1-x)^{1+\alpha} + x^{1+\alpha} - 1 \leq -C_\alpha (1-x) \quad 0 \leq x \leq 1. \]

Choosing \( x = \frac{\lambda}{M} \) and noticing that by Lemma 3.1 (a) \( h_M < 0 \), we have

\[ \mathcal{H}_C(f) - h_M \geq \left[ \left( 1 - \frac{\lambda}{M} \right)^{1+\alpha} + \left( \frac{\lambda}{M} \right)^{1+\alpha} - 1 \right] h_M - \frac{\lambda(M-\lambda)}{R} \]

\[ \geq -C_\alpha h_M \left( 1 - \frac{\lambda}{M} \right) \frac{\lambda^2}{M} - \frac{\lambda(M-\lambda)}{R} \]

\[ = -\left( \frac{C_\alpha h_M}{M^2} - \frac{1}{R} \right) (M-\lambda) \lambda \]

\[ = \left( \frac{1}{R_M} - \frac{1}{R} \right) m_f(R)(M-m_f(R)), \quad (3.5) \]

where

\[ R_M := -\frac{M^2}{C_\alpha h_M} > 0. \]

Now let \((f_j) \subset \mathcal{F}_M\) be a minimizing sequence of \( \mathcal{H}_C \), and assume the assertion of the lemma is wrong. Then there exist some \( R > R_M, \lambda > 0 \), and a subsequence called \((f_j)\) again, such that

\[ \lim_{j \to \infty} \int_{|x| > R} f_j \, dv \, dx = \lambda. \]

For every \( j \in \mathbb{N} \) we can choose \( R_j > R \) such that

\[ \lambda_j := \int_{|x| > R_j} f_j \, dv \, dx = \frac{1}{2} \int_{|x| > R_j} f_j \, dv \, dx. \]

Then

\[ \lim_{j \to \infty} \int_{|x| > R_j} f_j \, dv \, dx = \lim_{j \to \infty} \lambda_j = \frac{\lambda}{2} > 0. \]

Applying (3.5) to \([0,R_j]\) we get

\[ \mathcal{H}_C(f) - h_M \geq -\left( \frac{1}{R_M} - \frac{1}{R_j} \right) (M-\lambda_j) \lambda_j \]

\[ > -\left( \frac{1}{R_M} - \frac{1}{R} \right) (M-\lambda_j) \lambda_j \]

\[ = -\left( \frac{1}{R_M} - \frac{1}{R} \right) \left( M - \frac{\lambda}{2} \right) > 0, \quad n \to \infty, \]
since $0 < \lambda^2 < M$. This contradicts $(f_j)$ being a minimizing sequence. \hfill \square

4 Minimizers of $\mathcal{H}_C$

**Theorem 4.1** Let $M > 0$, $L_0 > 0$ and let $(f_j) \subset F_M$ be a minimizing sequence of $\mathcal{H}_C$. Then there is a minimizer $f_0$ and a subsequence $(f_{j_k})$ such that $\mathcal{H}_C(f_0) = h_M$, supp $f_0 \subset B_{R_M} \times \mathbb{R}^3$ with $R_M$ as in Lemma 3.2, and $f_{j_k} \to f_0$ weakly in $L^{k,l}$. For the induced potentials we have $\nabla U_{j_k} \to \nabla U_0$ strongly in $L^2(\mathbb{R}^3)$.

**Proof.** By Lemma 2.3, $E_{\text{kin}}(f_j) + C(f_j)$ is bounded and thus $(f_j)$ is bounded in $L^{k,l}$. Now there exists a weakly convergent subsequence, denoted by $(f_j)$ again:

$$f_j \to f_0 \text{ weakly in } L^{k,l}.$$  

Clearly, $f_0 \geq 0$ a.e. and $f_0(x,v) = 0$ a.e. for $0 \leq L < L_0$. By Lemma 3.2 we have

$$M = \lim_{j \to \infty} \int_{|v| \leq R_2} f_j \, dv \, dx + \lim_{j \to \infty} \int_{|v| \geq R_2} f_j \, dv \, dx$$

$$\leq \lim_{j \to \infty} \int_{|v| \leq R_2} f_j \, dv \, dx + \frac{C}{R_2^2},$$

where $R_1 > R_M$ and $R_2 > 0$ are arbitrary. This implies

$$\int_{|v| < R_2} f_0 \, dv \, dx = M$$

for every $R_1 > R_M$. This proves the assertion on supp $f_0$ and $\iint f_0 = M$. We also have

$$E_{\text{kin}}(f_0) \leq \limsup_{j \to \infty} E_{\text{kin}}(f_j) < \infty. \quad (4.1)$$

By Lemma 2.1, $(\rho_j) = (\rho_{f_j})$ is bounded in $L^{n,l}(\mathbb{R}^3)$. After choosing another subsequence, we conclude that

$$\rho_j \to \rho_0 \text{ weakly in } L^{n,l}.$$  

Thus, by Lemma 2.2 (a) (b), the strong convergence

$$\nabla U_j \to \nabla U_0 \text{ strongly in } L^2(\mathbb{R}^3), \quad (4.2)$$

follows, in particular,

$$E_{\text{pot}}(f_j) \to E_{\text{pot}}(f_0).$$

Indeed,

$$\frac{1}{4\pi} \int |\nabla U_j - \nabla U_0|^2 \, dx = \int_0^\infty \frac{m_j^2 - f_0}{r^2} \, dr$$

$$\leq \int_0^R \frac{m_j^2 - f_0}{r^2} \, dr + \frac{M^2}{R} \to 0.$$  

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Here the first term tends to zero because of the compactness of $T$, defined in Lemma 2.2 (b). It remains to show that $f_0$ actually is a minimizer, in particular $E_{\text{kin}}(f_0) + C(f_0) < \infty$. By weak covergence, we have

$$C(f_0) = \|f_0\|^{(k+1)/k}_{k,l} \leq \liminf_{j \to \infty} \|f_j\|^{(k+1)/k}_{k,l} < \infty.$$  
Together with (4.1) and (4.2) this implies

$$E_{\text{kin}}(f_0) + C(f_0) \leq \lim_{j \to \infty} (E_{\text{kin}}(f_j) + C(f_j)) < \infty,$$

note that the $\lim_{n \to \infty}$ in the above inequality exists. Finally, $H_C(f_0) = C(f_0) + E_{\text{kin}}(f_0) + E_{\text{pot}}(f_0) \leq \lim_{j \to \infty} (C(f_j) + E_{\text{kin}}(f_j) + E_{\text{pot}}(f_j)) = h_M$.

**Theorem 4.2** Let $f_0 \in \mathcal{F}_M$ be a minimizer of $\mathcal{D}$. Then there exists $E_0 < 0$ such that

$$f_0(x,v) = \begin{cases} \frac{k}{k+1} (E_0 - E)^{(k+1)/k}, & E_0 - E > 0 \\ 0, & E_0 - E \leq 0 \end{cases},$$

where

$$E := \frac{1}{2} v^2 + U_0(x)$$

(4.3) and $U_0$ is the potential induced by $f_0$. Moreover, $f_0$ is a steady state of the Vlasov-Poisson system (1.1)-(1.3).

**Proof.** Let $f_0$ be a minimizer. For $\epsilon > 0$ we define

$$K_\epsilon := \{ (x,v) \mid \epsilon < f_0(x,v) \leq \frac{1}{\epsilon}, \quad L_0 + \epsilon \leq L \leq L_0 + \frac{1}{\epsilon} \}.$$  

Let $g \in L^\infty(\mathbb{R}^6)$ with $\text{supp } g \subset K_\epsilon$, and

$$h := g - \frac{1}{|K_\epsilon|} \left( \iint g \, dv \, dx \right) \cdot \chi_{K_\epsilon}.$$  

Then for $\tau \in \mathbb{R}$ small enough we have $f_0 + \tau h \geq 0$ and $f_0 + \tau h \in \mathcal{F}_M$, indeed, $E_{\text{kin}}(f_0 + \tau h) < \infty$ and

$$C(f_0 + \tau h) = C(f_0) + \tau \iint Q'(f_0)(L - L_0)_+^{1/k} h + o(\tau) < \infty.$$  

Now we have

$$0 \leq \mathcal{H}_C(f_0 + \tau h) - \mathcal{H}_C(f_0) = \tau \iint \left( Q'(f_0)(L - L_0)_+^{1/k} + \frac{1}{2} v^2 + U_0(x) \right) h \, dv \, dx$$

$$+ o(\tau)$$

$$= \tau \iint \left( Q'(f_0)(L - L_0)_+^{1/k} + E \right) h \, dv \, dx + o(\tau),$$

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where we used Lemma 2.2 (c) to calculate the potential energy term. Since \(-h\) is also an admissible function, this implies
\[
\iint (Q'(f_0)(L-L_0)^{-1/k} + E) h \, dv \, dx = 0.
\]
Inserting the definition of \(h\) we get
\[
\iint \left[ (Q'(L-L_0)^{-1} f_0) + E - \frac{1}{|K_\epsilon|} \iint_{K_\epsilon} (Q'(L-L_0)^{-1} f_0) + E \right] g \, dv \, dx = 0.
\]
Consequently,
\[
Q'(L-L_0)^{-1} f_0 + E = E_0 \quad \text{a.e. on } K_\epsilon
\]
and we conclude
\[
Q'(L-L_0)^{-1} f_0 + E = E_0 \quad \text{a.e. on } \{(x,v)|f_0(x,v) > 0, L_0 \leq L\}.
\]  
(4.4)
Suppose now, there would exist a measurable set \(A \subset \{(x,v)|f_0(x,v) = 0, L_0 \leq L\}\) with
\[
E < E_0 \quad \text{a.e. on } A
\]
and \(0 < |A| < \infty\). We define
\[
h := \chi_A - \frac{1}{|K_\epsilon|} \left( \iint_{\chi A} \chi_{K_\epsilon} \right) \chi_{K_\epsilon}
\]
with \(K_\epsilon\) as above and small \(\epsilon > 0\). Then for \(\tau > 0\) sufficiently small we have
\[
f_0 + \tau h \in \mathcal{F}_M
\]
and again
\[
0 \leq \mathcal{H}(f_0 + \tau h) - \mathcal{H}(f_0) = \tau \iint (Q'(L-L_0)^{-1} f_0) + E) h \, dv \, dx + o(\tau).
\]
Plugging the definition of \(h\) in the above equation, we have
\[
0 \leq \iint (Q'(L-L_0)^{-1} f_0) \chi_A - E_0 \iint \chi_A - \iint_A (E - E_0) < 0,
\]
a contradiction and thus \(E \geq E_0\) a.e. on \(\{(x,v)|f_0(x,v) > 0, L_0 \leq L\}\). Together with (4.4) this implies that \(f_0\) is of the form given in the theorem.

Since \(f_0\) is a function of the microscopic energy \(E\) defined by (4.3) and \(L\), it is constant along solutions of the characteristic system
\[
\begin{aligned}
\dot{X} &= V \\
\dot{V} &= -\partial_x U_0(X)
\end{aligned}
\]
and thus \(f_0\) is a solution of the Vlasov equation, provided the potential \(U_0\) is sufficiently smooth. But one can indeed show that \(U_0 \in C^2(\mathbb{R}^3)\), cf. [3], and by construction
\[
\Delta U_0 = 4\pi \rho_0,
\]
so that \((f_0, \rho_0, U_0)\) is indeed a solution of the Vlasov-Poisson system. Since \(\rho_0\) has compact support and \(\lim_{|x| \to \infty} U_0(x) = 0\) we conclude that \(E_0 < 0\). \(\square\)
5 Dynamical stability

We investigate the nonlinear stability of $f_0$. For $f \in \mathcal{F}_M$,

$$
\mathcal{H}_C(f) - \mathcal{H}_C(f_0) = d(f, f_0) + E_{\text{pot}}(f - f_0) = d(f, f_0) - \frac{1}{8\pi} \int |\nabla U_f - \nabla U_{f_0}|^2 dx,
$$

(5.1)

where

$$
d(f, f_0) := \iint \left[ (f^{1+1/k}_0 - f_0^{1+1/k})(L - L_0)^{-1/k} + (E - E_0)(f - f_0) \right] dv dx,
$$

and we have $d(f, f_0) \geq 0$, $f \in \mathcal{F}_M$ with $d(f, f_0) = 0$, iff $f = f_0$. Indeed,

$$
d(f, f_0) \geq \iint \left[ Q'((L - L_0)^{-1} f_0) + (E - E_0) \right] (f - f_0) dv dx \geq 0,
$$

which is due to the convexity of $Q$, and on the support of $f_0$ the bracket vanishes. This fact allows us to use $d(., f_0)$ to measure the distance to the stationary solution $f_0$.

**Theorem 5.1** Assume that the minimizer $f_0$ is unique in $\mathcal{F}_M$. Then for all $\epsilon > 0$ there is $\delta > 0$ such that for any solution $f(t)$ of the Vlasov-Poisson system with $f(0) \in C^1_c(\mathbb{R}^6) \cap \mathcal{F}_M$,

$$
d(f(0), f_0) + \frac{1}{8\pi} \int |\nabla U_{f(0)} - \nabla U_{f_0}|^2 dx < \delta
$$

implies

$$
d(f(t), f_0) + \frac{1}{8\pi} \int |\nabla U_{f(t)} - \nabla U_{f_0}|^2 dx < \epsilon, \quad t \geq 0.
$$

**Proof.** We observe that $\mathcal{H}_C$ is conserved along any solution $f(t)$ of the Vlasov-Poisson system with $f(0) \in C^1_c(\mathbb{R}^6) \cap \mathcal{F}_M$. This follows from conservation of energy and the fact that both $f(t)$ and $L$ are conserved along characteristics. Assume the theorem were false. Then there exists $\epsilon_0 > 0, \quad t_j > 0$, and $f_j(0) \in C^1_c(\mathbb{R}^6) \cap \mathcal{F}_M$ such that

$$
d(f_j(0), f_0) + \frac{1}{8\pi} \int |\nabla U_{f_j(0)} - \nabla U_{f_0}|^2 dx \leq \frac{1}{j}
$$

and

$$
d(f_j(t_j), f_0) + \frac{1}{8\pi} \int |\nabla U_{f_j(t_j)} - \nabla U_{f_0}|^2 dx \geq \epsilon_0 > 0.
$$

From (5.1), we have

$$
\lim_{j \to \infty} \mathcal{H}_C(f_j(0)) = h_M,
$$

and because $\mathcal{H}_C(f_j(t))$ is conserved,

$$
\lim_{j \to \infty} \mathcal{H}_C(f_j(t_j)) = \lim_{j \to \infty} \mathcal{H}_C(f_j(0)) = h_M.
$$
Thus \((f_j(t_j)) \subset \mathcal{F}_M\) is a minimizing sequence of \(\mathcal{H}_C\) and with Theorem 4.1 we have
\[
\frac{1}{8\pi} \int |\nabla U_{f_j(t_j)} - \nabla U_{f_0}|^2 \, dx \to 0,
\]
which implies
\[
d(f_j(t_j), f_0) \to 0
\]
by (5.1), a contradiction.

**Corollary 5.2** If in Theorem 5.1 the assumption \(\|f(0)\|_{k,l} = \|f_0\|_{k,l}\) is added, then for any \(\epsilon > 0\) the parameter \(\delta > 0\) can be chosen such that the stability estimate
\[
\|f(t) - f_0\|_{k,l} < \epsilon, \quad t \geq 0
\]
holds.

**Proof.** We repeat the proof of Theorem 5.1 except that in the contradiction assumption have
\[
\|f_j(t_j) - f_0\|_{k,l} + d(f_j(t_j), f_0) + \frac{1}{8\pi} \int |\nabla U_{f_j(t_j)} - \nabla U_{f_0}|^2 \, dx \geq \epsilon_0 > 0.
\]
From the minimizing sequence \(f_j(t_j)\) we can now extract a subsequence which converges weakly in \(L^{k,l}\) to \(f_0\). But due to our additional restriction we have
\[
\|f_j(t_j)\|_{k,l} = \|f_0\|_{k,l}, \quad j \in \mathbb{N}.
\]
Now the lower semicontinuity of the norm and the uniform convexity of \(L^{k,l}(\mathbb{R}^6)\) imply \(f_j(t_j) \to f_0\) strongly in \(L^{k,l}\). Together with the rest of the proof of Theorem 5.1, the assertion follows.

**Final Remarks.**

(a) The uniqueness of the minimizer \(f_0\) subject to a fixed mass constraint can be shown by a scaling argument in the case \(L_0 = 0\). For \(L_0 > 0\), at least numerically the minimizer seems to be unique, but the scaling argument fails because of the translation in \(L\). We mention that, for the argument in Theorem 5.1, it would suffice if the minimizers of \(\mathcal{H}_C\) were isolated.

(b) The technical assumption \(f = 0\) a.e. for \(0 < L < L_0\) in the class of perturbations \(\mathcal{F}_M\), see (1.10), is needed for the scaling argument in Lemma 3.1 and it would be desirable to improve it to \(f = 0\) a.e. for \(0 < L < \gamma L_0\) for some \(0 < \gamma < 1\).

(c) We only obtain stability against spherically symmetric perturbations, because the quantity \(L\) is conserved by the characteristic flow only for spherically symmetric solutions. Stability against asymmetric perturbations is an open problem and more delicate mathematical tools have to be invented to address this question.
References


