Uniqueness and non-uniqueness in the non-axisymmetric direction problem

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Abstract

We consider the following nonlinear boundary-value problem in the exterior space \( \hat{V} = \{ x \in \mathbb{R}^3 : |x| > 1 \} \) of the unit sphere \( S \): given a vector field \( D : S \to \mathbb{R}^3 \), we ask for all harmonic vector fields \( B : \hat{V} \to \mathbb{R}^3 \) vanishing at infinity and parallel to \( D \) on \( S \), i.e. there is an amplitude \( a : S \to \mathbb{R} \) such that \( B = aD \). This ‘geomagnetic direction problem’ is related to the problem of reconstructing the geomagnetic field outside the earth from directional data measured on the earth’s surface.

Here we investigate this problem using the well-known multipole expansion \( B = \sum_{n=1}^{\infty} \sum_{|k|=-n}^{n} c_n^k B_n^k \) of exterior harmonic vector fields. In the case that only finite linear combinations of the \( B_n^k \) are admissible solutions we prove uniqueness of the problem. Without this restriction, non-uniqueness of the problem is well established in the axisymmetric case. In the non-axisymmetric case, we prove here uniqueness for all direction fields \( D_n^k := B_n^k|_S \) (\( n \in \mathbb{N} \), \( 0 < |k| \leq n \)), but give also an example for non-uniqueness.

Keywords: Nonlinear boundary value problem, geomagnetism, direction problem.

1 Introduction

The geomagnetic direction problem arises in the endeavour to determine the magnetic field \( B \) outside the earth (or any other celestial body) if only the direction \( D \) of the field is known on the earth’s surface. In fact, archaeomagnetic, palaeomagnetic, and even historical magnetic data sets up to the 19th century contain either exclusively information about the direction of the magnetic field vector or provide the directional information more reliably than information about the magnitude of the field vector (for more information about the significance of the direction problem for geomagnetism, we refer to [MM83, PG90, Ka10]). Even in the ideal case that deviations from the spherical shape \( S \) of the earth’s surface are neglected, that the exterior region \( \hat{V} \) is assumed to be insulating and free of sources of magnetic field, and that the directional data are everywhere on \( S \) available this boundary value problem is presently insufficiently understood.
Formally, the direction problem takes the form
\[
\begin{align*}
\nabla \times \mathbf{B} &= 0, \quad \nabla \cdot \mathbf{B} = 0 \quad \text{in } \hat{V}, \\
\mathbf{B}(\mathbf{x}) &\to 0 \quad \text{for } |\mathbf{x}| \to \infty, \\
\mathbf{B} \parallel \mathbf{D} &\quad \text{on } S.
\end{align*}
\]
(1.1)

It is the nonlinear boundary condition (1.1) that makes the problem different from the standard boundary value problems of potential theory, which specify either the normal component or the tangential components at the boundary. Existence and uniqueness results comparable to those which are well established for the standard problems (see, e.g., [Ma68, p. 221f and p. 240f]) are not yet known for the direction problem. So far only the axisymmetric situation is fairly well understood [Ka10, Ka11]: denoting the solution space of problem (1.1) for fixed direction field \( \mathbf{D} \) by \( \mathcal{L}(\mathbf{D}) \) and the rotation number of \( \mathbf{D} \) along a meridian on \( S \) by \( \varrho \) one finds \( \dim \mathcal{L}(\mathbf{D}) = \max\{\varrho - 1, 0\} \). So, uniqueness (up to a multiplicative constant) holds only for \( \varrho = 2 \), which describes for instance dipole-type direction fields. In the general (non-axisymmetric) situation, the only result of general character is an upper bound on \( \dim \mathcal{L}(\mathbf{D}) \) in terms of the number \( l_D \) of ‘poles’ of \( \mathbf{D} \) (loci on \( S \) with vanishing tangential components): \( \dim \mathcal{L}(\mathbf{D}) \leq l_D - 1 \) [HKL97]. So, this criterion predicts uniqueness for dipole-type direction fields; for more than two poles, however, the uniqueness question is still open. The aim of the present paper is to contribute to this uniqueness question.

In order to make problem (1.1) more precise we introduce the complex Hilbert space
\[
L^2(S) = \left\{ h : S \to \mathbb{C} \left| \int_S |h|^2 ds < \infty \right. \right\}
\]
with scalar product \( (h_1, h_2) := \int_S h_1 h_2^* ds \) and norm \( \|h\| := (h, h)^{1/2} \). It is, furthermore, convenient to introduce a potential \( \Psi \) for the harmonic field \( \mathbf{B} \), i.e. \( \mathbf{B} = \nabla \Psi \). Problem (1.1) can then be specified as follows.

**Direction problem:** Let \( \mathbf{D} \in L^2(S)^3 \). Determine \( L_D \), the space of all functions \( \Psi \in H^2(\hat{V}; \mathbb{C}) \) satisfying the boundary value problem
\[
\begin{align*}
\Delta \Psi &= 0 \quad \text{in } \hat{V}, \\
\Psi(\mathbf{x}) &\to 0 \quad \text{for } |\mathbf{x}| \to \infty, \\
\nabla \Psi &= a \mathbf{D} \quad \text{on } S, \\
\int_S \mathbf{x} \cdot \nabla \Psi ds &= 0
\end{align*}
\]
(1.2)

for some measurable almost everywhere non-vanishing function \( a : S \to \mathbb{R} \).

For \( \Psi \in H^2(\hat{V}) \), the gradient \( \nabla \Psi \) has a trace in \( L^2(S) \) and the boundary condition (1.2) is to be understood in the \( L^2(S) \)-sense. Note that the boundary condition fixes the direction of the solution (almost everywhere) on \( S \) but not its sign. Since the amplitude function is assumed to be real, real solutions of problem (1.2) are immediately obtained by taking real or imaginary parts. The complex formulation, however, allows the use of the spherical harmonics \( \{Y_n^k : n \in \mathbb{N}, |k| \leq n\} \), which form an orthonormal basis in \( L^2(S) \). Neglecting for a moment the boundary condition (1.2)\(_3\), solutions of (1.2)\(_{1,2,4}\) in spherical coordinates \( (r, \theta, \varphi) \) allow the well-known multipole expansion
\[
\Psi(r, \theta, \varphi) = \sum_{n=1}^{\infty} \sum_{k=-n}^{n} \frac{c_n}{\sqrt{\pi n + 1}} Y_n^k(\theta, \varphi)
\]
(1.3)
with coefficients $c_k^n$ satisfying $\sum_{n=1}^{\infty} \sum_{k=-n}^{n} (n+1)^2 |c_k^n|^2 < \infty$ in order that $\partial_r \Psi|_S \in L^2(S)$. The series (1.3) converges uniformly for any $r \geq r_0 > 1$, thus $\Psi$ is in fact an (exterior) harmonic function in $\hat{V}$. With

$$\nabla \Psi = \partial_r \psi \mathbf{e}_r + \frac{1}{r} \partial_\theta \psi \mathbf{e}_\theta + \frac{1}{r \sin \theta} \partial_\phi \psi \mathbf{e}_\phi,$$

the boundary condition (1.2) can equivalently be written in a form free of the amplitude function $a$:

$$\begin{align*}
\partial_r \psi|_S D_\theta - \partial_\theta \psi|_S D_r &= 0, \\
\sin \theta \partial_r \psi|_S D_\phi - \partial_\phi \psi|_S D_r &= 0,
\end{align*}$$

These equations have to be satisfied almost everywhere on $S$ and, of course, any two of them, if non-trivial, imply the third. To investigate uniqueness of the direction problem, it suffices to consider direction fields of the form $D = \nabla \Phi|_S$ with potentials $\Phi$ of type (1.3).

In the next section, we prove uniqueness of the direction problem for finite potentials (for the direction field as well as for the solution), i.e. there is $n_0 \in \mathbb{N}$ such that $c_n^k = 0$ for all $n > n_0$ in the representation (1.3). Interestingly, this problem has already been analysed [Ko76] and the author claimed to have proved uniqueness; subsequent investigators, however, either did not take notice or doubted the results and, be that as it may, described it as an open problem [PG90, HKL97]. In fact, Kono’s analysis contains a weak point: he shows correctly using even only inclination data that both potentials $\Phi$ and $\Psi$ have the same order $n_0$ and argues then with the relation

$$\partial_r \psi|_S = a \partial_r \Phi|_S,$$

that a non-constant amplitude would necessarily yield higher-than-$n_0$-order terms on the right-hand side. This argument, however, is obvious only if $a$ itself is of finite order. Otherwise, on the right-hand side there is no term of highest order which is easy to identify, and there is the possibility that the infinitely many higher-order terms cancel each other. So, because of this gap in the argument, it might seem desirable to have an independent proof avoiding (conditions on) the amplitude function.

As a second result, we prove in section 3 uniqueness for any direction field corresponding to a single non-axisymmetric multipole field, i.e. $D = D_n^k := \nabla (r^{-n-1} Y_n^k)|_S$, $n \in \mathbb{N}$, $0 < |k| \leq n$. This result is, in fact, not new (see [Ka05]). The present proof, however, is much simpler than that in [Ka05], which is based on a Hilbert space criterion formulated in [KN04]. In fact, [Ka05] used a slight modification of this criterion, which had not explicitly been proved. Moreover, it made use of some factorization properties of (associated) Legendre polynomials which seem to be commonly believed but not rigorously be proved.

In view of the previous results, one is seduced to assume generally uniqueness in the non-axisymmetric situation. This, however, is false as demonstrated in section 4. We give the example of a non-axisymmetric quadrupole-type direction field and prove the existence of two linearly independent solutions.
2 Uniqueness in the finite-dimensional direction problem

We deal in this section with exterior harmonic potentials \( \Psi \) and \( \Phi \) with finite multipole expansions

\[
\Psi(r, \theta, \varphi) = \sum_{n=0}^{n_0} \sum_{k=-n}^{n} a_n^k r^{-n-1} Y_n^k(\theta, \varphi) \tag{2.1}
\]

and

\[
\Phi(r, \theta, \varphi) = \sum_{\nu=0}^{\nu_0} \sum_{\kappa=-\nu}^{\nu} b_\nu^\kappa r^{-\nu-1} Y_\nu^\kappa(\theta, \varphi) \tag{2.2}
\]

of order \( n_0 \) and \( \nu_0 \), respectively, i.e. \( a_n^k \neq 0 \) for some \( k \) and \( b_\nu^\kappa \neq 0 \) for some \( \kappa \). We define then \( k_0 := \max\{ k : a_n^k \neq 0 \} \) and \( \kappa_0 := \max\{ \kappa : b_\nu^\kappa \neq 0 \} \). Monopole terms \( n = \nu = 0 \) here are also allowed. We then have

**Theorem 1** Let \( \Psi \) and \( \Phi : \hat{V} \to \mathbb{C} \) be non-trivial harmonic functions with finite multipole expansions and satisfying the boundary condition

\[
\nabla \Psi|_S = a \nabla \Phi|_S \tag{2.3}
\]

for some almost nowhere vanishing amplitude \( a : S \to \mathbb{R} \). Then

\[
\Psi = c \Phi \quad \text{in} \ \hat{V} \tag{2.4}
\]

for some \( c \in \mathbb{R} \setminus \{0\} \).

**Proof:** We exploit the boundary condition in the form (1.4) with \( D = \nabla \Phi|_S \). Inserting (2.1) and (2.2), this yields

\[
\sum_{n=0}^{n_0} \sum_{\nu=0}^{\nu_0} \sum_{(k,\kappa) \in \mathbb{Z}^2} a_n^k b_\nu^\kappa ((n+1)Y_n^k \partial_\theta Y_\nu^\kappa - (\nu+1)Y^k_\nu \partial_\theta Y_n^k) = 0 \tag{2.5}
\]

and

\[
\sum_{n=0}^{n_0} \sum_{\nu=0}^{\nu_0} \sum_{(k,\kappa) \in \mathbb{Z}^2} a_n^k b_\nu^\kappa ((n+1)Y_n^k \partial_\varphi Y_\nu^\kappa - (\nu+1)Y^k_\nu \partial_\varphi Y_n^k) = 0, \tag{2.6}
\]

where we have set \( a_n^k = 0 \) for \( |k| > n \) and \( b_\nu^\kappa = 0 \) for \( |\kappa| > \nu \) which spares us to specify the ranges for \( k \) and \( \kappa \). To exploit the polynomial character of the \( Y_n^k \) we make use of the following representations (cf. [KN04, App. A]):

\[
Y_n^k(\theta, \varphi) = d_n^k P_n^{|k|}(\cos \theta) e^{ik\varphi}, \quad d_n^k \neq 0, \quad n \in \mathbb{N}_0, \ |k| \leq n, \tag{2.7}
\]

\[
P_n^m(x) = (1 - x^2)^{m/2} \frac{d^m}{dx^m} P_n(x), \quad n, m \in \mathbb{N}_0, \tag{2.8}
\]

\[
P_n(x) = P_n^{(n)} x^n + O(x^{n-2}), \quad P_n^{(n)} \neq 0, \quad n \in \mathbb{N}_0. \tag{2.9}
\]

To relate the leading coefficient \( p_n^{(n)} \) of the Legendre polynomial \( P_n \) to those of the associated Legendre polynomial \( P_n^m \) and its derivative \( P_n^{m'} \) we distinguish between even and odd degree \( m \):

\[
P_n^m(x) = p_n^{(n,m)} x^n + O(x^{n-2}), \quad x P_n^{m'}(x) = n p_n^{(n,m)} x^n + O(x^{n-2}) \tag{2.10}
\]
with \( p_n^{(m)} = \frac{n!}{(n-m)!} (-1)^{m/2} p_n^{(m)} \neq 0 \) for even \( m \) and
\[
P_m^{\nu}(x) = \frac{(1 - x^2)^{1/2}}{x} (p_n^{(m)} x^n + O(x^{-2})) = \frac{(1 - x^2)^{1/2}}{x} \left( n p_n^{(m)} x^n + O(x^{-2}) \right),
\]
\( x P_m^{\nu}(x) = \frac{(1 - x^2)^{1/2}}{x} \left( n P_n^{(m)} x^n + O(x^{-2}) \right)
\]
(2.11)

with \( p_n^{(m)} = \frac{n!}{(n-m)!} (-1)^{(m-1)/2} p_n^{(m)} \neq 0 \) for odd \( m \). In particular, \( P_m^{\nu}(x) \) with even \( m \) and \( x(1 - x^2)^{-1/2} P_m^{\nu}(x) \) with odd \( m \) are again polynomials of order \( n \).

Introducing the variables \( x := \cos \theta \), \( y := e^{i\theta} \), and \( z := \tan \theta \) the parenthesis in (2.5) takes then with (2.7) and (2.8) the form (up to a factor \(-z\)):
\[
d_n^k d_{\nu}^{\kappa} \left( (n + 1) P_n^{[k]}(x) x P_{\nu}[k](x) - (\nu + 1) P_{\nu}[k](x) x P_n^{[k]}(x) \right) y^{k+\kappa} =: Q^{(n k \nu \kappa)}(x) y^{k+\kappa} L^{(k+\kappa)}(z)
\]
(2.12)

with
\[
L^{(k+\kappa)}(z) = \begin{cases} 
1 & k + \kappa \text{ even} \\
z & k + \kappa \text{ odd}
\end{cases}
\]

and the polynomial (of order at most \( n + \nu \)):
\[
Q^{(n k \nu \kappa)}(x) = \sum_{N=0}^{n+\nu} q_N^{(n k \nu \kappa)} x^N.
\]
(2.13)

\( Q \) satisfies the symmetry property
\[
Q^{(n k \nu \kappa)}(x) = -Q^{(\nu k \kappa n)}(x) \quad \iff \quad q_N^{(n k \nu \kappa)} = -q_N^{(\nu k \kappa n)} \quad \text{for} \quad N = 0, \ldots, n + \nu,
\]
(2.14)

and (2.10), (2.11) imply for its leading coefficient:
\[
q_n^{(n k \nu \kappa)} = (-1)^{k \kappa} d_n^k d_{\nu}^{\kappa} \left( (n + 1) \nu - (\nu + 1) n \right) p_n^{(n k \nu \kappa)} p_{\nu}^{(n k \nu \kappa)}.
\]
(2.15)

Now substituting (2.12), (2.13) into (2.5) yields after some rearrangement:
\[
0 = \sum_{N=0}^{n+\nu} \sum_{K=-(n+\nu)}^{n+\nu} \sum_{(n,\nu) \in \mathbb{N}_0^2, \nu = n + \nu} \sum_{k+\kappa = K} \sum_{a_n^k b_{n}^{\kappa} Q^{(n k \nu \kappa)}(x) y^{k+\kappa} L^{(K)}(z)}
\]
(2.16)

with
\[
r_N^{K} = \sum_{m=K}^{n+\nu} \sum_{(n,\nu) \in \mathbb{N}_0^2, \nu = n + \nu} \sum_{k+\kappa = K} \sum_{a_n^k b_{n}^{\kappa} q_N^{(n k \nu \kappa)}}.
\]
(2.17)

Analogously the parenthesis in (2.6) takes the form (up to a factor \( i \)):
\[
d_n^k d_{\nu}^{\kappa} \left( (n + 1) \nu - (\nu + 1) k \right) P_n^{[k]}(x) P_{\nu}[k](x) y^{k+\kappa} =: \hat{Q}^{(n k \nu \kappa)}(x) y^{k+\kappa} L^{(k+\kappa)}(z).
\]
(2.18)

\( \hat{Q} \) is again a polynomial (of order at most \( n + \nu \)),
\[
\hat{Q}^{(n k \nu \kappa)}(x) = \sum_{N=0}^{n+\nu} \hat{q}_N^{(n k \nu \kappa)} x^N,
\]
(2.19)
with leading coefficient

\[ q^{(n, k, \nu, \kappa)}_{n+\nu} = (-1)^k a_n^k d^k \left( (n+1)\kappa - (\nu+1)k \right) p^{(n|k)}_{n}\left(p^{(\nu|\kappa)}_{\nu}\right), \]

and satisfying the symmetry property (2.14). Substituting (2.18), (2.19) into (2.6) yields again a polynomial of type (2.16):

\[ 0 = \sum_{N=0}^{n_0+\nu_0} \sum_{k_0+\kappa_0} r^K_N z^N y^K L(K) \]

with coefficients of type (2.17) with \( q^{(n, k, \nu, \kappa)}_N \), however, replaced by \( \hat{q}^{(n, k, \nu, \kappa)}_N \).

Now, (2.16) and (2.21) clearly imply

\[ r^K_N = \hat{r}^K_N = 0 \quad \text{for} \quad 0 \leq N \leq n_0 + \nu_0, \quad -(n_0 + \nu_0) \leq K \leq k_0 + \kappa_0. \]

By definition we have \( a_{n_0}^{k_0} \neq 0 \) and \( a_{n_0}^k = 0 \) for \( n > n_0 \) or \( n = n_0, k > k_0 \), and analogous results hold for \( b_{\nu_0}^k \). We set \( c := a_{n_0}^{k_0} b_{\nu_0}^{k_0} \neq 0 \) and, by systematically exploiting (2.22), we show \( a_n^k = c b_n^k \) for all \( n \in \mathbb{N}, k \in \mathbb{Z} \), which is the assertion.

(i) We show first \( (n_0, k_0) = (\nu_0, \kappa_0) \), hence \( a_{n_0}^{k_0} = c b_{\nu_0}^{k_0} \). By (2.17) we have

\[ 0 = r^{k_0+\kappa_0}_{n_0+\nu_0} = a_{n_0}^{k_0} b_{\nu_0}^{k_0} q^{(n_0, k_0, \nu_0, \kappa_0)}_{n_0+\nu_0} \]

and by (2.15) we conclude \( n_0 = \nu_0 \). Similarly, \( \hat{r}^{k_0+\kappa_0}_{2n_0} = 0 \) implies \( \hat{q}^{(n_0, k_0, n_0, \kappa_0)}_{2n_0} = 0 \) and by (2.20) we obtain \( k_0 = \kappa_0 \).

(ii) We show next by induction \( a_{n_0}^k = c b_{n_0}^k \) for all \( k \in \mathbb{Z} \). By (i) we have \( a_{n_0}^0 = c b_{n_0}^0 \). We assume now for some \( l \in \mathbb{N} \)

\[ a_{n_0}^{k_0-i} = c b_{n_0}^{k_0-i} \quad \text{for} \quad 0 \leq i \leq l - 1 \]

and show \( a_{n_0}^{k_0-l} = c b_{n_0}^{k_0-l} \). We have by (2.17), (2.14), and (2.23):

\[ 0 = \hat{r}^{2k_0-l}_{2n_0} = a_{n_0}^{k_0-l} b_{n_0}^{k_0-l} \hat{q}^{(n_0, k_0-l, n_0, \kappa_0)}_{2n_0} + a_{n_0}^{k_0} b_{n_0}^{k_0-l} \hat{q}^{(n_0, k_0, n_0, \kappa_0)}_{2n_0} - b_{n_0}^{k_0} \left( a_{n_0}^{k_0-l} - c b_{n_0}^{k_0-l} \right) \hat{q}^{(n_0, k_0-l, n_0, \kappa_0)}_{2n_0} + c b_{n_0}^{k_0} b_{n_0}^{k_0-l} \hat{q}^{(n_0, k_0, n_0, \kappa_0)}_{2n_0} \]

The assertion follows since by (2.20) \( \hat{q}^{(n_0, k_0-l, n_0, \kappa_0)}_{2n_0} \neq 0 \).

(iii) To prove (2.4) we proceed again by induction. By (ii) we have \( a_{n_0}^k = c b_{n_0}^k \) for all \( k \in \mathbb{Z} \). We assume now for some \( m \) with \( 1 \leq m \leq n_0 \):

\[ a_{n_0-j}^k = c b_{n_0-j}^k \quad \text{for} \quad 0 \leq j \leq m - 1, \quad k \in \mathbb{Z}, \]

(2.24)
and show $a_{n_0-m}^k = c b_{n_0-m}^k$ for all $k \in \mathbb{Z}$. Let $k_m := \max\{k : a_{n_0-m}^k \neq 0 \text{ or } b_{n_0-m}^k \neq 0\}$ and $l \in \mathbb{N}_0$. By (2.17), (2.14) and (2.24) we have:

$$0 = r_0^{k_0+k_m-l} = \sum_{(n,\nu) \in \mathbb{N}_0^2} \sum_{n+\nu=2n_0-m} \sum_{k+\kappa=k_0+k_m-l} a_{n}^{k_{\kappa}} b_{\nu}^{(n \kappa \nu)} q_{2n_0-m}$$

$$+ \sum_{j=0}^{m-1} \sum_{(n,\nu) \in \mathbb{N}_0^2} \sum_{n+\nu=2n_0-m} \sum_{k+\kappa=k_0+k_m-l} a_{n}^{k_{\kappa}} b_{\nu}^{(n \kappa \nu)} q_{2n_0-m}$$

$$= \sum_{(k,\kappa) \in \mathbb{Z}^2} (a_{n_0}^{k} b_{n_0-m}^k q_{2n_0-m}^{(n_0 k n_0-m k_0)}) + a_{n_0-m}^k b_{n_0}^k q_{2n_0-m}^{(n_0-m k n_0 \kappa)}$$

$$+ \sum_{j=0}^{m-1} \sum_{(n,\nu) \in \mathbb{N}_0^2} \sum_{n+\nu=2n_0-m} \sum_{k+\kappa=k_0+k_m-l} c b_{n}^{k_{\kappa}} b_{\nu}^{(n \kappa \nu)} q_{2n_0-m}$$

$$= \sum_{(k,\kappa) \in \mathbb{Z}^2} (c b_{n_0}^k b_{n_0-m}^k q_{2n_0-m}^{(n_0 k n_0-m k_0)}) + a_{n_0-m}^k b_{n_0}^k q_{2n_0-m}^{(n_0-m k n_0 \kappa)}.$$  (2.25)

In the special case $l = 0$ again by (2.14) this amounts to:

$$0 = b_{n_0}^k (c b_{n_0-m}^k - a_{n_0-m}^k) q_{2n_0-m}^{(n_0 k n_0-m k_0)}$$

and hence $a_{n_0-m}^k = c b_{n_0-m}^k$ since by (2.15) $q_{2n_0-m}^{(n_0 k n_0-m k_0)} \neq 0$. The rest is proved by induction over $l$. So, let

$$a_{n_0-m}^{k-m-i} = c b_{n_0-m}^{k-m-i} \quad \text{for} \quad 0 \leq i \leq l - 1. \quad (2.26)$$

By (2.25), (2.26) we obtain

$$0 = r_0^{k_0+k_m-l} = c b_{n_0}^k b_{n_0-m}^{k-m-l} q_{2n_0-m}^{(n_0 k n_0-m k_0-m-l)} + a_{n_0-m}^{k-m-l} b_{n_0}^k q_{2n_0-m}^{(n_0-m k n_0-l n_0 k_0)}$$

$$+ \sum_{(k,\kappa) \in \mathbb{Z}^2, k \neq k_0} c b_{n_0}^k b_{n_0-m}^k q_{2n_0-m}^{(n_0 k n_0-m \kappa)} + \sum_{(k,\kappa) \in \mathbb{Z}^2, k \neq k_0} c b_{n_0}^k b_{n_0-m}^k q_{2n_0-m}^{(n_0-m k n_0 \kappa)}.$$  (3.1)

Again by (2.14) the two sums cancel and the first two terms yield $a_{n_0-m}^{k-m-l} = c b_{n_0-m}^{k-m-l}$. This concludes the proof.  

\[\square\]

3 Uniqueness for single non-axisymmetric multipole fields

The potential $\Phi$ of a single non-axisymmetric multipole field $D = \nabla \Phi |_S$ is represented by $b_{n_0}^k = \delta_{n_00} \delta_{k0}$, $n_0 \in \mathbb{N}_0$, $0 < k_0 \leq n_0$ in the expansion (2.2), i.e.

$$\Phi(r, \theta, \varphi) = r^{-n_0-1} Y_{n_0}^{k_0}(\theta, \varphi).$$  (3.1)

In that case holds
Theorem 2  The direction problem (1.2) with a single non-axisymmetric multipole field as direction field is uniquely solvable.

Proof: We make use of the boundary condition in the form (1.4)\textsubscript{1,2}. With (1.3) and (3.1) these take the form

\[ \sum_{n=1}^{\infty} \sum_{k \in \mathbb{Z}} c_n^k ((n+1)Y_n^k \partial_{\theta} Y_{n0}^{k0} - (n_0+1)Y_n^{k0} \partial_{\phi} Y_n^k) = 0 \]  

and

\[ \sum_{n=1}^{\infty} \sum_{k \in \mathbb{Z}} c_n^k ((n+1)Y_n^k \partial_{\phi} Y_{n0}^{k0} - (n_0+1)Y_n^{k0} \partial_{\theta} Y_n^k) = 0, \]  

where we have set again \( c_n^k = 0 \) for \( |k| > n \). By (2.7) eq. (3.3) takes the form

\[ \left\{ \sum_{n=1}^{\infty} \sum_{k \in \mathbb{Z}} c_n^k ((n+1)k_0 - (n_0+1)k) Y_n^k \right\} Y_{n0}^{k0} = 0. \]  

Equation (3.4) implies the term in braces to vanish almost everywhere on \( S \), and hence for all \( n \in \mathbb{N}, k \in \mathbb{Z} \):

\[ c_n^k = 0 \quad \text{or} \quad \frac{n+1}{n_0+1} = \frac{k}{k_0}, \]  

which means, in particular, that given \( k \in \mathbb{Z} \) there is at most one number \( n = n(k) \in \mathbb{N} \) with \( c_{n(k)}^k \neq 0 \).

By (2.7) and using the variables \( x := \cos \theta, y := e^{i\phi}, z := \tan \theta \), and the notation introduced in (2.12) and (2.13), eq. (3.2) can be put into the form

\[ -z \sum_{n=1}^{\infty} \sum_{k \in \mathbb{Z}} c_n^k Q^{(n,k,n_0,k_0)}(x) y^{k+k_0} L^{(k+k_0)}(z) \]
\[ = -z \left\{ \sum_{k \in \mathbb{Z}} \left[ \sum_{n=1}^{\infty} c_n^k Q^{(n,k,n_0,k_0)}(x) \right] y^k L^{(k+k_0)}(z) \right\} y^{k_0} = 0. \]

So, vanishing of the term in braces implies the bracket to vanish for any \( k \in \mathbb{Z} \), and by (3.5):

\[ c_{n(k)}^k Q^{(n(k),k,n_0,k_0)}(x) = 0, \]

where \( k \in \mathbb{Z} \) takes only those values such that \( n(k) \in \mathbb{N} \) and \( c_{n(k)}^k \neq 0 \). By (2.13) \( Q^{(n,k,\nu,\kappa)} \) is a polynomial with leading coefficient \( q_{n+\nu,\kappa}^{(n,k,\nu,\kappa)} \) given in (2.15). Therefore,

\[ d_n^{(n(k),k,n_0,k_0)} = (-1)^{kk_0} d_{n(k)}^{k} c_n^{k0} (n(k) + 1) n_0 - (n_0 + 1)n(k)) p_{n(k)}^{(n(k),k)} p_{n_0}^{(n(k),k_0)} = 0 \]

implies \( n(k) = n_0 \) and by (3.5) \( k = k_0 \). This is the assertion. \( \square \)
4 Non-uniqueness: A non-axisymmetric example

To find a non-axisymmetric direction field allowing more than one solution we rely on the most simple axisymmetric example for non-uniqueness: the axisymmetric quadrupole field $\nabla (r^{-3} Y_0^2)\big|_S$, for which exactly one additional (axisymmetric) solution in terms of an infinite series of spherical harmonics has been established [KN04]. In standard spherical coordinates the symmetry axis of $Y_0^2$ coincides with the $z$-axis of the coordinate system $CS$. When rotating $CS$ into a coordinate system $CS'$ such that the $z'$-axis lies perpendicular to the $z$-axis, $Y_0^2$ may be described in primed coordinates by

$$Y_0^2 + a(Y_2^2 + Y_2^{-2}) =: \phi_a \quad (4.1)$$

with $|a| = \sqrt{3/2}$. $\phi_{\pm} \sqrt{3/2}$ is no longer axisymmetric with respect to the $z$-axis, but has still an additional solution which is given by an infinite series involving the parameter $a$. To find a direction field which has no symmetry axis at all we consider the linear combination (4.1) with $|a|$ in some neighbourhood of $\sqrt{3/2}$ and argue that the infinite series still converges.

We start by describing the rotation of a spherical harmonic by means of Wigner $D$-functions (cf. [VMK88, p. 72 and p. 141]). Let $0 \leq \alpha \leq 2\pi$, $0 \leq \beta \leq \pi$, and $0 \leq \gamma \leq 2\pi$ be the Euler angles transforming the coordinate system $CS \triangleq \{e_x, e_y, e_z\}$ into $CS' \triangleq \{e'_x, e'_y, e'_z\}$ by successive rotations with angle $\gamma$ around $e_z$, with $\beta$ around $e_y$, and with $\alpha$ again around $e_z$. A spherical harmonic $Y_n^k$ expressed in primed coordinates is then related to $Y_n^k$ in unprimed coordinates by

$$Y_n^k(\theta, \varphi) := Y_n^k(\theta', \varphi') = \sum_{|\kappa| \leq n} Y_n^\kappa(\theta, \varphi) D_n^\kappa k(\alpha, \beta, \gamma), \quad (4.2)$$

where $D_n^\kappa k$ denote the (Wigner) $D$-functions satisfying the unitarity condition

$$\sum_{|\kappa| \leq n} D_n^\kappa k(\alpha, \beta, \gamma) D_n^{\kappa'} k'(\alpha, \beta, \gamma) = \delta^{kk'} \quad (4.3)$$

Equation (4.2) means for $Y_2^0$

$$Y_2^0(\theta, \varphi) = \sum_{k=-2}^{2} Y_2^k(\theta, \varphi) D_2^k 0(\alpha, \beta) \quad (4.4)$$

with

$$D_2^0 0(\alpha, \beta) = \frac{1}{2} (3 \cos^2 \beta - 1),$$

$$D_2^{\pm 1} 0(\alpha, \beta) = \mp \frac{3}{2} \sin \beta \cos \beta e^{\mp i\alpha},$$

$$D_2^{\pm 2} 0(\alpha, \beta) = \frac{3}{8} \sin^2 \beta e^{\pm 2i\alpha}.$$

Note that axisymmetric spherical harmonics $Y_n^0$ are invariant under $\gamma$-rotations; therefore, the $D_n^0$ do not depend on $\gamma$. The rotation ($\beta = \pi/2$, $\alpha = 0$) yields

$$Y_2^0|_{\beta=\pi/2, \alpha=0} = -\frac{1}{2} \left( Y_2^0 - \sqrt{\frac{3}{2}} (Y_2^2 + Y_2^{-2}) \right) \quad (4.5)$$
On the other hand, requiring the right-hand side in (4.4) to be a linear combination of type (4.1) yields \( \beta = \pi/2 \) and \( \alpha = 0 \) or \( \alpha = \pi/2 \), which means \( a = -\sqrt{3}/2 \) or \( a = \sqrt{3}/2 \), respectively. So, given the linear combination (4.1), only the cases \( a = 0 \) and \( a = \pm \sqrt{3}/2 \) represent axisymmetric functions.

Let now

\[
\Psi(r, \theta) = \sum_{n=1}^{\infty} c_{2n-1} r^{-2n} Y_{2n-1}^0(\theta), \quad \sum_{n=1}^{\infty} 4n^2 |c_{2n-1}|^2 < \infty \tag{4.6}
\]

be the additional solution for the axisymmetric quadrupole field established in [KN04].\(^1\) Rotating the coordinate system by \((\beta, 1)\) be the additional solution for the axisymmetric quadrupole field established in [KN04].\(^1\)

Thus, \( \Psi \) is a convergent series solving the direction problem for the direction field \( \nabla (r^{-3} Y_2^{0'}) \big|_S \) with \( Y_2^{0'} \) given in (4.5).

We derive next recurrence relations for the coefficients of \( \Psi \) when the direction field is of type \( \nabla (r^{-3} \phi_a) \big|_S \) with \( \phi_a \) given in (4.1). Using for simplicity complex notation \( \phi_a \) is explicitly (and up to a factor) the real part of

\[
\tilde{\phi}_a = 1 - 3 \cos^2 \theta + \tilde{a} \sin^2 \theta e^{2i\varphi}, \quad \tilde{a} = -\sqrt{6} a.
\]

Therefore, \( \tilde{D}_a := \nabla (r^{-3} \tilde{\phi}_a) \big|_S \) takes the explicit form

\[
\tilde{D}_a = -3(1 - 3 \cos^2 \theta + \tilde{a} \sin^2 \theta e^{2i\varphi}) e_r + 2 \sin \theta \cos \theta (3 + \tilde{a} e^{2i\varphi}) e_\theta + 2i \tilde{a} \sin \theta e^{2i\varphi} e_\varphi. \tag{4.8}
\]

Inserting (1.3) and (4.8) into (1.4) yields

\[
\sum_{n=1}^{\infty} \sum_{k=-n}^{n} c_n^k [2i\tilde{a}(n+1)Y_n^k(\theta, \varphi) \sin^2 \theta e^{2i\varphi} - 3ik Y_n^k(\theta, \varphi) (1 - 3 \cos^2 \theta + \tilde{a} \sin^2 \theta e^{2i\varphi})] = 0 \tag{4.9}
\]

and

\[
\sum_{n=1}^{\infty} \sum_{k=-n}^{n} c_n^k [2i\tilde{a} \partial_\theta Y_n^k(\theta, \varphi) \sin^2 \theta e^{2i\varphi} - 2ik Y_n^k(\theta, \varphi) \sin \theta \cos \theta (3 + \tilde{a} e^{2i\varphi})] = 0. \tag{4.10}
\]

\(^{1}\)In fact, in [KN04] the solution is given in terms of the amplitude function \( a \) in the representation (1.2)\(_3\), which, of course, determines \( \Psi \) uniquely.
Expressing all functions in the brackets exclusively by spherical harmonics (of various order and degree), rearranging the resulting series again in the form $\sum_{n,k} C^k_n Y^k_n$, and setting $C^k_n = 0$ yields the desired recurrence relations. We obtain from (4.9)

$$\tilde{a}(r_- c_n^{k-1} + r_0 c_n^{k-1} + r_+ c_n^{k-1}) + r_- c_n^{k+1} + r_0 c_n^{k+1} + r_+ c_n^{k+1} = 0,$$

(4.11)

and from (4.10)

$$\tilde{a}(t_- c_n^{k-1} + t_0 c_n^{k-1} + t_+ c_n^{k-1}) + t_- c_n^{k+1} + t_0 c_n^{k+1} + t_+ c_n^{k+1} = 0.$$

(4.12)

The twelve coefficients $r_{\pm,0}^k = r_{\pm,0}^k(n,k)$, $t_{\pm,0}^k = t_{\pm,0}^k(n,k)$ in the recurrence relations (4.11), (4.12) depend still on $n$ and $k$, but not on $\tilde{a}$. Explicit expressions together with some useful relations between spherical harmonics for their computation can be found in appendix A.

A direct solution of (4.11), (4.12) seems to be a hard business – even for this simple direction field. On the other side, the structure of the recurrence relations reveals that, once a solution $\{c_{2n-1}^k(\tilde{a}) : n \in \mathbb{N}, k \in \mathbb{Z}\}$ of (4.11), (4.12) for some value $\tilde{a}_0$ is known, one obtains immediately a whole family of (formal) solutions $\{c_{2n-1}^k(\tilde{a}) : n \in \mathbb{N}, k \in \mathbb{Z}\}$ by setting $c_{2n-1}^{2k-1}(\tilde{a}) := (\tilde{a}/\tilde{a}_0)^k c_{2n-1}^{2k-1}(\tilde{a}_0)$. So, starting with

$$c_{2n-1}^{2k-1}(3) := c_{2n-1} D_{2n-1}^{2k-10}(0, \frac{\pi}{2}),$$

which corresponds to the axisymmetric solution (4.7), any set $\{(\tilde{a}/3)^k c_{2n-1}^{2k-1}(3) : n \in \mathbb{N}, k \in \mathbb{Z}\}$ with

$$\sum_{n=1}^{\infty} 4n^2 |c_{2n-1}|^2 \sum_{k=-n+1}^{n} \left| \frac{\tilde{a}}{3} \right|^{2k} \left| D_{2n-1}^{2k-10}(0, \frac{\pi}{2}) \right|^2 < \infty$$

(4.13)

represents for $|\tilde{a}| \neq 3$ a non-axisymmetric solution of the direction problem with direction field (4.8). By the estimate

$$\sum_{k=-n+1}^{n} \left| \frac{\tilde{a}}{3} \right|^{2k} \left| D_{2n-1}^{2k-10}(0, \frac{\pi}{2}) \right|^2 \leq \max \left\{ \left( \frac{\tilde{a}}{3} \right)^{2n}, \left( \frac{3}{\tilde{a}} \right)^{2n} \right\} \sum_{k=-n+1}^{n} \left| D_{2n-1}^{2k-10}(0, \frac{\pi}{2}) \right|^2$$

$$= \max \left\{ \left( \frac{\tilde{a}}{3} \right)^{2n}, \left( \frac{3}{\tilde{a}} \right)^{2n} \right\} =: A^{2n}$$

convergence of the series

$$\sum_{n=1}^{\infty} 4n^2 |c_{2n-1}|^2 A^{2n}, \quad A > 1$$

(4.14)

implies (4.13) in the range $3/A < \tilde{a} < 3A$. To check the convergence of (4.14) we make use of the recurrence relation

$$s_- \tilde{c}_{2n-1} + s_0 \tilde{c}_{2n+1} + s_+ \tilde{c}_{2n+3} = 0, \quad n \in \mathbb{N}_0$$

(4.15)

with coefficients $s_{\pm,0} = s_{\pm,0}(n)$. The $\tilde{c}_{2n-1}$ are related to $c_{2n-1}$ by

$$2n c_{2n-1} = C(\tilde{s}_- \tilde{c}_{n-3} + \tilde{s}_0 \tilde{c}_{2n-1} + \tilde{s}_+ \tilde{c}_{2n+1}), \quad n \in \mathbb{N},$$

(4.16)

where the coefficients $\tilde{s}_{\pm,0}$ depend again on $n$. In particular, there is $s_0(0) = s_- (0) = \tilde{s}_-(1) = 0$. Up to a constant factor eqs. (4.15), (4.16) determine uniquely the set $\{c_{2n-1} : n \in \mathbb{N}\}$
representing the axisymmetric solution (4.6) (for explicit expressions of $s_{\pm,0}$, $\tilde{s}_{\pm,0}$ and other details see appendix B). Now relations (4.15), (4.16) can easily be rescaled to describe the coefficients $\tilde{c}_{2n-1}(A) := c_{2n-1}A^n$ and $c_{2n-1}(A) := c_{2n-1}A^n$:

$$s_- A^2 \tilde{c}_{2n-1}(A) + s_0 A \tilde{c}_{2n+1}(A) + s_+ \tilde{c}_{2n+3}(A) = 0, \quad n \in \mathbb{N}_0$$

(4.17)

and

$$2n c_{2n-1}(A) = C(\tilde{s}_- A \tilde{c}_{n-3}(A) + \tilde{s}_0 \tilde{c}_{2n-1}(A) + \tilde{s}_+ A^{-1} \tilde{c}_{2n+1}(A)), \quad n \in \mathbb{N}.$$  

(4.18)

According to Lemma 4.1 in [KN04] the asymptotic values $s_{\pm,0}^\infty := \lim_{n \to \infty} s_{\pm,0}(n)$ are decisive for convergence: if all complex zeros of the polynomial

$$s_+^{\infty} X^4 + s_0^{\infty} A X^2 + s_-^{\infty} A^2 = 0$$

lie within the unit circle, then $\sum_{n=1}^{\infty} |\tilde{c}_{2n-1}(A)|^2$ converges for arbitrary initial values. For bounded coefficients $\tilde{s}_{\pm,0}(n)$ the series (4.14) then converges by (4.18). From (B.2), (B.3) one deduces the boundedness of $\tilde{s}_{\pm,0}(n)$ and from (B.1), (B.2):

$$s_-^\infty = \frac{1}{4}, \quad s_0^\infty = \frac{1}{2}, \quad s_+^\infty = \frac{5}{4}.$$  

Thus, convergence holds (at least) for $A < \sqrt{5}$ or, equivalently, $3/\sqrt{5} < a < 3\sqrt{5}$.

5 Conclusions

In this paper the geomagnetic direction problem is considered for a spherical surface in $\mathbb{R}^3$ without the assumption of axisymmetry. Using multipole expansions of exterior harmonic vector fields the uniqueness question has been settled for finite multipole expansions, a result that previously seemed to be in doubt. From a practical point of view, this result is of limited use since further approximate solutions are not excluded which become better the larger the truncation level in the multipole expansion. Allowing infinite expansions, non-uniqueness is well known in the axisymmetric direction problem. We demonstrate by example that non-uniqueness holds also in the non-axisymmetric direction problem. This contrasts with another finding of this paper, viz. uniqueness for any single non-axisymmetric multipole field as direction field. This result demonstrates, in particular, that the ‘pole-criterion’ [HKL97], which gives an upper bound on the dimension of the solution space, is too rough to predict exactly the dimension. In the case of axisymmetry the rotation number of the direction field, which can equivalently be obtained by counting the ‘signed poles’ of the direction field, turns out to be the right quantity to predict exactly the dimension. Unfortunately, a corresponding quantity that is decisive in the fully three-dimensional case is not yet known. In this situation, any criterion is desirable predicting uniqueness for a larger (or any other) class of direction fields. From an applied point of view, such information could be relevant in the interpretation of palaeomagnetic data records which can well display more than two poles.

Appendix A

This appendix translates the boundary conditions (4.9), (4.10) into the recurrence relations (4.11), (4.12). Here, we make use of the following expansions in spherical harmonics (see
Equation (4.10) is first multiplied by the recurrence relation (4.11) with $k$:

\[
\begin{align*}
\cos^2 \theta Y_n^k &= N_{n-1}^k N_n^k Y_{n-2}^k + (N_n^k + N_{n+1}^k)^2 Y_n^k + N_n^k N_{n+2}^k Y_{n+2}^k, \\
\sin^2 \theta e^{2i\varphi} Y_n^k &= M_{n-1}^k M_n^k Y_{n-2}^k - \frac{2(n+1)}{(2n-1)(2n+3)} M_{n}^k M_{n+1}^k Y_{n+2}^k \\
&\quad + M_{n+1}^k M_{n+2}^k Y_{n+2}^k,
\end{align*}
\]

(A.1)

\[
\begin{align*}
\sin \theta \cos \theta e^{\pm i\varphi} y_n^k &= \pm N_n^k M_{n-1}^k Y_{n-2}^k \mp (N_n^k M_{n+1}^k - N_{n+1}^k M_{n+1}^k) Y_{n+1}^k \\
&\quad \mp N_n^k M_{n+1}^k Y_{n+2}^k, \\
\sin^2 \theta e^{i\varphi} \partial_\theta Y_n^k &= -(n+1)N_n^k M_{n-1}^k Y_{n-2}^k + ((n+1)N_n^k M_{n+1}^k \\
&\quad + n N_n^k M_{n+1}^k) Y_{n+1}^k - n N_n^k M_{n+1}^k Y_{n+2}^k,
\end{align*}
\]

(A.2)

where

\[
N_n^k := \sqrt{\frac{(n-k)(n+k)}{(2n-1)(2n+1)}}, \quad M_n^k := \sqrt{\frac{(n+k)(n+k-1)}{(2n-1)(2n+1)}}.
\]

Inserting (A.1) into (4.9) and rearranging the resulting series yields after a shift $k \to k + 1$ the recurrence relation (4.11) with

\[
\begin{align*}
 r_- &= (2n - 3k + 1)M_{n-1}^k M_{n+1}^k, \\
 r_0^- &= -(2n - 3k + 5) M_{n+1}^k M_{n+2}^k, \\
 r_0^+ &= 9(k+1)N_{n+1}^k N_{n-1}^k, \\
 r_+^+ &= 9(k+1)N_{n+1}^k N_{n-1}^k, \\
 r_0^+ &= 9(k+1) \left( (N_{n+1}^k)^2 + (N_{n-1}^k)^2 - \frac{1}{3} \right).
\end{align*}
\]

Equation (4.10) is first multiplied by $e^{-i\varphi}$ and then substituted by (A.2) resulting in the recurrence relation (4.12) with

\[
\begin{align*}
t_- &= -(n-k-1)N_{n-1}^k M_n^k, \\
t_- &= -(n-k+1)N_{n+1}^k M_{n+1}^k, \\
t_0^- &= (n+k)N_{n-1}^k M_{n}^k + (n-k+1)N_{n+1}^k M_{n+1}^k, \\
t_0^+ &= 3(k+1)N_{n-1}^k M_{n}^k, \\
t^+ &= 3(k+1)N_{n+1}^k M_{n+1}^k, \\
t_0^+ &= -3(k+1) \left( N_{n}^k M_{n}^k - N_{n+1}^k M_{n+1}^k \right).
\end{align*}
\]

**Appendix B**

This appendix substantiates the relations (4.15) and (4.16) representing the additional solutions for the axisymmetric quadrupole field established in [KN04, section 4.1]. The recurrence relation (4.10) with

\[
\begin{align*}
 s_- &= 3\alpha_{2n-1} \alpha_{2n} - 2\alpha_{2n-1} \beta_{2n+1}, \\
 s_0 &= 3\alpha_{2n}^2 + 5\alpha_{2n+1}^2 - 2\alpha_{2n} \beta_{2n+1} - 1, \\
 s_+ &= 5\alpha_{2n+1} \alpha_{2n+2},
\end{align*}
\]

(B.1)
\[ \alpha_n := \frac{n + 1}{\sqrt{(2n + 1)(2n + 3)}} \] \[ \beta_n := \frac{n + 1}{\sqrt{(2n - 1)(2n + 1)}} \]
\( n \in \mathbb{N}_0 \), \( \alpha_{-1} = 0 \)
\( \beta_{-1} := \frac{n + 1}{\sqrt{(2n - 1)(2n + 1)}} \), \( n \in \mathbb{N} \) (B.2)

corresponds to eq. (4.12) in [KN04], and its solution \( \{ \tilde{c}_{2n-1} : n \in \mathbb{N} \} \) determines the amplitude function \( a \) in the boundary condition (1.2) by

\[
a = \sum_{n=1}^{\infty} \tilde{c}_{2n-1} \tilde{P}_{2n-1},
\]

where \( \tilde{P}_n \) denotes the normalized Legendre polynomial \( \sqrt{(2n + 1)/2} P_n \). To relate the \( \tilde{c}_{2n-1} \) to \( c_{2n-1} \) in the axisymmetric representation \( \Psi(r, \theta) = \sum_{n=1}^{\infty} c_n r^{n-1} Y_n^0(\theta) \) we use the radial component of (1.2) in the form

\[
\sum_{n=1}^{\infty} 2n c_{2n-1} \tilde{P}_{2n-1}(x) = \left( \sum_{n=1}^{\infty} \tilde{c}_{2n-1} \tilde{P}_{2n-1}(x) \right) \left( 3x^2 - 1 \right).
\]

Expressing the right-hand side, by means of

\[
x \tilde{P}_n(x) = \alpha_{n-1} \tilde{P}_{n-1}(x) + \alpha_n \tilde{P}_{n+1}(x),
\]

again by (normalized) Legendre polynomials, we find

\[
2n c_{2n-1} = C \left[ \alpha_{2n-2} \alpha_{2n-3} \tilde{c}_{2n-3} + \left( \alpha_{2n-1}^2 + \alpha_{2n-2}^2 - \frac{1}{3} \right) \tilde{c}_{2n-1} + \alpha_{2n} \alpha_{2n-1} \tilde{c}_{2n+1} \right] \quad (B.3)
\]

with some constant \( C \neq 0 \), which is (4.16).

References


