On the robustness of the toroidal velocity theorem

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Abstract

In dynamo theory the toroidal velocity theorem in its classical version (Elsasser 1946, Bullard & Gellman 1954) rules out dynamo action in a spherical conducting volume provided that the fluid is incompressible, the conductivity is uniform, and the velocity field is purely toroidal. We prove in this note that this result is robust in the sense that slight compressibility of the fluid, small non-radial variations and even large radial variations in conductivity, and the presence of a small non-toroidal velocity component do not invalidate the theorem. Moreover, by proper choice of the conductivity distribution modelling the conducting volume, small deviations from spherical symmetry of the conductor can also be taken into account.

Key Words: Dynamo theory, Antidynamo theorem, Toroidal velocity theorem

1 Introduction

Antidynamo theorems provide rough but general restrictions on the dynamo process in that they rule out dynamo action for certain classes of magnetic fields or flow fields. The prototypical theorem of the latter kind is the toroidal velocity theorem\(^1\), which restricts the solenoidal velocity field to its toroidal component. In spherical geometry this restriction implies, in particular, that there is no radial motion of the fluid. In applications this theorem is often invoked for claiming the necessity of radial motions in the interiors of planets and stars for generation of magnetic fields in these celestial objects. Motions with a certain strength of the radial component are typically driven by buoyancy forces and dynamo action is thus usually associated with convection in the interiors of those objects. However, magnetic fields are also found in celestial bodies without any indication of buoyancy driven motions in their interiors, which is at first sight puzzling. In this context the question arises whether the validity of the toroidal velocity theorem is restricted to the ideal case when the poloidal velocity component is exactly zero or whether it remains valid in the presence of a sufficiently small poloidal component. The same question may be asked with respect to the other assumptions of the

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\(^{1}\)We prefer the more detailed terminology “toroidal velocity theorem” as against “toroidal theorem” in order to prevent confusion with “toroidal magnetic field theorem” (Kaiser 2009a).
theorem. No real-world fluid is absolutely incompressible. Rotating objects exhibit necessarily deviations from spherical symmetry. Electrical conductivity is a material property that depends on temperature and pressure and varies with the composition of the fluid. The relevance of the toroidal velocity theorem for applications thus crucially depends on a robust formulation that allows small deviations from the ideal situation described by the theorem’s classical version.

An important first step in this direction has been done by Proctor (2004), who proved the validity of the theorem in the presence of a non-zero poloidal velocity component satisfying an upper bound that depends on the toroidal component. The following theorem can be seen as a fully robust version of the toroidal velocity theorem.

**Theorem** Let $DV^*$, $V_{t}^{*}$, and $V_{nt}^{*}$ be non-dimensional maxima (in space and time) of the divergence of the velocity field and its toroidal and non-toroidal components, respectively, and let, moreover, $\delta_t^{*}\lambda$, $\delta_r^{*}\lambda$, and $\delta_{nr}^{*}\lambda$ be non-dimensional maxima of the temporal, radial, and non-radial variations of the magnetic diffusivity of the fluid, respectively. If these quantities are small in the sense of (42) or (43), then the energy of any solution of the induction equation decays (in time) exponentially fast to zero.

Some preliminary comments on the theorem are in order.
1. The precise definitions of the maxima are given in eqs. (22),(23), and (34).
2. Not all the maxima need to be simultaneously small; in particular, inequalities (42) or (43) can be satisfied for arbitrarily large values of $V_{t}^{*}$ (which is the essence of the toroidal velocity theorem) and of $\delta_r^{*}\lambda$ (which is physically plausible (Busse & Proctor 2007) but lacked so far a rigorous proof).
3. The theorem is formulated for a spherical fluid volume. Nevertheless small deviations from sphericity are implicit in the theorem by a suitable choice of the diffusivity distribution. For example, a diffusivity distribution of the type

$$\lambda(r) = \left[ g(|r|) + \frac{1}{\epsilon} \left( 1 + \tanh \frac{|r| - 1}{\epsilon} \right) \right] (1 + \epsilon^2 h(r/|r|)),$$

where $\epsilon \ll 1$ and $g$ and $h$ are functions that together with their derivatives are of order 1, models an approximately spherical fluid volume of radius 1. Note that $\delta_t^{*}\lambda$ becomes necessarily large but $\delta_{nr}^{*}\lambda$ is of order $\epsilon$.
4. The method of proof of the theorem is elementary in the sense that only energy balances (of the poloidal and toroidal magnetic scalars and of the magnetic field itself) are considered. A properly chosen combination of these variables defines a “generalized energy” that decays exponentially provided that the sufficient conditions (42) or (43) are satisfied. Generalized energies are well-known in hydrodynamic stability theory, where they can provide stability in parameter regions that are undecided by the ordinary energy method (see, e.g., Galdi & Padula 1990). In particular, in non-normal problems (as the one considered here), where transient growth can precede the ultimate decay of a quantity, a suitable generalized energy can still exhibit exponential decay whereas monotonous decay of the ordinary energy can no longer be proved.
2 Basic equations, representations, and estimates

Starting point is the kinematic dynamo problem (see, e.g., Moffatt 1978), which can be summarized as follows:

\[
\begin{align*}
\partial_t \mathbf{B} &= \nabla \times (\mathbf{v} \times \mathbf{B}) - \nabla \times (\lambda \nabla \times \mathbf{B}), \quad \nabla \cdot \mathbf{B} = 0 \quad \text{in } B_1 \times (0, \infty), \\
\nabla \times \mathbf{B} &= 0, \quad \nabla \cdot \mathbf{B} = 0 \quad \text{in } \hat{B}_1 \times (0, \infty), \\
\mathbf{B} \text{ continuous} & \quad \text{in } \mathbb{R}^3 \times (0, \infty), \\
|\mathbf{B}(\mathbf{r}, \cdot)| &\to 0 \quad \text{for } |\mathbf{r}| \to \infty, \\
\mathbf{B}(\cdot, 0) &= \mathbf{B}_0, \quad \nabla \cdot \mathbf{B}_0 = 0 \quad \text{on } B_1 \times \{t = 0\}. 
\end{align*}
\]

(2)

The conducting region is the unit ball \(B_1\) with boundary \(S_1\), \(\hat{B}_1\) denotes the vacuum region, and time \(t\) varies in the interval \((0, \infty)\). The velocity field \(\mathbf{v}\) and the diffusivity \(\lambda > 0\) are prescribed quantities, which, for the present, are assumed to be arbitrary, smooth functions of space and time. In particular, \(\mathbf{v}\) need not be solenoidal and \(\lambda\) need not be spherically symmetric. The kinematic dynamo problem asks then for solutions \(\mathbf{B}\) of the initial-value problem (2), which do not decay in time (so-called dynamo solutions). In particular, if \(\mathbf{v}\), \(\lambda\), and the conducting region are such that no dynamo solutions exist, the configuration fails as a dynamo (there is no magnetic regeneration or dynamo action).

In the following we make ample use of the poloidal/toroidal decomposition of solenoidal vector fields (see, e.g., Backus 1958). For the magnetic field we use it in the form

\[
\mathbf{B} = \mathbf{B}_p + \mathbf{B}_t = -\nabla \times \Lambda S - \Lambda T = \nabla \times (\nabla \times S \mathbf{r}) + \nabla \times T \mathbf{r},
\]

(3)

where \(\Lambda := \mathbf{r} \times \nabla\) is a non-radial-derivative operator; its square \(\Lambda \cdot \Lambda =: \mathcal{L}\) is the Laplace-Beltrami-operator on \(S_1\). The poloidal scalar \(S\) and the toroidal one \(T\) are uniquely determined by

\[
\mathbf{r} \cdot \mathbf{B} = -\mathcal{L} S, \quad \Lambda \cdot \mathbf{B} = -\mathcal{L} T,
\]

(4)

provided that \(S\) and \(T\) have vanishing mean-value over spheres \(S_r\) of radius \(r\),

\[
\langle S \rangle_r = 0, \quad \langle T \rangle_r = 0, \quad \langle \ldots \rangle_r := \frac{1}{4\pi r^2} \int_{S_r} \ldots \, ds
\]

(5)

and if \(\mathbf{B}\) is sufficiently regular. This justifies the usage (that we follow here) to refer to \(P := \mathbf{r} \cdot \mathbf{B} = -\mathcal{L} S\) instead of \(S\) as poloidal scalar. From (2)_{2–4} follow the boundary conditions

\[
T|_{S_1} = 0, \quad [P]_{S_1} = [\mathbf{r} \cdot \nabla P]_{S_1} = 0,
\]

(6)

where \([\ldots]_{S_1}\) denotes the jump over \(S_1\). In the vacuum region \(\hat{B}_1\) we have \(T \equiv 0\), whereas \(P\) is a harmonic function that vanishes \(\sim O(1/|\mathbf{r}|^2)\) at infinity.

Concerning the velocity field we combine the Helmholtz decomposition with the poloidal/toroidal decomposition to obtain:

\[
\mathbf{v} = \mathbf{v}_0 + \nabla \pi = \mathbf{v}_t + \mathbf{v}_p + \nabla \pi = \mathbf{v}_t + \mathbf{v}_{nt},
\]

(7)

\footnote{The radius \(R = 1\) has been chosen for reasons of simplicity. In the subsequent formulae radii \(R \neq 1\) can easily be restored by dimensional considerations.}

\footnote{Note that the representation (3) differs from that used by Backus (1958) by a sign.}
where \( v_0 \) is the divergence-free part of \( v \) and \( \nabla \pi \) is the curl-free part; the non-toroidal part \( v_{nt} \) of the velocity field comprises the poloidal component \( v_p \) and \( \nabla \pi \). On \( S_1 \) we assume the boundary condition

\[
\mathbf{r} \cdot \mathbf{v} \big|_{S_1} = 0. \tag{8}
\]

In the following it is useful to introduce derivative operators adapted to spherical geometry, viz. radial and non-radial derivative operators:

\[
\partial_r := \frac{\mathbf{r}}{r} \cdot \nabla, \quad \nabla_{nr} := \nabla - \left( \frac{\mathbf{r}}{r} \right) \partial_r, \quad r := |\mathbf{r}|. \tag{9}
\]

\( \nabla_{nr} \) and \( \Lambda \) are related by

\[
\Lambda = \mathbf{r} \times \nabla = \mathbf{r} \times \nabla_{nr}, \quad \nabla_{nr} = -\frac{1}{r^2} (\mathbf{r} \times \Lambda). \tag{10}
\]

\( \mathcal{L} \) can now equivalently be expressed by

\[
\mathcal{L} = \Lambda \cdot \Lambda = r^2 \nabla_{nr} \cdot \nabla_{nr} \tag{11}
\]

and is related to the Laplacian by

\[
\Delta = \nabla \cdot \nabla = \frac{1}{r} \partial_r^2 r + \frac{1}{r^2} \mathcal{L}. \]

Note that \( \partial_r \) interchanges with \( \Lambda \) and with \( r \nabla_{nr} \), but \( \Lambda \) and \( \nabla_{nr} \) do not.\(^4\)

We summarize, finally, some variational inequalities, which we need in the subsequent analysis. Well-known are the following Poincaré-type inequalities (Backus 1958):

\[
\begin{align*}
\pi^2 \int_{\mathbb{R}^3} \mathbf{B}^2 dv & \leq \int_{B_1} |\nabla \times \mathbf{B}|^2 dv, \\
\pi^2 \int_{B_1} P^2 dv & \leq \int_{\mathbb{R}^3} |\nabla P|^2 dv, \\
c_i^2 \int_{B_1} T^2 dv & \leq \int_{B_1} |\nabla T|^2 dv.
\end{align*} \tag{12}
\]

Here \( \mathbf{B}, P, \) and \( T \) are supposed to be sufficiently regular, mean-value-free, and to satisfy the boundary conditions (2)\(_{2-4}\) and (6), respectively; \( c_i^2 = 20.19 \) holds approximately.

Recall next, that \( -\mathcal{L} \) is a positive, symmetric and hence invertible operator with lower bound

\[
2 \int_{S_1} f^2 ds \leq \int_{S_1} f (-\mathcal{L}) f ds \tag{13}
\]

on the space of all sufficiently regular, mean-value-free functions \( f \) defined on \( S_1 \) or, more generally, on any spherically symmetric domain. Thus, the definition \( H := (-\mathcal{L})^{-1} T \) makes sense (cf. Proctor 2004) and we obtain by (repeated) integration by parts the estimates:

\[
\int_{B_1} |\partial_r (rH)|^2 dv \overset{\text{def}}{=} \int_{B_1} |\partial_r (r(-\mathcal{L})^{-1} T)|^2 dv
\]

\[
= \int_{B_1} \partial_r (rT) (-\mathcal{L})^{-2} \partial_r (rT) dv \leq \frac{1}{4} \int_{B_1} |\partial_r (rT)|^2 dv, \quad \tag{14}
\]

\( ^4 \)Further useful relations between these operators may be found in the appendix of (Kaiser 2009b).
\[
\int_{B_1} |\nabla n_\tau \partial_\tau (rH)|^2 dv = \int_{B_1} \nabla n_\tau \partial_\tau (r(-\mathcal{L})^{-1}T) \cdot \nabla n_\tau \partial_\tau (r(-\mathcal{L})^{-1}T) dv \\
= \int_{B_1} \frac{1}{r^2} \partial_\tau (rT)(-\mathcal{L})^{-1} \partial_\tau (rT) dv \leq \frac{1}{2} \int_{B_1} \left| \frac{1}{r} \partial_\tau (rT) \right|^2dv,
\]
(15)
\[
\int_{B_1} |\nabla \times \Lambda H|^2 dv = \int_{B_1} (\nabla \times \Lambda H) \cdot (\nabla \times \Lambda H) dv = -\int_{B_1} \Lambda H \cdot \Lambda \Delta H dv \\
= -\int_{B_1} T \Delta (-\mathcal{L}^{-1}T dv = \int_{B_1} \nabla T \cdot (-\mathcal{L}^{-1} \nabla T dv \leq \frac{1}{2} \int_{B_1} |\nabla T|^2 dv.
\]
(16)

Finally, we estimate \(B_p\) and \(B_t\) by the gradients of the corresponding scalars. For \(B_p\) we have by a calculation similar to (16) (cf. Busse 1975):
\[
\int_{R^3} \|B_p\|^2 dv = \int_{R^3} |\nabla \times S|^2 dv \leq \frac{1}{2} \int_{R^3} \|
abla P\|^2 dv,
\]
(17)
and for \(B_t\):
\[
\int_{B_1} \|B_t\|^2 dv = \int_{B_1} |\Lambda T|^2 dv \leq \int_{B_1} |\nabla n_\tau T|^2 dv \leq \int_{B_1} |\nabla T|^2 dv.
\]
(18)

At last, we recollect Young’s inequality in the form
\[
a b \leq \frac{\epsilon}{2} a^2 + \frac{1}{2 \epsilon} b^2, \quad \epsilon > 0.
\]
(19)

3 Energy-balances and -estimates

This section presents the energy-balances of the magnetic field \(B\) and of the poloidal and toroidal magnetic scalars \(P\) and \(T\). These quantities are measured in the energy norm, abbreviated here by\(^5\)
\[
\left( \int_{B_1} \cdot \cdot^2 dv \right)^{1/2} =: \| \cdot \| \quad \text{and} \quad \left( \int_{R^3} \cdot \cdot^2 dv \right)^{1/2} =: \| \cdot \|_{\infty}.
\]
(20)

The coefficients \(\lambda\) and \(\mathbf{v}\) in the dynamo equation (2)\(_1\) and those of their variations, which play a role in the subsequent analysis, are measured by the corresponding space-time-maxima. Especially for \(\lambda\) we introduce the abbreviations:
\[
\min_{B_1 \times (0, \infty)} \lambda =: \lambda_0
\]
(21)
and
\[
\max_{B_1 \times (0, \infty)} \frac{\partial_\tau \lambda}{\lambda} =: \delta_\lambda , \quad \max_{B_1 \times (0, \infty)} \frac{\partial_\tau \lambda}{\lambda} =: \delta_\lambda , \quad \max_{B_1 \times (0, \infty)} \frac{\nabla n_\tau \lambda}{\lambda_0} =: \delta_{n\tau} \lambda.
\]
(22)

Concerning \(\mathbf{v}\) the following maxima turn out to be useful:
\[
\max_{B_1 \times (0, \infty)} \mid \mathbf{v}_t \mid =: V_t , \quad \max_{B_1 \times (0, \infty)} \mid \mathbf{v}_{nt} \mid =: V_{nt} , \quad \max_{B_1 \times (0, \infty)} \mid \nabla \cdot \mathbf{v} \mid =: DV.
\]
(23)

\(^5\)For comparison: Proctor (2004) uses the notation \(\int_{B_1} \cdot dv = \langle \cdot \rangle\) and \(\int_{R^3} \cdot dv = \langle \cdot \rangle_{\infty}\). Note that we here reserved the angular bracket for the spherical mean (5).
We start with the $B$-balance, which is quite standard (Backus 1958)\footnote{Note that Backus considers the case $\lambda = \text{const}$. His manipulations, however, work for space-time-dependent $\lambda$ as well.}:

\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} |B|^2 dv = \int_{B_1} [v \times B \cdot (\nabla \times B) - \lambda |\nabla \times B|^2] dv. \tag{24}
\]

Using (3), (7), (20), (21), and (23) we obtain from (24) the estimate

\[
\frac{1}{2} \frac{d}{dt} \|B\|^2_\infty \leq (V_t + V_{nt}) \|B_p + B_t\| \|\nabla \times B\| - \lambda_0 \|\nabla \times B\|^2
\]

and by (17) and (18):

\[
\frac{1}{2} \frac{d}{dt} \|B\|^2_\infty \leq (V_t + V_{nt}) \left( \frac{1}{\sqrt{2}} \|\nabla P\|_\infty + \|\nabla T\| \right) \|\nabla \times B\| - \lambda_0 \|\nabla \times B\|^2. \tag{25}
\]

We consider next the poloidal scalar $P$. Multiplying (2) by $r$ one obtains by (4) the following evolution equation for $P$:

\[
\partial_t P = (-\nabla \cdot v B - v \cdot \nabla B + B \cdot \nabla v) \cdot r - \Lambda \cdot (\lambda \nabla \times B)
\]

\[
= -\nabla \cdot (v P) + B \cdot \nabla (v \cdot r) - \Lambda \lambda \cdot \nabla \times B + \lambda \Delta P. \tag{26}
\]

A suitable $P$-balance is then obtained by multiplying (26) by $P/\lambda$ and integrating over $B_1$:

\[
\int_{B_1} \partial_t PP/\lambda dv = - \int_{B_1} \nabla \cdot (v P) P/\lambda dv + \int_{B_1} B \cdot \nabla (v \cdot r) P/\lambda dv
\]

\[
- \int_{B_1} \Lambda \lambda \cdot \nabla \times B P/\lambda dv + \int_{B_1} \Delta PP dv. \tag{27}
\]

The various terms in (27) may be rewritten as follows:

\[
\int_{B_1} \partial_t PP/\lambda dv = \frac{1}{2} \frac{d}{dt} \int_{B_1} P^2/\lambda dv + \frac{1}{2} \int_{B_1} (\partial_1 \lambda/\lambda) P^2/\lambda dv,
\]

\[
\int_{B_1} \nabla \cdot (v P) P/\lambda dv = \frac{1}{2} \int_{B_1} \nabla \cdot (v P^2/\lambda) dv + \frac{1}{2} \int_{B_1} [\nabla \cdot v + v_r \partial_r \lambda/\lambda + v \cdot \nabla n r \lambda/\lambda] P^2/\lambda dv,
\]

\[
\int_{B_1} B \cdot \nabla (v \cdot r) P/\lambda dv = \int_{B_1} \nabla \cdot (B v_r P/\lambda) dv - \int_{B_1} B \cdot \nabla P r v_r/\lambda dv
\]

\[
+ \int_{B_1} v_r (\partial_r \lambda/\lambda) P^2/\lambda dv + \int_{B_1} r v_r \cdot (v n r \lambda/\lambda) P/\lambda dv,
\]

\[
\int_{B_1} \Lambda \lambda \cdot \nabla \times B P/\lambda dv = \int_{B_1} r \times (v n r \lambda/\lambda) \cdot (\nabla \times B) P dv,
\]

and (cf. Backus 1958)

\[
\int_{B_1} \Delta PP dv = - \int_{\mathbb{R}^3} |\nabla P|^2 dv.
\]
Thus, by Gauss’ law and (8) and using the definitions (20) - (23) the P-balance may be estimated as follows:

\[
\frac{1}{2} \frac{d}{dt} \|P/\lambda^{1/2}\|^2 \leq \frac{1}{2} \left[ DV + \delta \lambda + \delta_r \lambda V_{nt} + \delta_n \lambda (V_t + V_{nt}) \right] \|P/\lambda^{1/2}\|^2 \\
+ V_{nt} \left( \frac{\delta_n \lambda}{\lambda_0^{1/2}} \|P/\lambda^{1/2}\| + \frac{1}{\lambda_0} \|\nabla P\| \right) \|B_p + B_t\| \\
+ \delta_n \lambda \|\nabla B\| \|P\| - \|\nabla P\|^2. 
\]

By (12)_2, (17), and (18) we can further estimate:

\[
\frac{1}{2} \frac{d}{dt} \|P/\lambda^{1/2}\|^2 \leq \left\{ \frac{1}{2\pi^2 \lambda_0} \left[ DV + \delta b \lambda + \delta_r \lambda V_{nt} + \delta_n \lambda (V_t + V_{nt}) \right] \\
+ \frac{V_{nt}}{\sqrt{2} \lambda_0 \pi} \left( 1 + \frac{\delta_n \lambda}{\pi} \right) - 1 \right\} \|\nabla P\|^2 \|T\| \\
+ \frac{V_{nt}}{\lambda_0 \pi} \left( 1 + \frac{\delta_n \lambda}{\pi} \right) \|\nabla P\| \|\nabla T\| + \frac{\delta_n \lambda}{\pi} \|\nabla P\| \|\nabla \times B\|. 
\]

Note that the variable \(P/\lambda^{1/2}\) as against merely \(P\) on the left-hand side of (28) has the advantage that large radial \(\lambda\)-variations are admissible, since on the right-hand side \(\delta_r \lambda\) is multiplied by \(V_{nt}\), which can be made small.

Finally, the \(T\)-balance is obtained by applying \(\Lambda\) on (2)_1, multiplying by \(H\), and integrating over \(B_1\):

\[
\int_{B_1} \Lambda \cdot \partial_t B H dv = \int_{B_1} \nabla \times (v \times B) H dv - \int_{B_1} \Lambda \cdot \nabla \times \left( \lambda \nabla \times B \right) H dv. 
\]

By (3), (4)_2, (7), and twice integrating by parts (using the boundary condition \(\Lambda H|_{S_1} = 0\), which follows from (6)_1) (29) assumes the form

\[
\frac{1}{2} \frac{d}{dt} \int_{B_1} T^2 dv = - \int_{B_1} (\nabla \times \Lambda H) \cdot (v_t \times B_t) dv \\
- \int_{B_1} (\nabla \times \Lambda H) \cdot (v_{nt} \times B_t + v \times B_p) dv \\
+ \int_{B_1} \lambda (\nabla \times \Lambda H) \cdot \nabla \times B dv. 
\]

The first integral on the right-hand side of (30) vanishes (cf. Proctor 2004); this is the basic effect in the \(T\)-balance and explains the choice of the variable \(H\) in (29). The last integral in (30) may be rewritten as follows:

\[
\int_{B_1} \lambda (\nabla \times \Lambda H) \cdot \nabla \times B dv = - \int_{B_1} \lambda \left[ \frac{1}{r^2} (r H + \nabla_{nr} \partial_r (r H)) \right. \\
\left. \cdot \left[ (r/2)(-\mathcal{L}) T + \nabla_{nr} \partial_r (r T) + \nabla \times B_p \right] dv \\
= - \int_{B_1} \lambda \left[ \frac{1}{r^2} T (-\mathcal{L}) T + \nabla_{nr} \partial_r (r H) \cdot \nabla_{nr} \partial_r (r T) + \nabla_{nr} \partial_r (r H) \cdot \nabla \times B_p \right] dv. 
\]
Here we made use of (10), (11) and the representation (cf. Kaiser 2009b)
\[ \nabla \times \Lambda = (r/r^2) \mathcal{L} \cdot -\nabla_n r \partial_t (r \cdot). \]

Integrating by parts with (6) and (11) yields
\[
\int_{B_t} \lambda (\nabla \times \Lambda H) \cdot \nabla \times B \, dv = - \int_{B_t} [\lambda|\nabla_n T|^2 + \nabla_n \lambda \cdot \nabla_n T T \\
+ \lambda \left| \frac{1}{r} \partial_r (r T) \right|^2 - \nabla_n \lambda \cdot \nabla_n \partial_r (r H) \partial_r (r T) - \partial_r (r H) \nabla_n \lambda \cdot \nabla \times B_p] \, dv
\]

Thus, by (14) - (16) and by (20) - (23), eq. (30) can be estimated as follows:
\[
\frac{1}{2} \frac{d}{dt} \| T \|^2 \leq \frac{1}{\sqrt{2}} \| \nabla T \| [\| V_{nt} ||B_t|| + (V_t + V_{nt}) ||B_p||] - \lambda_0 \left[ \frac{1}{r} |\nabla (r T)| \right]^2
\]
\[
+ \lambda_0 \delta_n \lambda \left[ \| \nabla_n T \| \| T \| + \frac{1}{\sqrt{2}} \left| \frac{1}{r} \partial_r (r T) \right|^2 \right] + \frac{1}{2} \| \partial_r (r T) \| \| \nabla \times B_p \|
\]

and with (17), (18) one obtains
\[
\frac{1}{2} \frac{d}{dt} \| T \|^2 \leq \lambda_0 \left( \frac{V_{nt}}{\sqrt{2} \lambda_0} + \frac{\delta_n \lambda}{\sqrt{2}} - 1 \right) \| \nabla T \|^2 + \frac{1}{2} (V_t + V_{nt}) \| \nabla T \| \| \nabla P \|_\infty
\]
\[
+ \frac{1}{2} \lambda_0 \delta_n \lambda \| \nabla T \| \| \nabla \times B \|
\]

where we used \( \| T \| \leq (1/\sqrt{2}) \| \nabla_n T \| \) and \( \| \partial_r (r T) \| \leq \| (1/r) \partial_r (r T) \| = \| \partial_r T \| \leq \| \nabla T \| \).

4 A decaying magnetic functional

We apply in the following a kind of generalized energy method, i.e. we combine the magnetic field variables in such a way into a (magnetic energy dominating) functional that its time-derivative can be proved to be negative.

The following functional turns out to be suitable:
\[
\mathcal{F}[P, T, B] := \| P / \lambda^{1/2} \|^2 + 2 \mu \lambda_0 \| T \|^2 + \frac{\nu}{\lambda_0} \| B \|^2_\infty,
\]
where \( \mu, \nu > 0 \) are parameters yet to be determined. A differential inequality of type
\[
\frac{d}{dt} \mathcal{F} \leq -C \mathcal{F}
\]
with some constant \( C > 0 \) implies then exponential decay of \( \mathcal{F} \) to zero. In order to prove (33) it is expedient to standardize the energy-balances (25),(28), and (31) as much as possible; in particular, we introduce the scaled and non-dimensional maxima\(^7\)
\[
\begin{align*}
\delta_n \lambda &=: \delta^*_n \lambda, & \delta_n \lambda &=: \delta^*_n \lambda, & \delta_n \lambda &=: \delta^*_n \lambda, \\
V_t &=: V^*_t, & V_{nt} &=: V^*_t, & \frac{D V}{2 \pi^2 \lambda_0} &=: \langle D V \rangle.
\end{align*}
\]

\(^7\)Note that \( V^*_t \) and \( V^*_t \) can be interpreted (after restoring the radius) as toroidal and non-toroidal magnetic Reynolds numbers, respectively. The numerical factors have been inserted to ease the comparison with (Proctor 2004).
and we split the mixed terms by (19) introducing thereby further parameters. Inequality (28) takes then the form

\[
\frac{1}{2} \frac{d}{dt} ||P/\lambda^{1/2}||^2 \leq \left[DV^* + \delta^*_n \lambda + \delta^*_n V_{nt}^* + \left(1 + \frac{\epsilon_1}{2}\right)(1 + 2\delta^*_n \lambda) V_{nt}^* + \frac{1}{2} \delta^*_n \lambda (V_t^* + V_{nt}^*) + \eta_1 \delta^*_n \lambda - 1 \right] ||\nabla P||^2_{\infty} + \frac{1}{2\epsilon_1} (1 + 2\delta^*_n \lambda) V_{nt}^* 2 ||\nabla T||^2 + \frac{1}{\eta_1} \delta^*_n \lambda ||\nabla \times B||^2,
\]

(35)

where we estimated (for simplicity) \(\pi \geq 2\sqrt{2}\) and introduced \(\epsilon_1, \eta_1 > 0\). Analogously, (31) takes the form

\[
\frac{1}{\lambda_0} \frac{d}{dt} ||T||^2 \leq \frac{\epsilon_2}{2} (V_t^* + V_{nt}^*) ||\nabla P||^2_{\infty} + \left[\frac{V_t^* + 4 \delta^*_n \lambda + \frac{1}{2\epsilon_2} (V_t^* + V_{nt}^*)}{2} \right] \|\nabla T\|^2 + \frac{2}{\omega_1} \delta^*_n \lambda \|\nabla \times B\|^2
\]

(36)

with \(\epsilon_2, \omega_1 > 0\). For (25) we obtain, finally,

\[
\frac{1}{2\lambda_0} \frac{d}{dt} ||B||^2_{\infty} \leq \frac{\eta_2}{2} (V_t^* + V_{nt}^*) ||\nabla P||^2_{\infty} + \left[\frac{V_t^* + 4 \delta^*_n \lambda + \frac{1}{2\epsilon_2} (V_t^* + V_{nt}^*)}{2} \right] \|\nabla \times B\|^2 + \left[\frac{\epsilon_2}{2\eta_2} + \frac{\epsilon_2}{2\omega_2} (V_t^* + V_{nt}^*) - 1 \right] \|\nabla \times B\|^2
\]

(37)

with \(\eta_2, \omega_2 > 0\). The combination of (35) - (37) according to (32) yields

\[
\frac{d}{dt} \mathcal{F} \leq A \|\nabla P\|^2_{\infty} + B \|\nabla T\|^2 + C \|\nabla \times B\|^2,
\]

which, when estimated by (12), is of type (33) provided

\[
A, B, C < 0.
\]

(38)

Thus, (38) constitutes a sufficient criterion for decay of \(\mathcal{F}\). Explicitly, (38) reads

\[
DV^* + \delta^*_n \lambda + \eta_1 \delta^*_n \lambda + \delta^*_n V_{nt}^* + \left(1 + \frac{\epsilon_1}{2}\right)(1 + 2\delta^*_n \lambda) V_{nt}^* + \frac{1}{2} \delta^*_n \lambda (V_t^* + V_{nt}^*) + \eta_1 \delta^*_n \lambda - 1 \leq 1,
\]

\[
2(\omega_1 + 2)\delta^*_n \lambda + V_{nt}^* + \frac{1}{2\epsilon_1 \mu} (1 + 2\delta^*_n \lambda) V_{nt}^* + \left(1 + \frac{\omega_2}{2\mu} \nu \right)(V_t^* + V_{nt}^*) < 1,
\]

\[
\left(1 + \frac{2\mu}{\eta_1 \nu} \right) \delta^*_n \lambda + \left(1 + \frac{2\omega_2}{\nu} \right)(V_t^* + V_{nt}^*) < 1.
\]

(39)

To fix the parameters observe that the three parentheses multiplied by \(V_t^*\) must simultaneously become small for large \(V_t^*\). This can be achieved by the following choice of parameters that leaves just one, viz. \(x > 0\), free:

\[
\epsilon_1 = \epsilon_2 = \frac{1}{\sqrt{\mu}}, \quad \eta_1 = \eta_2 = \frac{1}{\sqrt{\nu}}, \quad \omega_1 = \omega_2 = \sqrt{\mu/\nu}, \quad \nu = \mu^2 =: x^4.
\]

(40)
Conditions (39) then take the form

\[
DV^* + \delta^*_{n^*} + \frac{1}{x^2} \delta^*_{n^*} + \delta^*_{t^*} V^*_{n^t} + \left(1 + \frac{1}{2x}\right)(1 + 2\delta^*_{n^*} V^*_{n^t}) + \left(\frac{1}{2} \delta^*_{n^*} + \frac{x^2}{2}\right)(V^*_t + V^*_n) < 1,
\]

\[
2\left(\frac{1}{x^2} + 2\right)\delta^*_{n^*} + V^*_n + \frac{1}{2x}(1 + 2\delta^*_{n^*}) V^*_{n^t} + x(V^*_t + V^*_n) < 1,
\]

\[
\left(\frac{1}{x^2} + \frac{2}{x}\right)\delta^*_{n^*} + \frac{1}{2}(x^2 + x)(V^*_t + V^*_n) < 1.
\]

For \(x \leq 1\) these three conditions may be replaced by the following single sufficient condition:

\[
DV^* + \delta^*_{t^*} + \frac{6}{x^2} \delta^*_{n^*} + \delta^*_{n^*} V^*_{n^t} + \left(1 + \frac{1}{2x}\right)(1 + 2\delta^*_{n^*} V^*_{n^t}) + \left(\frac{1}{2} \delta^*_{n^*} + x\right)(V^*_t + V^*_n) < 1,
\]

or, by rearrangement,

\[
DV^* + \delta^*_{t^*} + (1 + \delta^*_{n^*}) V^*_{n^t} + \frac{1}{2} \delta^*_{n^*}(V^*_t + 5V^*_n)
\]

\[
+ \frac{6}{x^2} \delta^*_{n^*} + \frac{1}{x} \delta^*_{n^*} V^*_{n^t} + \frac{1}{2x} V^*_n + x(V^*_t + V^*_n) < 1. \tag{41}
\]

The choice

\[
x := \frac{1}{\sqrt{2}} \sqrt{\frac{V^*_n}{V^*_t + V^*_n}}
\]

equilibrates the last and the second last terms in (41) with the result:

\[
DV^* + \delta^*_{t^*} + (1 + \delta^*_{n^*}) V^*_{n^t} + \delta^*_{n^*}\left[12 + \frac{1}{2}(V^*_t + 5V^*_n)\right]
\]

\[
+ (1 + \delta^*_{n^*}) [2V^*_n(V^*_t + V^*_n)]^{1/2} + 12 \delta^*_{n^*} \lambda \left\frac{V^*_t}{V^*_n} < 1. \tag{42}
\]

Condition (42) can clearly be met for given (arbitrarily large) values of \(V^*_t\) and \(\delta^*_{t^*}\), if only \(DV^*, \delta^*_{t^*}, \delta^*_{n^*}\), and \(V^*_{n^t}\) are small enough; there is, however, a hierarchy between \(\delta^*_{n^*}\) and \(V^*_{n^t}\), as exhibited by the last term in (42): \(\delta^*_{n^*}\) must be small compared to \(V^*_{n^t}\). Another relation between \(\delta^*_{n^*}\) and \(V^*_{n^t}\) is elucidated by the choice

\[
x := \left(\frac{12 \delta^*_{n^*} \lambda}{V^*_t + V^*_n}\right)^{1/3},
\]

which equilibrates the last and the fourth last terms in (41) with the result:

\[
DV^* + \delta^*_{t^*} + (1 + \delta^*_{n^*}) V^*_{n^t} + \frac{1}{2} \delta^*_{n^*}(V^*_t + 5V^*_n)
\]

\[
+ 3\left(\frac{3}{2} \delta^*_{n^*}\right)^{1/3} (V^*_n + V^*_n)^{2/3} + \left[V^*_n(\delta^*_{n^*})^{2/3} + \frac{V^*_n(\delta^*_{n^*})^{1/3}}{(\delta^*_{n^*})^{1/3}}\right] \left(\frac{1}{12} (V^*_t + V^*_n)\right)^{1/3} < 1. \tag{43}
\]

Here, the last term requires that \(V^*_n\) is small compared to \((\delta^*_{n^*})^{1/3}\). Of course, other choices of \(x\) in (41) yield other explicit conditions expressing the robustness of the toroidal velocity.
The inclusion of $\delta V$ and (43) requires then $\lambda$. When the diffusivity problem), the diffusivity can indeed regenerate magnetic fields (see Busse & Wicht 1992 for a planar model remembered, however, that purely toroidal flows when acting in fluids with laterally variable other side, inclusion of $\parallel V$ is a trace of the additional (and for $\lambda$) for large $V$.

b) In the case $DV^* = \delta^*_V = \delta^*_n = \delta^*_m = 0$, condition (42) reduces to

$$V_t^* + \sqrt{2} \sqrt{V_p^*(V_p^* + V_t^*)} < 1,$$

which is (up to the factor $\sqrt{2}$, see the first comment below) the condition in (Proctor 2004).

c) In the case $DV^* = V_{nt}^* = 0$, condition (43) reduces to

$$\delta_t^* + \left(\frac{3}{2} \delta_m^*\right)^{1/3} V_t^{*2/3} \left[3 + \frac{1}{3} \left(\frac{3}{2} \delta_m^*\right)^{2/3} V_t^{*1/3}\right] < 1,$$

which, for large $V_t^*$, amounts to

$$\delta_t^* \lesssim \frac{2((1 - \delta_t^*)/3)^3}{3 V_t^{*2}} \quad \text{for} \quad V_t^* \gg 1.$$

d) For large $V_t^*$, say $V_t^* \sim 1/\epsilon$, $\epsilon \ll 1$, the second last and last terms in (42) require $V_{nt}^* \lesssim \epsilon$, $\delta_m^* \lesssim \epsilon^2$ and (42) simplifies to

$$DV^* + \delta_t^* \lambda + \delta_m^* V_{nt}^* + (2 V_t^* V_{nt}^*)^{1/2} + 12 \delta_m^* \frac{V_t^*}{V_{nt}^*} < 1;$$

whereas (43) in the large-$V_t^*$-limit reads

$$DV^* + \delta_t^* \lambda + \delta_m^* V_{nt}^* + 3 \left(\frac{3}{2} V_t^{*2} \delta_m^*\right)^{1/3} + \left(\frac{V_t^* V_{nt}^*}{12 \delta_m^*}\right)^{1/3} < 1.$$  

We close with some more comments:

1. The inclusion of $\|B\|_\infty^2$ in (32) appears to be redundant at first sight. The energy-balances of $P$ and $T$, however, contain second-order radial derivatives of these variables, which cannot be controlled by their energy-balances alone. These terms are associated with $\delta_m^* \lambda$ and are not present in the case of constant diffusivity as in (Proctor 2004). The differing factor $\sqrt{2}$ in (44) is a trace of the additional (and for $\delta_m^* = 0$ truly redundant) $B$-balance. On the other side, inclusion of $\|B\|_\infty^2$ in (32) automatically guarantees decay of the magnetic field itself and not only of its representing scalars.\textsuperscript{8}

2. The criterion (45) specializing on purely toroidal flows might appear void. It must be remembered, however, that purely toroidal flows when acting in fluids with laterally variable diffusivity can indeed regenerate magnetic fields (see Busse & Wicht 1992 for a planar model problem).

3. When the diffusivity $\lambda$ is of type (1) modelling an approximately spherical fluid volume we find $\delta_\lambda^* \sim 1/\epsilon^2$ and $\delta_m^* \sim \epsilon$ at the “boundary” $r \approx 1$. The third term in both criteria (42) and (43) requires then $V_{nt}^* \lesssim \epsilon^2$ at $r \approx 1$, which is the equivalent of the boundary condition

\textsuperscript{8}Decay of higher derivatives of the magnetic scalars, especially of the magnetic field $B$ and of the electric current $J = \nabla \times B$, can in effect be deduced from the dynamo equation once decay of the scalars in the energy norm is established (see Kaiser 2015).
Note that criterion (43) allows large toroidal flow $V_t^*$ at $r \approx 1$, whereas criterion (42) does not.

4. In this note we did not focus on obtaining optimal estimates or the most detailed criteria. Although, most of the basic estimates in section 2 are known to be optimal, the estimates of the energy-balances in section 3 are at some places not as sharp as possible. Moreover, finer criteria can supposedly be formulated if radial and non-radial components of $B_p, \nabla P,$ and $\nabla T$ are separately considered, or if one distinguishes between the various components of $V_{nt}$. Finally, as already noted, by different choices of the parameters different weights can be given to the various perturbations collected in condition (39).

5. As to applications one should distinguish between stellar and planetary dynamos with typically very large magnetic Reynolds numbers, numerical simulations of these objects using effective Reynolds numbers, and laboratory experiments with Reynolds numbers of order one. In the first case numbers $V_t^* \gg 1$ and $V_{nt}^* > 1$ clearly violate conditions (42) and (43), i.e. the toroidal velocity theorem is not applicable. So, in the astrophysical context arguments based on the toroidal velocity theorem (like that mentioned in the introduction) should be taken with care. This may be different when considering numerical simulations of planetary and stellar dynamos. These often work with eddy diffusivities leading to “effective” Reynolds numbers that are significantly smaller than the “true” numbers. Here conditions of type (42) or (43) can provide valuable information for the design and interpretation of computational models. The same may be said about dynamo experiments, where these conditions can be helpful in the design of efficient dynamos, i.e. velocity fields with as low as possible critical magnetic Reynolds numbers. Finally, it should be remembered that our theorem formulates only necessary conditions for the applicability of the toroidal velocity theorem. The range of applicability could well be much larger but this is not guaranteed.

6. The question of robust versions may, of course, be asked also for other antidynamo theorems. The non-radial-velocity theorem, for example, is a natural generalization of the toroidal velocity theorem, in that the radial component of an otherwise unrestricted velocity field vanishes (Ivers & James 1988). One might thus expect a criterion of type (42) (without $DV^*$) distinguishing between (small) radial and (large) non-radial velocity components. The proof of such a result, however, requires refined energy estimates as can be concluded from the rigorous proof of the classical theorem (Kaiser 2007). The situation is similar for the axisymmetric (or Cowling’s) theorem. The proof of its present quite general form requires a variety of mathematical methods, which go beyond energy estimates (see, e.g., Kaiser & Tilgner 2014). A robust version of this theorem thus seems to require more than mere energy-balances. Work on this problem is in progress.

References


