Well-posedness of the kinematic dynamo problem

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Abstract

In the framework of magnetohydrodynamics the generation of magnetic fields by the prescribed motion of a liquid conductor in a bounded region $G \subset \mathbb{R}^3$ is described by the induction equation, a linear system of parabolic equations for the magnetic field components. Outside $G$ the solution matches continuously to some harmonic field which vanishes at spatial infinity. The kinematic dynamo problem seeks to identify those motions which lead to nondecaying (in time) solutions of this evolution problem.

In this paper the existence problem of classical (decaying or not) solutions of the evolution problem is considered for the case that $G$ is a ball and for sufficiently regular data. The existence proof is based on the poloidal/toroidal representation of solenoidal fields in spherical domains and on the construction of appropriate basis functions for a Galerkin procedure.

Key Words: Magnetohydrodynamics, dynamo theory, poloidal/toroidal representation.

1 Introduction

Formally, the kinematic dynamo problem reads [2]:

$$
\begin{align*}
\partial_t B + \nabla \times (\eta \nabla \times B) &= \nabla \times (v \times B), & \nabla \cdot B = 0 & \text{in } G \times \mathbb{R}_+, & (1.1a) \\
\nabla \times B &= 0, & \nabla \cdot B &= 0 & \text{in } \widehat{G} \times \mathbb{R}_+, & (1.1b) \\
B &= \text{continuous} & \text{in } \mathbb{R}^3 \times \mathbb{R}_+, & (1.1c) \\
B(x,t) &\to 0 & \text{for } |x| \to \infty, t \in \mathbb{R}_+, & (1.1d) \\
B(\cdot,0) &= B_0 & \text{on } G \times \{t=0\}. & (1.1e)
\end{align*}
$$

Here, $G \subset \mathbb{R}^3$ is a bounded domain with (sufficiently) smooth boundary $\partial G$, and $\widehat{G}$ denotes the exterior region $\mathbb{R}^3 \setminus \overline{G}$. The flow field $v$ and the magnetic diffusivity $\eta$ are prescribed (sufficiently smooth) functions of $x \in G$ and $t \in \mathbb{R}_+$; $\eta$ is, moreover, bounded from below by a positive constant. The (solenoidal) initial-value $B_0$ for the magnetic field $B$ is prescribed on $G$ only.

The induction equation (1.1a) constitutes a system of parabolic equations for the magnetic field components coupled by the flow field and a variable diffusivity. This becomes more transparent by rewriting (1.1a)$_1$ in the form

$$
\partial_t B - \eta \Delta B = - (\nabla \eta \nabla) B + (\nabla v)^T B - (v - \nabla \eta) \cdot \nabla B - \nabla \cdot v B,
$$

where $(\nabla \eta \nabla)$ and $(\nabla v)$ denote the matrices $(\partial_i \eta \partial_j)_{ij}$ and $(\partial_i v_j)_{ij}$, respectively, and $(\cdot)^T$ means transposition, or by using index notation:

$$
\partial_t B_i - \eta \Delta B_i = - \sum_{j=1}^3 \left( \partial_j \eta \partial_i B_j - \partial_j v_i B_j \right) - \sum_{j=1}^3 \left( (v_j - \partial_j \eta) \partial_j B_i + \partial_j v_j B_i \right), \quad i = 1, 2, 3.
$$
It is the coupling of this parabolic problem to the elliptic problem (1.1b), (1.1d) in \( \hat{G} \) at \( \partial G \) via the matching condition (1.1c) which prevents the straightforward application of standard parabolic existence results (see, e.g., [7] or [10]) on problem (1.1).

In astrophysical applications solutions of the dynamo problem (1.1) model the magnetic field generated by the flow field in (approximately) spherical cosmic bodies like stars or planets. So, of primary interest is the case where \( G \) is a ball, as it is assumed in this paper. Early on much effort has been spent on the analytical (exemplary: [2]) or numerical (starting with [3]) study of (1.1) with those special flow fields, which promised non-decaying solutions. In contrast to these studies it is the aim of the present paper to consider the more basic problem to establish the well-posedness of problem (1.1), i.e. to prove the existence of a unique solution for arbitrary (sufficiently regular) flow fields and conductivity distributions.

The basic idea of our treatment of problem (1.1) is to solve (1.1a) in \( G \) by a Galerkin procedure using basis functions which satisfy already eqs. (1.1b) – (1.1d), i.e., which match continuously to harmonic ‘extensions’ outside \( G \). In the construction of these basis functions we make use of the so-called poloidal/toroidal representation of solenoidal fields in spherical domains [2, 14]:

\[
B = P + T = \nabla \times (\nabla \times \phi x) + \nabla \times \psi x.
\]

The first part is the poloidal field determined by the poloidal scalar \( \phi \) and the second is the toroidal field with toroidal scalar \( \psi \). This representation incorporates the divergence constraint (and is local in contrast to the usual projection method) and facilitates the harmonic extension onto \( \hat{G} \). In fact, only the poloidal field has a nontrivial harmonic (vector field) extension, whereas the toroidal field has a vanishing normal component at \( \partial G \) and hence vanishes altogether in \( \hat{G} \). The basis functions are then constructed by solving suitable scalar eigenvalue problems for \( \psi \), which is standard, and for \( \phi \), which is nonstandard but has already been done [8]. Once the basis functions are established the weak solution of the evolution problem uses more or less standard Hilbert space methods; higher regularity of the weak solution depends, as usual, on the regularity of the data and suitable compatibility conditions.

## 2 Poloidal/toroidal representation

In this section \( G \) denotes a (not necessarily bounded) spherically symmetric domain in \( \mathbb{R}^3 \); in particular, \( G \) may be a ball \( B_R \) of radius \( R \), the exterior \( \bar{B}_R \) of \( B_R \), a spherical shell \( B_R \setminus \bar{B}_r \), \( R > r > 0 \), or all space \( \mathbb{R}^3 \). \( \mathbf{a} \cdot \mathbf{b} \) denotes the euclidean scalar product of \( \mathbf{a}, \mathbf{b} \in \mathbb{R}^3 \) and \( \mathbf{a} \times \mathbf{b} \) denotes the vector product in \( \mathbb{R}^3 \). The symbols for function spaces \( C_0^\infty (G) \), \( L^2(G) \), \( H^k_0(G), H^k(G), k \in \mathbb{N}_0 \) have their usual meaning. \( \mathcal{D}'(G) \) denotes the space of distributions, i.e. continuous linear functionals on \( C_0^\infty (G) \) equipped with the usual topology. A vector field\(^1\) \( B \in L^2(G) \) is called solenoidal iff

\[
\nabla \cdot B = 0 \quad \text{in} \quad \mathcal{D}'(G) \quad \text{and} \quad \int_G B \cdot \hat{x} \, \mathrm{d}x = 0,
\]

where \( \hat{x} := x/|x| \). The second condition is equivalent to the more familiar surface-integral condition

\[
\int_{|x|=r} B \cdot \hat{x} \, \mathrm{d}s = 0 \quad \text{for a.e.} \quad r = |x|, \ x \in G.
\]

\(^1\)We use the notation \( B \in L^2(G) \) (instead of \( B \in (L^2(G))^3 \)) to indicate that any component of the vector field \( B \) is element of \( L^2(G) \). We use throughout capital letters for vector fields and lower case letters for scalar functions; so, confusion should not arise.
The spherical mean of a function \( f \in \mathcal{L}^2(G) \) is well-defined for a.e. \( r = |x| \), \( x \in G \) by

\[
\langle f \rangle := \langle f \rangle(r) := \frac{1}{4\pi r^2} \int_{|x|=r} f \, ds.
\]

In the following we make extensive use of the non-radial derivative operator \( \Lambda \) and its square \( \mathcal{L} \):

\[
\Lambda := x \times \nabla = -\nabla \times x, \quad \mathcal{L} := \Lambda \cdot \Lambda.
\]

In fact, there is \( x \cdot \Lambda f = 0 \) for any (sufficiently smooth) function \( f \). Note, furthermore,

\[
\nabla \cdot \Lambda f = 0, \quad \Lambda \Delta f = \Delta \Lambda f.
\] (2.1)

To realize the latter equality recall the vector analysis identity

\[
\Delta F = -\nabla \times (\nabla \times F) + \nabla (\nabla \cdot F).
\] (2.2)

Writing \( \mathcal{L} \) in spherical coordinates \((r, \theta, \phi)\) we find

\[
\mathcal{L} = (x \times \nabla) \cdot (x \times \nabla) = \frac{1}{\sin \theta} \partial_\theta (\sin \theta \partial_\theta) + \frac{1}{\sin^2 \theta} \partial^2_\phi,
\]

which is just the Laplace-Beltrami operator on the unit sphere. We are now prepared to make the representation (1.3) precise for \( \mathcal{L}^2 \)-functions. The following characterization follows ref. [14, definition and lemma 3.5]:

**Lemma 2.1** Let \( G \in \mathbb{R}^3 \) be a spherically symmetric domain and let \( P, T \in \mathcal{L}^2(G) \) be solenoidal vector fields. Then

(i) \( P \) is called poloidal iff one of the following equivalent conditions holds:

- a) \( \Lambda \cdot P = 0 \) in \( \mathcal{D}'(G) \),
- b) There is \( \phi \in \mathcal{L}^2(G) : P = -\nabla \times \Lambda \phi \) in \( \mathcal{D}'(G) \).

The poloidal scalar \( \phi \) is uniquely determined by the conditions

\[
-\mathcal{L} \phi = P \cdot x \text{ in } \mathcal{D}'(G), \quad \langle \phi \rangle = 0 \text{ for a.e. } r = |x|, x \in G.
\] (2.3)

(ii) \( T \) is called toroidal iff one of the following equivalent conditions holds:

- a) \( T \cdot x = 0 \) a.e. in \( G \),
- b) There is \( \psi \in \mathcal{L}^2(G) : T = -\Lambda \psi \) in \( \mathcal{D}'(G) \).

The toroidal scalar \( \psi \) is uniquely determined by the conditions

\[
-\mathcal{L} \psi = \Lambda \cdot T \text{ in } \mathcal{D}'(G), \quad \langle \psi \rangle = 0 \text{ for a.e. } r = |x|, x \in G.
\] (2.4)

Let us define the closed subspaces of \( \mathcal{L}^2 \): \(^2\)

\[
\mathcal{B} = \{ F \in \mathcal{L}^2 : F \text{ solenoidal } \},
\]

\[
\mathcal{P} = \{ B \in \mathcal{B} : B \text{ poloidal } \},
\]

\[
\mathcal{T} = \{ B \in \mathcal{B} : B \text{ toroidal } \}.
\]

\(^2\)Where the domain under consideration is clear from the context we sometimes use the symbols \( \mathcal{L}^2, \mathcal{H}, \mathcal{B}, \mathcal{P}, \) and \( \mathcal{T} \) without specified domain.
With the standard \( L^2 \)-scalar product \( (B_1, B_2)_{L^2} := \int_G B_1 \cdot B_2 \, dx \), \( B \) becomes a real Hilbert space, and \( \mathcal{P} \) and \( \mathcal{T} \) turn out to be orthogonal subspaces. In fact, with (2.2) and the further vector analysis identity

\[
F \cdot \nabla \times G - (\nabla \times F) \cdot G + \nabla \cdot (F \times G) = 0
\]  

(2.5)

one obtains for \( P = -\nabla \times \Lambda \phi, \phi \in C^3(\overline{G}) \) and \( T = -\Lambda \psi, \psi \in C^1(\overline{G}) \):

\[
(P, T)_{L^2} = \int_G \left( \nabla \times \left( \nabla \times \phi x \right) \right) \cdot \left( \nabla \times \psi x \right) \, dx = \int_G \left( \nabla \times \left( \nabla \times \nabla \times \phi x \right) \right) \cdot \psi x \, dx
= -\int_{\partial G} \left( \nabla \times \nabla \times \phi x \right) \cdot \psi x \cdot n \, ds + \int_G (\nabla \Delta \phi) \cdot \psi x \, dx = 0 ,
\]

(2.6)

where \( n \) denotes the exterior normal at \( \partial G \), i.e. \( n = \pm \hat{x} \).

Now given a vector field \( B \in \mathcal{B} \), solutions of the problems (2.3) and (2.4) provide associated scalars \( \phi \) and \( \psi \), respectively, which allow the representation (1.3) of \( B \). Following [14, theorem 3.6] we have:

\textbf{Lemma 2.2} Let \( G \subset \mathbb{R}^3 \) be a spherically symmetric domain and \( B \in \mathcal{B} \). Then there holds the orthogonal decomposition

\[
\mathcal{B} = \mathcal{P} \oplus \mathcal{T} , \quad B = P + T = -\nabla \times \Lambda \phi - \Lambda \psi , \quad \langle \phi \rangle = \langle \psi \rangle = 0
\]

(2.7)

with uniquely determined elements \( P \in \mathcal{P}, T \in \mathcal{T}, \) and \( \phi, \psi \in L^2 \).

As to higher regularity we have according to [14, propositions 2.6, 3.3, and theorem 3.6]:

\textbf{Lemma 2.3} Let \( G \subset \mathbb{R}^3 \) be a spherically symmetric domain and let \( B \in \mathcal{B} \cap H^k \), \( k \in \mathbb{N} \) be decomposed according to (2.7). Then, in fact, there is \( P \in \mathcal{P} \cap H^k, T \in \mathcal{T} \cap H^k \), and \( \phi, \Lambda \phi, \) and \( \psi \in H^k \); and the following estimates hold with constants \( \tilde{C}, \tilde{C} \) depending only on \( k \):

\[
\| \phi \|_{H^k} + \| \Lambda \phi \|_{H^k} \leq \tilde{C} \| P \cdot x \|_{H^k} \leq \tilde{C} \| B \cdot x \|_{H^k} , \quad k \in \mathbb{N}_0 ,
\]

(2.8)

\[
\| \psi \|_{H^k} \leq \tilde{C} \| T \|_{H^k} \leq \tilde{C} \| B \|_{H^k} , \quad k \in \mathbb{N}_0 .
\]

(2.9)

In view of problem (1.1) we are interested in special solenoidal fields, namely those which allow a unique ‘harmonic extension’ outside \( G \). If \( G \) is a ball \( B_R \) this notion can be made precise for poloidal fields \( P \in \mathcal{P}(B_R) \cap H^1(B_R) \) using the trace of \( \hat{x} \cdot P \) on \( \partial G \). We mention before an auxiliary result:

\textbf{Lemma 2.4 (A boundary-value problem for exterior harmonic fields)} Let \( G \subset \mathbb{R}^3 \) be a bounded domain with simply connected \( C^1 \)-boundary \( \partial G \) and exterior normal \( n \) at \( \partial G \). Let, furthermore, \( \hat{G} := \mathbb{R}^3 \setminus \overline{G} \) and \( g \in H^{1/2}(\partial G) \) with \( \int_{\partial G} g \, ds = 0 \), where \( H^{1/2}(\partial G) \) is the trace space associated to \( H^1(\hat{G}) \). Then, the boundary-value problem

\[
\begin{align*}
\nabla \times B &= 0 , \quad \nabla \cdot B &= 0 \quad \text{a.e. in } \hat{G} , \\
n \cdot B &= g \quad \text{in the trace sense at } \partial G
\end{align*}
\]

(2.10a)

(2.10b)

has a unique solution \( B \in H^1(\hat{G}) \), and we have the estimate

\[
\| B \|_{H^1(\hat{G})} \leq C \| g \|_{H^{1/2}(\partial G)}
\]

(2.11)

with a constant \( C \) depending on \( G \).
Proof: The lemma is a consequence of the well-known Neumann problem for (exterior) harmonic functions:

\[
\begin{align*}
\Delta u &= 0 \quad \text{a.e. in } \hat{G}, \\
n \cdot \nabla u &= g \quad \text{in the trace sense at } \partial G,
\end{align*}
\]

which has a unique solution \( u \in H^2(\hat{G}) \), together with the estimate

\[
\| \nabla u \|_{H^1(\hat{G})} \leq C \| g \|_{H^{1/2}(\partial G)}
\]

(see, e.g., [11]). In fact, \( B := \nabla u \), where \( u \) solves (2.12), is a solution of (2.10). Uniqueness follows with the unique solvability of (2.12) in the simply connected domain \( \hat{G} \).

Remark 2.5 As is well-known there is \( u \in C^\infty(\hat{G}) \) and hence \( B \in C^\infty(\hat{G}) \), and with the representation

\[
u(x) = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \frac{c_{nm}}{|x|^{n+1}} Y_{nm}(x/|x|)
\]

of an exterior harmonic function \( u \in H^1(\hat{G}) \) by spherical harmonics \( Y_{nm} \) (see, e.g., [8, Appendix C]) follows the asymptotic behaviour \( |u| = O(|x|^{-2}) \) and hence

\[|B| = O(|x|^{-3}) \quad \text{for } |x| \to \infty.\]

Note that if \( G \) is a ball \( B_R \), \( u \) and hence \( B = \nabla u \) have vanishing spherical mean \( \langle u \rangle = \langle B \rangle = 0 \) for all \( r > R \).

Lemma 2.6 (H\(^1\) Harmonic extension) Any poloidal field \( P \in \mathcal{P}(B_R) \cap H^1(B_R) \) has a unique harmonic extension \( \tilde{P} \in \mathcal{P}(\mathbb{R}^3) \), i.e. \( \hat{x} \cdot \tilde{P} \in H^1(\mathbb{R}^3) \), \( \tilde{P} = P \) a.e. in \( B_R \), and \( \tilde{P} \) is harmonic in \( \hat{B}_R \) and satisfies (2.15). Moreover, we have the following norm-equivalences

\[
\| \hat{x} \cdot P \|_{H^1(B_R)} \sim \| \hat{x} \cdot \tilde{P} \|_{H^1(\mathbb{R}^3)}, \quad \| P \|_{H^1(B_R)} \sim \| \tilde{P} \|_{H^1(\hat{B}_R \cup \hat{B}_R)}. \]

Proof: Given \( P \in \mathcal{P}(B_R) \cap H^1(B_R) \) the component \( \hat{x} \cdot P \) has a trace \( g \in H^{1/2}(S_R) \) on \( S_R = \partial B_R \), which satisfies \( \int_{S_R} g \, ds = 0 \). Thus, lemma 2.4 provides a harmonic (hence poloidal) exterior field \( B =: \tilde{P} |_{\hat{B}_R} \), which satisfies (2.15) and

\[
\hat{x} \cdot B = g = \hat{x} \cdot P \quad \text{in the trace sense at } S_R.
\]

Property (2.17) implies \( \hat{x} \cdot \tilde{P} \in H^1(\mathbb{R}^3) \). The equivalences (2.16) follow with the estimate

\[
\| \tilde{P} \|_{H^1(\hat{B}_R)} \leq C \| g \|_{H^{1/2}(S_R)} \leq \tilde{C} \| \hat{x} \cdot P \|_{H^1(B_R)},
\]

where we used (2.11) and a standard trace estimate (see [1, p. 217]).

In the following it will be convenient to have harmonic extensions of any element \( P \in \mathcal{P}(B_R) \). This can be achieved by realizing that \( P \) is the curl of a toroidal field \( T \in T(B_R) \cap H^1(B_R) \). Since \( x \cdot T = 0 \) a.e. in \( G \), the harmonic (vector field) extension of \( T \) is of course trivial. The following lemma, however, associates to \( T \) a ‘scalar harmonic extension’:

Lemma 2.7 (scalar harmonic extension) Any toroidal field \( T \in T(B_R) \cap H^1(B_R) \) has a unique scalar harmonic extension \( \tilde{T} \in T(\mathbb{R}^3) \cap H^1(\mathbb{R}^3) \), i.e. \( \tilde{T} = T \) a.e. in \( B_R \) and \( \Delta \tilde{T} = 0 \) together with \( |\tilde{T}| = O(|x|^{-2}) \) for \( |x| \to \infty \) in \( \hat{B}_R \). Moreover, we have the norm-equivalence

\[
\| T \|_{H^1(B_R)} \sim \| \nabla \tilde{T} \|_{L^2(\mathbb{R}^3)}.
\]

5
PROOF: Let \( T \in T(B_R) \cap H^1(B_R) \) and \( \psi \in H^1(B_R) \) be the associated toroidal scalar according to the lemmata 2.1 and 2.3. By theorem 3.3 in \(^3\) \( \psi \) has a unique harmonic extension \( \tilde{\psi} \in H^1(\mathbb{R}^3) \). Setting \( \tilde{T} := -\Lambda \tilde{\psi} \) we have clearly \( \tilde{T} = T \) in \( B_R \) and \( \Delta \tilde{T} = 0 \) in \( \hat{B}_R \). The asymptotics follows with (2.14). In order to prove \( \tilde{T} \in H^1(\mathbb{R}^3) \) let \( F \in C_0^\infty(\mathbb{R}^3) \) and \( i \in \{1, 2, 3\} \). As \( \Lambda \) involves only non-radial derivatives we have the identity
\[
\int_{\mathbb{R}^3} \partial_i \tilde{\psi} \Lambda \cdot F \, dx = - \int_{\mathbb{R}^3} \Lambda \partial_i \tilde{\psi} \cdot F \, dx.
\]
By \( \Lambda \psi \in H^1(B_R) \) and \( \Lambda \tilde{\psi}\big|_{\hat{B}_R} \in H^1(\hat{B}_R) \) we have \( \Lambda \partial_i \tilde{\psi} \in L^2(\mathbb{R}^3) \), i.e. \( \partial_i \tilde{\psi} \) is weakly non-radially differentiable in all of \( \mathbb{R}^3 \). This implies the assertion. As to relation (2.18) observe that \( \tilde{T} \) is component-wise the unique harmonic extension of \( T \); so applying (component-wise) the equivalence relation (3.12) \( k=1 \) from the next section yields
\[
\|T\|_{H^1(B_R)} \sim \|T\|_{1/2} = \|
abla \tilde{T}\|_{L^2(\mathbb{R}^3)},
\]
which is (2.18).

\[\square\]

Remark 2.8 We note as a rule for later use: For any function \( u \in H^k(B_R) \) with \( \Lambda u \in H^k(B_R) \) and with harmonic extension \( \tilde{u} \in H^1(\mathbb{R}^3) \), \( l \leq k \) we have also \( \Lambda \tilde{u} \in H^1(\mathbb{R}^3) \), i.e. non-radial derivatives do not deteriorate the regularity of the harmonic extension over the boundary.

**Lemma 2.9 (L^2 Harmonic extension)** Any poloidal field \( P \in \mathcal{P}(B_R) \) with representation \( P = -\nabla \times \Lambda \phi \) has a harmonic extension \( \tilde{P} := -\nabla \times (\Lambda \tilde{\phi}) \in \mathcal{P}(\mathbb{R}^3) \), where \( \Lambda \tilde{\phi} \) denotes the scalar harmonic extension of \( \Lambda \phi \in T(B_R) \). It holds \( \tilde{P} = P \ a.e. \ in \ B_R, \ \nabla \times \tilde{P} = 0 \ in \ \hat{B}_R \) and the asymptotics (2.15). If, additionally, \( P \in H^1(B_R) \), \( \tilde{P} \) coincides with the \( H^1 \) harmonic extension from lemma 2.6.

PROOF: The definition of \( \tilde{P} \) makes sense, as \( \Lambda \phi \in L^2(B_R), \ \nabla \cdot \Lambda \phi = 0, \ \nabla \times \Lambda \phi \in L^2(B_R), \) and \( \tilde{\phi} \cdot \Lambda \phi = 0 \) on \( S_R \) imply \( \Lambda \phi \in T(B_R) \cap H^1(B_R) \) (cf. [4, p. 358]), hence lemma 2.7 applies and provides \( \tilde{\Lambda} \phi \in H^1(\mathbb{R}^3) \) with \( \Delta \tilde{\Lambda} \phi|_{\hat{B}_R} = 0 \). Therefore,
\[
\nabla \times \tilde{P} = -\nabla \times \nabla \times (\Lambda \tilde{\phi}) = \Delta \tilde{\Lambda} \phi = 0 \quad \text{in} \ \hat{B}_R,
\]
i.e. \( \tilde{P}|_{\hat{B}_R} \) is an exterior harmonic vector field with asymptotics (2.15). In the case that \( P \in H^1(B_R) \) we find by \( \tilde{\Lambda} \phi \in H^1(\mathbb{R}^3) \) and remark 2.8
\[
x \cdot \tilde{P} = -x \cdot \nabla \times (\Lambda \tilde{\phi}) = -\Lambda \cdot (\tilde{\Lambda} \phi) \in H^1(\mathbb{R}^3),
\]
i.e. \( \tilde{P} \) is in fact the unique \( H^1 \) harmonic extension of \( P \).

\[\square\]

Remark 2.10 There are attempts to generalize the poloidal/toroidal representation in \( \mathbb{R}^3 \) by introducing generalized coordinates \( (\xi_1, \xi_2, \xi_3) \) such that (1.3) holds in the form
\[
B = P + \Delta = \nabla \times (\nabla \times \phi \nabla \xi_1) + \nabla \times \psi \nabla \xi_1.
\]
These attempts are in parts successful, for instance, the representation (2.19) with uniquely determined scalars \( \phi \) and \( \psi \) has been established for smooth functions \( \xi_1 \) [6]. The coordinate surfaces \( \xi_1 = \text{const} \) play here the role of the spheres \( |x| = \text{const} \) in (1.3); the toroidal field

\[\text{Cited in lemma 3.4 below.}\]
\( T \) is again tangential to these surfaces. However, not all features of (1.3) are preserved. The determining equations for \( \phi \) and \( \psi \) are no longer as simple as eqs. (2.3) and (2.4), respectively; \( \nabla \times P \) is generally no longer a toroidal field. In fact, the latter condition singles out just two cases [12]: The spherical case \( \xi_1 = |x| \) employed here and the Cartesian case \( \xi_1 = x_1 \), for which also an elaborate theory is available [15]. For all other “geometries” the diffusive coupling of \( \nabla \xi_1 \cdot B \) with the other components of \( B \) seems to prevent representations as useful as (1.3).

### 3 Basis fields

For the dynamo problem (1.1) in a ball \( B_R \) appropriate basis fields are solutions of the following eigenvalue problem:

\[
\begin{align*}
\nabla \times \nabla \times B &= \lambda B, \quad \text{in } B_R, \quad (3.1a) \\
\nabla \times B &= 0, \quad \nabla \cdot B = 0 \quad \text{in } \hat{B}_R, \quad (3.1b) \\
B \text{ continuous} &\text{ in } \mathbb{R}^3, \quad (3.1c) \\
B(x) &\to 0 \quad \text{for } |x| \to \infty. \quad (3.1d)
\end{align*}
\]

Having in mind the poloidal/toroidal representation (1.3), the solution of (3.1) makes use of the solutions of the following two scalar eigenvalue problems

\[
\begin{align*}
-\Delta v &= \mu v \quad \text{in } B_R, \quad (3.2a) \\
\Delta v &= 0 \quad \text{in } \hat{B}_R, \quad (3.2b) \\
v \text{ and } \nabla v \text{ continuous} &\text{ in } \mathbb{R}^3, \quad (3.2c) \\
\langle v \rangle &= 0 \quad \text{in } \mathbb{R}_+, \quad (3.2d) \\
v(x) &\to 0 \quad \text{for } |x| \to \infty. \quad (3.2e)
\end{align*}
\]

and

\[
\begin{align*}
-\Delta w &= \nu w \quad \text{in } B_R, \quad (3.3a) \\
\langle w \rangle &= 0 \quad \text{for } r < R, \quad (3.3b) \\
w &= 0 \quad \text{on } S_R. \quad (3.3c)
\end{align*}
\]

Except for the zero-spherical-mean condition the latter problem is of standard type (see, e.g., [5, p. 334]) and the solution of the former problem can be found in [8, theorem 3.1]. The zero-spherical-mean condition just eliminates some eigenfunctions of the problem without this condition (cf. [8, appendix D]). We summarize the results in

**Lemma 3.1** (i) Problem (3.2) has a countable set of eigensolutions \( \{ (\tilde{v}_m, \mu_m) : m \in \mathbb{N} \} \), where \( \tilde{v}_m \in C^\infty(B_R \cup \hat{B}_R) \cap C^1(\mathbb{R}^3) \) and \( (\mu_m)_{m \in \mathbb{N}} \) is a non-decreasing sequence of real, positive numbers with \( \lim_{m \to \infty} \mu_m = \infty \). The set \( \{ \tilde{v}_m : m \in \mathbb{N} \} \) is orthogonal with respect to the ‘gradient scalar product’

\[
(\nabla \tilde{v}_m, \nabla \tilde{v}_n)_{L^2(\mathbb{R}^3)} = \mu_m \delta_{m,n}.
\]

Restricting \( \tilde{v}_m \) onto \( B_R \) the set \( \{ v_m = \tilde{v}_m|_{B_R} : m \in \mathbb{N} \} \) constitutes an orthonormal basis of \( L^2(B_R) \).

(ii) Problem (3.3) has a countable set of eigensolutions \( \{ (w_n, \nu_n) : n \in \mathbb{N} \} \), where \( w_n \in C^\infty(\hat{B}_R) \) and \( (\nu_n)_{n \in \mathbb{N}} \) is a non-decreasing sequence of real, positive numbers with \( \lim_{n \to \infty} \nu_n = \infty \). The set \( \{ w_n : n \in \mathbb{N} \} \) constitutes an orthonormal basis of \( L^2(B_R) \).
As $-\Delta$ and $-\mathcal{L}$ are commuting operators, $v_m$ and $w_n$ are eigenfunctions of the Laplace-Beltrami operator as well:

$$-\mathcal{L} \tilde{v}_m = \tau_m \tilde{v}_m, \quad -\mathcal{L} w_n = \sigma_n w_n \quad (3.5)$$

with $(\tau_m)_{m\in\mathbb{N}}$ and $(\sigma_n)_{n\in\mathbb{N}}$ denoting sequences of real positive numbers. Explicit representations of $\tilde{v}_m$, $\mu_m$, $\tau_m$ and $w_n$, $\nu_n$, $\sigma_n$ can be found in [2].

Now, setting

$$\tilde{P}_m := -\mu_m^{-1/2} \tau_m^{-1/2} \nabla \times \Lambda \tilde{v}_m, \quad \tilde{T}_n := -\sigma_n^{-1/2} \Lambda \tilde{w}_n, \quad (3.6)$$

where $\tilde{w}_n$ is the trivial extension of $w_n$ onto $\mathbb{R}^3$, we have clearly $\tilde{P}_m$, $\tilde{T}_n \in C^\infty(B_R \cup \hat{B}_R) \cap C(\mathbb{R}^3)$ and we find with (2.1), (2.2), and (2.14) the pairs $(\tilde{P}_m, \mu_m)$, $(\tilde{T}_n, \nu_n)$ to be solutions of the eigenvalue problem (3.1). If the union of poloidal and toroidal eigenfunctions is ordered according to non-decreasing eigenvalues, we use the notation $(\hat{B}_l, \lambda_l)$, i.e. it holds

$$\{(\tilde{P}_m, \mu_m) : m \in \mathbb{N}\} \cup \{(\tilde{T}_n, \nu_n) : n \in \mathbb{N}\} = \{(\hat{B}_l, \lambda_l) : l \in \mathbb{N}\}.$$ 

Besides the spaces $\mathcal{P}(B_R) = \mathcal{P}$, $\mathcal{T}(B_R) = \mathcal{T}$, and $\mathcal{B}(B_R) = \mathcal{B}$ it is now convenient to introduce the spaces

$$\hat{\mathcal{P}} := \{\hat{P} \in \mathcal{P}(\mathbb{R}^3) : \hat{P} \text{ is the } L^2 \text{ harmonic extension of some } P \in \mathcal{P}\},$$
$$\hat{\mathcal{T}} := \{\hat{T} \in \mathcal{T}(\mathbb{R}^3) : \hat{T} \text{ is the trivial extension of some } T \in \mathcal{T}\},$$

and $\hat{\mathcal{B}} := \hat{\mathcal{P}} \oplus \hat{\mathcal{T}}$.

We then have

**Theorem 3.2** Problem (3.1) has a countable set of eigensolutions $\{(\hat{B}_l, \lambda_l) : l \in \mathbb{N}\}$, where $\hat{B}_l \in C^\infty(B_R \cup \hat{B}_R) \cap C(\mathbb{R}^3)$ and $(\lambda_l)_{l \in \mathbb{N}}$ is a non-decreasing sequence of real, positive numbers with $\lim_{l \to \infty} \lambda_l = \infty$. The set $\{\hat{B}_l : l \in \mathbb{N}\}$ constitutes an orthonormal basis of $\hat{\mathcal{B}}$; in particular, $\{\tilde{P}_m : m \in \mathbb{N}\}$ and $\{\tilde{T}_n : n \in \mathbb{N}\}$ are orthonormal bases of $\hat{\mathcal{P}}$ and $\hat{\mathcal{T}}$, respectively. Moreover, $\{P_m : m \in \mathbb{N}\}$ is orthogonal with respect to the ‘curled’ scalar product

$$\langle \nabla \times P_m, \nabla \times P_n \rangle_{L^2(B_R)} = \mu_m \delta_{m,n}. \quad (3.7)$$

**Proof:** It remains to show the orthonormality of the set $\{\hat{B}_l : l \in \mathbb{N}\}$, the completeness of the sets $\{\tilde{P}_m : m \in \mathbb{N}\}$ and $\{\tilde{T}_n : n \in \mathbb{N}\}$ or - equivalently - $\{T_n : n \in \mathbb{N}\}$, and relation (3.7). By (2.6) we find immediately $(\tilde{P}_m, \tilde{T}_n)_{L^2(\mathbb{R}^3)} = (P_m, T_n)_{L^2(B_R)} = 0$ for any $\tilde{P}_m \in \hat{\mathcal{P}}$ and $\tilde{T}_n \in \hat{\mathcal{T}}$. For $\tilde{T}_n$, $\tilde{T}_m \in \hat{\mathcal{T}}$ we obtain by (2.5), (3.5), and Lemma 3.1

$$\sigma_n^{1/2} \sigma_m^{1/2} \langle \nabla \times \tilde{T}_n, \nabla \times \tilde{T}_m \rangle_{L^2(\mathbb{R}^3)} = \sigma_n^{1/2} \sigma_m^{1/2} \langle T_n, T_m \rangle_{L^2(B_R)} = \int_{B_R} \langle \nabla \times w_n x, \nabla \times w_m x \rangle \mathrm{d}x$$

$$= \int_{B_R} \langle \nabla \times (\nabla \times w_n x), w_m x \rangle \mathrm{d}x - \int_{S_R} \langle \nabla \times w_n x, \nabla \times w_m x \rangle \mathrm{d}S$$

$$= \int_{B_R} (-\mathcal{L} w_n) w_m \mathrm{d}x = \sigma_n \delta_{m,n}.$$ 

$^4$For toroidal fields the notation is consistent with that introduced in lemma 2.7: Because of $T_n|_{S_R} = 0$, $\tilde{T}_n$ is just the scalar harmonic extension of $\tilde{T}_n = -\sigma_n^{-1/2} \Lambda w_n$.
For \( \tilde{P}_n, \tilde{P}_m \in \tilde{T} \) we calculate similarly

\[
\mu_n^{1/2} \mu_m^{1/2} r_n^{1/2} r_m^{1/2} \langle P_n, P_m \rangle_{L^2(B_R)} = \int_{B_R} \left( \nabla \times (\nabla \times v_n) \right) : \left( \nabla \times (\nabla \times v_m) \right) \, dx
\]

\[
= \int_{B_R} \left( \nabla \times (\nabla \times (\nabla \times v_n)) \right) : \left( \nabla \times v_m \right) \, dx - \int_{S_R} \left( \nabla \times (\nabla \times v_n) \right) \times (\nabla \times v_m) \cdot \hat{x} \, ds
\]

and

\[
\mu_n^{1/2} \mu_m^{1/2} r_n^{1/2} r_m^{1/2} \langle \tilde{P}_n, \tilde{P}_m \rangle_{L^2(B_R)} = \int_{B_R} \left( \nabla \times (\nabla \times \tilde{v}_n) \right) : \left( \nabla \times (\nabla \times \tilde{v}_m) \right) \, dx
\]

\[
= \int_{B_R} \left( \nabla \times (\nabla \times (\nabla \times \tilde{v}_n)) \right) : \left( \nabla \times \tilde{v}_m \right) \, dx + \int_{S_R} \left( \nabla \times (\nabla \times \tilde{v}_n) \right) \times (\nabla \times \tilde{v}_m) \cdot \hat{x} \, ds.
\]

Thus, by (2.1), (2.2), (2.5), (3.5), and lemma 3.1:

\[
\mu_n^{1/2} \mu_m^{1/2} r_n^{1/2} r_m^{1/2} \langle \tilde{P}_n, \tilde{P}_m \rangle_{L^2(\mathbb{R}^3)} = \int_{B_R} \left( \nabla \times (\nabla \times \tilde{v}_n) \right) : \left( \nabla \times \tilde{v}_m \right) \, dx - \int_{S_R} \left( \nabla \times \tilde{v}_n \right) \times \tilde{v}_m \cdot \hat{x} \, ds
\]

\[
= \mu_n \int_{B_R} (\nabla \times P_n) \cdot (\nabla \times P_m) \, dx - \int_{S_R} (\nabla \times v_n) \times (\nabla \times v_m) \cdot \hat{x} \, ds
\]

\[
= \mu_n \tau_n \delta_{nm}.
\]

The orthogonality relation (3.7) follows analogously:

\[
\mu_n^{1/2} \mu_m^{1/2} r_n^{1/2} r_m^{1/2} \langle \nabla \times P_n, \nabla \times P_m \rangle_{L^2(\mathbb{R}^3)} = \int_{B_R} (\Lambda(-\Delta) v_n) \cdot (\Lambda(-\Delta) v_m) \, dx
\]

\[
= \mu_n \mu_m \int_{B_R} (-\mathcal{L} v_n) v_m \, dx = \mu_n \mu_m \tau_n \delta_{nm}.
\]

Note that orthonormality of the \( \tilde{P}_m \) with respect to the \( L^2 \)-scalar product refers to \( \mathbb{R}^3 \), whereas orthonormality of the \( \tilde{T}_n \) holds for \( \mathbb{R}^3 \) as well as for \( B_R \).

As to completeness of \( \{ T_n : n \in \mathbb{N} \} \) let \( T \in \mathcal{T} \) with representation \( T = -\Lambda \psi \) in \( D(B_R) \) (according to lemma 2.1) by \( \psi \in L^2(B_R) \) with expansion \( \psi = \sum_{j=1}^{\infty} d_j w_j, \sum_{j=1}^{\infty} |d_j|^2 < \infty \) (according to lemma 3.1). \( T = -\Lambda \psi \) in \( D(B_R) \) means by definition

\[
\int_{B_R} T \cdot F \, dx = \int_{B_R} \psi \Lambda \cdot F \, dx
\]

for any \( F \in C_0^\infty(B_R) \). So, choosing a sequence \( (F_n^{(i)})_{i \in \mathbb{N}} \subset C_0^\infty(B_R) \) with \( \|F_n^{(i)} - T_n\|_{H^1(B_R)} \to 0 \) for \( i \to \infty \) we can calculate for any \( n \in \mathbb{N} \)

\[
\int_{B_R} T \cdot T_n \, dx = \lim_{i \to \infty} \int_{B_R} T \cdot F_n^{(i)} \, dx = \lim_{i \to \infty} \int_{B_R} \psi \Lambda \cdot F_n^{(i)} \, dx
\]

\[
= \int_{B_R} \psi \Lambda \cdot T_n \, dx = \sigma_n^{-1/2} \int_{B_R} \psi (\mathcal{L} w_n) \, dx
\]

\[
= \sigma_n^{1/2} \sum_{j=1}^{\infty} d_j (w_j, w_n)_{L^2(B_R)} = \sigma_n^{1/2} d_n,
\]
where we used (3.5) and (3.6). So, \((T, T_n)_L^2(B_R) = 0\) for any \(n \in \mathbb{N}\) implies \(T = 0\), hence \(\{T_n : n \in \mathbb{N}\}\) is complete in \(T\).

Similarly in the poloidal case, let \(\tilde{P} \in \tilde{P}\) with representation \(\tilde{P} = -\nabla \times (\tilde{\Lambda} \phi)\) (according to lemma 2.9) with scalar harmonic extension \(\tilde{\Lambda} \phi = \Lambda \tilde{\phi} \in H^1(\mathbb{R}^3)\) and expansion \(\tilde{\phi} = \sum_{i=1}^{\infty} c_i \tilde{v}_i\), \(\sum_{i=1}^{\infty} |c_i|^2 < \infty\). By (2.5), (3.2), and again (3.5), (3.6) we obtain now

\[
\int_{\mathbb{R}^3} \tilde{P} \cdot \tilde{P}_m \, dx = \mu_m^{1/2} r_m^{-1/2} \int_{B_R} \Lambda \phi \cdot \Lambda (\nabla v_m) \, dx + \int_{B_R} \Lambda \tilde{\phi} \cdot \Lambda (\nabla \tilde{v}_m) \, dx
\]

\[
= \mu_m^{1/2} r_m^{-1/2} \int_{B_R} \phi (-\nabla v_m) \, dx = \mu_m^{1/2} r_m^{-1/2} \sum_{i=1}^{\infty} c_i (v_i, v_m)_{L^2(B_R)}
\]

\[
= \mu_m^{1/2} r_m^{-1/2} c_m.
\]

Note that \(\tilde{\Lambda} \phi\) and \(\tilde{P}_m\) have well-defined traces on \(S_R\) so that boundary terms do not appear in (3.8). Therefore, \((\tilde{P}, \tilde{P}_m)_{L^2(\mathbb{R}^3)} = 0\) for any \(m \in \mathbb{N}\) implies again \(\tilde{P} = 0\), and hence \(\{\tilde{P}_m : m \in \mathbb{N}\}\) is complete in \(\tilde{P}\).

**Remark 3.3** Due to the divergence constraint the radial component of \(\tilde{P}_m\) is even more regular over the boundary. In fact, writing \(\nabla \cdot \tilde{P} = 0\) in the form

\[
\Lambda \cdot (x \times \tilde{P}) = -\langle \tilde{x}, \nabla \rangle (|x| \tilde{P} \cdot x)
\]

and observing that for \(\tilde{P} = \tilde{P}_m\) the left-hand side is continuous in \(\mathbb{R}^3\), so is the right-hand side, which implies \(\tilde{P}_m \cdot \tilde{x} \in C^1(\mathbb{R}^3)\). More generally, \(P \in H^2(B_R)\) with \(\tilde{P} \in H^1(\mathbb{R}^3)\) implies by remark 2.8 \(\Lambda \cdot (x \times \tilde{P}) \in H^1(\mathbb{R}^3)\) and hence \(\tilde{P} \cdot \tilde{x} \in H^2(\mathbb{R}^3)\).

Improved regularity of functions, expanded in a suitable set of eigenfunctions shows up in an improved convergence behaviour of their Fourier coefficients. In order to measure the convergence behaviour of functions expanded in the eigenfunctions introduced in lemma 3.1 let us define the spaces of ‘formal series’,

\[
\mathcal{V} := \left\{ \sum_{m=1}^{\infty} c_m v_m : c_m \in \mathbb{R} \right\} \quad \text{and} \quad \mathcal{W} := \left\{ \sum_{n=1}^{\infty} d_n w_n : d_n \in \mathbb{R} \right\}
\]

with subspaces

\[
\mathcal{V}_\alpha := \left\{ v \in \mathcal{V} : \|v\|_\alpha < \infty \right\} \quad \text{and} \quad \mathcal{W}_\alpha := \left\{ w \in \mathcal{W} : \|w\|_\alpha < \infty \right\},
\]

where \(\|\cdot\|_\alpha\) with \(\alpha \in \mathbb{R}\) denotes the \(\alpha\)-norm

\[
\|v\|_\alpha := \left( \sum_{m=1}^{\infty} \mu_m^{2\alpha} |c_m|^2 \right)^{1/2} \quad \text{for} \quad v = \sum_{m=1}^{\infty} c_m v_m \in \mathcal{V}
\]

and

\[
\|w\|_\alpha := \left( \sum_{n=1}^{\infty} \nu_n^{2\alpha} |d_n|^2 \right)^{1/2} \quad \text{for} \quad w = \sum_{n=1}^{\infty} d_n w_n \in \mathcal{W},
\]

respectively. We then have the ordering \(\mathcal{V}_\alpha \subset \mathcal{V}_\beta\) for \(\alpha \geq \beta\) and, in particular, \(\mathcal{V}_\alpha \subset \mathcal{V}_0 = L^2(B_R)\) for any \(\alpha \geq 0\). With the pairing \(\langle v, \overline{v} \rangle = \sum_{m=1}^{\infty} c_m \overline{c}_m\) for \(v = \sum_{m=1}^{\infty} c_m v_m \in \mathcal{V}_\alpha\) and \(\overline{v} = \sum_{m=1}^{\infty} \overline{c}_m v_m \in \mathcal{V}_{-\alpha}\), \(\mathcal{V}_{-\alpha}\) is the dual space of \(\mathcal{V}_\alpha\). Analogous results hold for \(\mathcal{W}_\alpha\).
Defining, furthermore, the linear operator $A^\alpha$, $\alpha \in \mathbb{R}$ on $v = \sum_{m=1}^{\infty} c_m v_m \in V$ and $w = \sum_{n=1}^{\infty} d_n w_n \in W$ by
\[
A^\alpha v = \sum_{m=1}^{\infty} \mu_m^\alpha c_m v_m \quad \text{and} \quad A^\alpha w = \sum_{n=1}^{\infty} \nu_n^\alpha d_n w_n, \tag{3.9}
\]
respectively, we have obviously $A^0(\mathcal{V}_\beta) = \mathcal{V}_{\beta - \alpha}$ and, in particular, $A^0(\mathcal{V}_\alpha) = L^2(B_R)$, and analogous relations for $W_\beta$. $A^0$ is the identity map and $A^1 = A$ with $D(A) = \mathcal{V}_1$, resp. $D(A) = \mathcal{W}_1$ are extensions of the Laplacian with boundary conditions (3.2b-e), resp. (3.3c).

The following lemma characterizes regularity and boundary behaviour of elements of $\mathcal{V}_{k/2}$ and $\mathcal{W}_{k/2}$, $k \in \mathbb{N}$ (for part (i) see [9, theorem 2], which is a corrected version of [8, theorem 3.3], and [13, p. 183f] for part (ii); the present version differs from these references in that only zero-spherical-mean functions are admitted).

Lemma 3.4 Let $k \in \mathbb{N}$ and $[r] := \max\{i \in \mathbb{N} : i \leq r\}$ the integer part of $r \in \mathbb{R}$.

(i) For $k = 1$ holds
\[
\mathcal{V}_{1/2} = \{v \in H^1(B_R) : \langle v \rangle = 0\}. \tag{3.10}
\]
Any $v = \sum_{m=1}^{\infty} c_m v_m \in \mathcal{V}_{1/2}$ has a unique harmonic extension $\tilde{v} := \sum_{m=1}^{\infty} \tilde{c}_m \tilde{v}_m \in H^1(\mathbb{R}^3)$, i.e. $\tilde{v}|_{B_R} = v$ and $\tilde{v}|_{\partial B_R}$ is harmonic.

For $k > 1$ holds
\[
\mathcal{V}_{k/2} = \{v \in H^k(B_R) : \langle v \rangle = 0 \text{ and } \tilde{\Delta} v \in H^2(\mathbb{R}^3) \text{ for } i = 0, \ldots, [k/2] - 1\}, \tag{3.11}
\]
where $\tilde{\Delta} v$ denotes again the harmonic extension of $\Delta^i v$. On $\mathcal{V}_{k/2}$, $k \in \mathbb{N}$ we have the equivalence of norms:
\[
\|v\|_{k/2} = \|\Delta^{k/2} v\|_{L^2(B_R)} \sim \|v\|_{H^k(B_R)}, \quad k \text{ even},
\]
\[
\|v\|_{k/2} = \|\nabla(\tilde{\Delta} (\Delta^{(k-1)/2} v))\|_{L^2(\mathbb{R}^3)} \sim \|v\|_{H^k(B_R)}, \quad k \text{ odd}. \tag{3.12}
\]

(ii) For $k \in \mathbb{N}$ holds
\[
\mathcal{W}_{k/2} = \{w \in H^k(B_R) : \langle w \rangle = 0 \text{ and } \Delta^j w \in H^1_0(B_R) \text{ for } j = 0, \ldots, [(k-1)/2]\}, \tag{3.13}
\]
and on $\mathcal{W}_{k/2}$ we have the equivalence of norms
\[
\|w\|_{k/2} = \|(-\Delta)^{k/2} w\|_{L^2(B_R)} \sim \|w\|_{H^k(B_R)}. \tag{3.14}
\]
These results will now be lifted to poloidal and toroidal fields just by replacing the sets $\{v_m : m \in \mathbb{N}\}$ and $\{w_n : n \in \mathbb{N}\}$ by $\{P_m : m \in \mathbb{N}\}$ and $\{T_n : n \in \mathbb{N}\}$, respectively, introduced in theorem 3.2. So, let us define
\[
\mathcal{P}_\alpha := \left\{P = \sum_{m=1}^{\infty} C_m P_m : \|P\|_\alpha < \infty\right\}, \quad \mathcal{T}_\alpha := \left\{T = \sum_{n=1}^{\infty} D_n T_n : \|T\|_\alpha < \infty\right\},
\]
where
\[
\|P\|_\alpha := \left(\sum_{m=1}^{\infty} \mu_m^{2\alpha} |C_m|^2\right)^{1/2}, \quad \|T\|_\alpha := \left(\sum_{n=1}^{\infty} \nu_n^{2\alpha} |D_n|^2\right)^{1/2},
\]
and $\alpha, C_m, D_n \in \mathbb{R}$. For $\mathcal{P}_\alpha$ and $\mathcal{T}_\alpha$ hold the same relations as for $\mathcal{V}_\alpha$ and $\mathcal{W}_\alpha$, respectively; in particular, we have
\[
\mathcal{P}_0 = \{P \in L^2(B_R) : P \text{ poloidal }\} = \mathcal{P}, \quad \mathcal{T}_0 = \{T \in L^2(B_R) : T \text{ toroidal }\} = \mathcal{T}.
\]
$A^\alpha$ operates on $P \in \mathcal{P}_\beta$ and $T \in \mathcal{T}_\beta$ analogously to (3.9). The half-integer spaces $\mathcal{P}_{k/2}$ and $\mathcal{T}_{k/2}$, $k \in \mathbb{N}$ are characterized in the following theorem.
Theorem 3.5  (i) Let $k \in \mathbb{N}$ and $\tilde{P}$ denoting the harmonic extension of the poloidal field $P$, then

$$\mathcal{P}_{k/2} = \{ P \in \mathcal{P} \cap H^k(B_R) : \Delta^i P \in H^1(\mathbb{R}^3) \text{ for } i = 0, \ldots, [(k-1)/2] \}$$

(3.15)

and on $\mathcal{P}_{k/2}$ we have the equivalence of norms:

$$\|P\|_{k/2} = \|\Delta^{k/2}P\|_{L^2(\mathbb{R}^3)} \sim \|P\|_{H^k(B_R)}, \quad k \text{ even,}$$

$$\|P\|_{k/2} = \|\nabla \times (\Delta^{(k-1)/2}P)\|_{L^2(B_R)} \sim \|P\|_{H^k(B_R)}, \quad k \text{ odd.}$$

(3.16)

(ii) For $k \in \mathbb{N}$ holds

$$\mathcal{T}_{k/2} = \{ T \in \mathcal{T} \cap H^k(B_R) : \Delta^j T \in H^1_0(B_R) \text{ for } j = 0, \ldots, [(k-1)/2] \}$$

(3.17)

and on $\mathcal{T}_{k/2}$ we have the equivalence of norms

$$\|T\|_{k/2} = \|(-\Delta)^{k/2}T\|_{L^2(B_R)} \sim \|T\|_{H^k(B_R)}.$$  

(3.18)

Proof: (I) We begin with the (simpler) toroidal case. Let $T \in \mathcal{T} \cap H^k(B_R)$ with $\Delta^j T|_{S_R} = 0$ in the trace sense for $j = 0, \ldots, [(k-1)/2]$. According to Lemma 2.3 there is a unique $\psi \in H^k(B_R)$ with $\langle \psi \rangle = 0$ such that $T = -\Lambda \psi$ in $B_R$ and

$$\Lambda \Delta^j \psi|_{S_R} = \Delta^j \Lambda \psi|_{S_R} = 0 \quad \text{in } L^2(S_R) \text{ for } j = 0, \ldots, [(k-1)/2].$$

(3.19)

Property (3.19) together with $\langle \psi \rangle = 0$ implies $\Delta^j \psi|_{S_R} = 0$ in $L^2(S_R)$ for $j = 0, \ldots, [(k-1)/2]$, and by (3.13) $\psi$ has a representation

$$\psi = \sum_{n=1}^{\infty} d_n w_n, \quad \sum_{n=1}^{\infty} \nu_n^k|d_n|^2 < \infty, \quad d_n := (\psi, w_n)_{L^2(B_R)}.$$

On the other side, according to theorem 3.2, $T$ has in $L^2(B_R)$ the representation $T = \sum_{n=1}^{\infty} D_n T_n$, whose coefficients $D_n$ are related to $d_n$ by

$$D_n = (T, T_n)_{L^2(B_R)} = \sigma_n^{-1/2}(\Lambda \psi, \Lambda w_n)_{L^2(B_R)} = \sigma_n^{-1/2}(\sigma_n^{-1/2}(\psi, (-\mathcal{L}) w_n)_{L^2(B_R)}) = \sigma_n^{-1/2}d_n.$$

Since $T = -\Lambda \psi \in H^k(B_R)$ with property (3.19) we can (component-wise) apply (3.14) and can, thus, calculate

$$\infty \geq \|T\|_{H^k(B_R)}^2 \sim \|(-\Delta)^{k/2}\psi\|_{L^2(B_R)}^2 = \|(-\Delta)^{k/2}\psi, (-\mathcal{L}) \psi\|_{L^2(B_R)}^2 = \left( \sum_{n=1}^{\infty} d_n \nu_n^k w_n \right) \left( \sum_{m=1}^{\infty} \sigma_m m w_m \right)_{L^2(B_R)} = \sum_{n=1}^{\infty} \nu_n^k \sigma_n |d_n|^2 = \sum_{n=1}^{\infty} \nu_n^k |D_n|^2,$$

(3.20)

which means $T \in \mathcal{T}_{k/2}$.

To prove the opposite inclusion let $T \in \mathcal{T}_{k/2}$ with representation

$$T = \sum_{n=1}^{\infty} D_n T_n, \quad \sum_{n=1}^{\infty} \nu_n^k |D_n|^2 < \infty, \quad D_n := (T, T_n)_{L^2(B_R)}.$$

Obviously, for any component $T_n^{(i)}$, $i = 1, 2, 3$ of $T_n$ hold $T_n^{(i)} \in H^k(B_R)$ and the boundary condition (3.19), so $T_n^{(i)} \in \mathcal{W}_{k/2}$ and the equivalence (3.14) applies (component-wise) to $T_n$.  

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This argument holds for the finite sum \( S_N := \sum_{n=1}^{N} D_n T_n \), \( N \in \mathbb{N} \) as well, and we can calculate by (3.14) similarly to (3.20):

\[
\|S_N\|_{H^k(B_R)}^2 \sim \|(-\Delta)^{k/2} S_N\|_{L^2(B_R)}^2 = \left\| (-\Delta)^{k/2} \left( \sum_{n=1}^{N} \sigma_n^{-1/2} D_n \Lambda w_n \right) \right\|_{L^2(B_R)}^2 = \sum_{n=1}^{N} \nu_n |D_n|^2 \leq \|T\|_{k/2} < \infty.
\]

Therefore, \( \sum_{n=1}^{N} D_n T_n \in H^k(B_R) \) and hence \( T = \sum_{n=1}^{\infty} D_n T_n \in H^k(B_R) \); the boundary conditions \( \Delta^j : |S_R| = 0 \) in \( L^2(S_R) \) for \( j = 0, \ldots, [(k-1)/2] \) clearly hold for any \( S_N \) and hence for \( T \). This proves (3.17). Relation (3.18) is implied by (3.20).

\( \text{(II) In the poloidal case we prove first the cases } k = 1 \text{ and } k = 2, \text{ which exhibit already the main difficulties, before proving the general case.} \)

\( \text{(i) So, let } P \in \mathcal{P} \cap H^1(B_R) \text{ with harmonic extension } \tilde{P} \in H^1(\mathbb{R}^3) \text{ and representation (according to theorem 3.2) } P = \sum_{n=1}^{\infty} C_m P_m \text{ with } \sum_{n=1}^{\infty} |C_m|^2 = \sum_{n=1}^{\infty} |(\tilde{P}, \tilde{P}_m)|_{L^2(\mathbb{R}^3)}^2 < \infty. \)

With \( \tilde{P} \in H^1(\mathbb{R}^3) \) the restricted fields \( P \) and \( \tilde{P}|_{B_R} \) have well-defined traces on \( S_R \), which coincide, and we can compute

\[
(\nabla \times P, \nabla \times P_m)_{L^2(B_R)} = (P, \nabla \times \nabla \times P_m)_{L^2(B_R)} + \int_{S_R} P \times (\nabla \times P_m) \cdot \hat{x} \, ds = \mu_m (P, P_m)_{L^2(B_R)} - \mu_m^{1/2} \tau_m^{-1/2} \int_{S_R} P \times \Lambda v_m \cdot \hat{x} \, ds,
\]

and, similarly,

\[
\mu_m (\tilde{P}, \tilde{P}_m)_{L^2(B_R)} = -\mu_m^{1/2} \tau_m^{-1/2} (\tilde{P}, \nabla \times \Lambda \tilde{v}_m)_{L^2(B_R)} = -\mu_m^{1/2} \tau_m^{-1/2} (\tilde{P}, \Lambda \tilde{v}_m)_{L^2(B_R)} + \mu_m^{1/2} \tau_m^{-1/2} \int_{S_R} \tilde{P} \times \Lambda \tilde{v}_m \cdot (\hat{x}) \, ds = -\mu_m^{1/2} \tau_m^{-1/2} \int_{S_R} P \times \Lambda v_m \cdot \hat{x} \, ds.
\]

We have, thus, for every \( m \in \mathbb{N} \):

\[
(\nabla \times P, \nabla \times P_m)_{L^2(B_R)} = \mu_m (P, P_m)_{L^2(B_R)} + \mu_m (\tilde{P}, \tilde{P}_m)_{L^2(B_R)} = \mu_m (\tilde{P}, \tilde{P}_m)_{L^2(\mathbb{R}^3)} = \mu_m C_m.
\]

With (3.21) and the completeness of the set \( \{\tilde{P}_m : m \in \mathbb{N}\} \) follows for the orthogonal set \( \{\nabla \times P_m : m \in \mathbb{N}\} \) the property: \( (\nabla \times P, \nabla \times P_m)_{L^2(B_R)} = 0 \) for every \( m \in \mathbb{N} \) implies \( \nabla \times P = 0. \) Thus, \( \nabla \times P \) has a representation in the set \( \{\nabla \times P_m : m \in \mathbb{N}\}. \) By (3.7) we find

\[
\nabla \times P = \sum_{m=1}^{\infty} \frac{1}{\mu_m} (\nabla \times P, \nabla \times P_m)_{L^2(B_R)} \nabla \times P_m = \sum_{m=1}^{\infty} C_m \nabla \times P_m.
\]

Now, computing the \( \| \cdot \|_{1/2} \)-norm of \( P \) we find

\[
\| \nabla \times P \|^2_{L^2(B_R)} = \sum_{m,n=1}^{\infty} C_m C_n (\nabla \times P_m, \nabla \times P_n)_{L^2(B_R)} = \sum_{m=1}^{\infty} \mu_m |C_m|^2 = \|P\|^2_{1/2},
\]

hence \( P \in \mathcal{P}_{1/2}. \)

To prove the opposite inclusion we show for \( P \in \mathcal{P}_{1/2} \) that \( \tilde{P} \in H^1(\mathbb{R}^3) \) together with the identity

\[
\| \nabla \times P \|_{L^2(B_R)} = \| \nabla \tilde{P} \|_{L^2(\mathbb{R}^3)},
\]
which – in physical terms – equates the total dissipation of the field \( \tilde{P} \) with the Ohmic loss of the current \( \nabla \times P \) in \( B_R \). We start by proving (3.24) to hold for eigenfunctions \( \tilde{P}_m \). By (2.5), (2.2), and integration by parts we obtain

\[
\| \nabla \times P_m \|_{L^2(B_R)}^2 = \int_{B_R} (\nabla \times P_m) \cdot (\nabla \times P_m) \, dx
\]

\[
= \int_{B_R} (\nabla \times \nabla \cdot P_m) \cdot P_m - \int_{S_R} (\nabla \times P_m) \cdot \hat{x} \, ds
\]

\[
= \int_{B_R} (-\Delta P_m) \cdot P_m - \int_{S_R} (P_m \cdot \nabla P_m) \cdot \hat{x} \, ds
\]

\[
= \int_{B_R} \left( \sum_{i,j=1}^3 \partial_i P_m^{(j)} \partial_i P_m^{(j)} \right) - \int_{S_R} (P_m \cdot \nabla P_m) \cdot \hat{x} \, ds
\]

\[
= \| \nabla P_m \|_{L^2(B_R)}^2 - \int_{S_R} \left( P_m \cdot \nabla (P_m \cdot \hat{x}) - \sum_{i,j=1}^3 P_m^{(i)} P_m^{(j)} \partial_i \hat{x}_j \right) \, ds,
\]

where we used in the surface integral the identity \( (\nabla \times F) \times F = (F \cdot \nabla) F - \nabla (F \cdot F)/2 \). An analogous calculation in \( \tilde{B}_R \) yields

\[
0 = \| \nabla \times \tilde{P}_m \|_{L^2(\tilde{B}_R)}^2 = \| \nabla \tilde{P}_m \|_{L^2(\tilde{B}_R)}^2 + \int_{S_R} \left( \tilde{P}_m \cdot \nabla (\tilde{P}_m \cdot \hat{x}) - \sum_{i,j=1}^3 \tilde{P}_m^{(i)} \tilde{P}_m^{(j)} \partial_i \hat{x}_j \right) \, ds.
\]

Summing up while observing remark 3.3 yields (3.24) for \( \tilde{P}_m \). Now let \( P \in \mathcal{P}_{1/2} \) with representation \( P = \sum_{m=1}^\infty C_m P_m \). Applying (3.24) on the finite sum \( \tilde{S}_M = \sum_{m=1}^M C_m \tilde{P}_m \) and using (3.22) yields for every \( M \in \mathbb{N} \)

\[
\| \nabla \tilde{S}_M \|_{L^2(\mathbb{R}^3)}^2 = \| \nabla \times S_M \|_{L^2(B_R)}^2 = \sum_{m=1}^M \mu_m |C_m|^2 \leq \sum_{m=1}^\infty \mu_m |C_m|^2 < \infty,
\]

which implies \( (\nabla \tilde{S}_M)_M \in \mathbb{N} \) to converge in \( L^2(\mathbb{R}^3) \). Thus, we have \( \nabla \tilde{P} \in L^2(\mathbb{R}^3) \) together with (3.24). \( \tilde{P} \in L^2(\mathbb{R}^3) \) follows by (2.16) and the Poincaré-type inequality

\[
\| P \|_{L^2(B_R)} \leq C_P \| \nabla P \|_{L^2(B_R)}
\]

(3.25) with \( C_P = C_P(R) > 0 \), valid for zero-spherical-mean fields in the ball \( B_R \). So, we can conclude: \( \tilde{P} \in H^1(\mathbb{R}^3) \). This proves (3.15)\(_{k=1}\). The norm-equivalence (3.16)\(_{k=1}\) follows by (3.23), (3.24), (3.25), and (2.16)\(_2\).

(ii) In the case \( k = 2 \) let \( P \in \mathcal{P} \cap H^2(B_R) \) and \( \tilde{P} \in H^1(\mathbb{R}^3) \) with expansion \( \tilde{P} = \sum_{m=1}^\infty C_m \tilde{P}_m \). We show first

\[
\| P \|_1 = \| \nabla \times (\nabla \times P) \|_{L^2(\mathbb{R}^3)} = \| \tilde{\Delta} P \|_{L^2(\mathbb{R}^3)},
\]

(3.26) with \( \tilde{\Delta} P \) represented (according to lemma 2.9) by the curl of the scalar harmonic extension of the toroidal field \( -\nabla \times P \in H^1(B_R) \). Since \( \nabla \times P \in H^1(\mathbb{R}^3) \), no boundary terms arise in the subsequent computation:

\[
(\nabla \times (\nabla \times P), \tilde{P}_m)_{L^2(\mathbb{R}^3)} = (\nabla \times P, \nabla \times \tilde{P}_m)_{L^2(\mathbb{R}^3)} = (\nabla \times P, \nabla \times P_m)_{L^2(B_R)} = \mu_m (\tilde{P}, \tilde{P}_m)_{L^2(\mathbb{R}^3)} = \mu_m C_m.
\]

(3.27) In the second line we used (3.21). Inserting (3.27) into the expansion of \( \nabla \times (\nabla \times P) \in \tilde{P} \) yields

\[
\nabla \times (\nabla \times P) = \sum_{m=1}^\infty (\nabla \times (\nabla \times P), \tilde{P}_m)_{L^2(\mathbb{R}^3)} \tilde{P}_m = \sum_{m=1}^\infty \mu_m C_m \tilde{P}_m,
\]

(3.28)
and hence
\[
\|\nabla \times (\nabla \times \tilde{P})\|_{L^2(\mathbb{R}^3)}^2 = \sum_{m,n=1}^{\infty} \mu_m \mu_n C_m C_n (\tilde{P}_m, \tilde{P}_n)_{L^2(\mathbb{R}^3)} = \sum_{m=1}^{\infty} \mu_m^2 |C_m|^2 = \|\tilde{P}\|_1^2.
\]

Combining, finally, (3.26) with (2.18) yields
\[
\|P\|_1 = \|\nabla \times (\nabla \times P)\|_{L^2(\mathbb{R}^3)} \leq C \|\nabla \times P\|_{H^1(B_R)} \leq \tilde{C} \|P\|_{H^2(B_R)} < \infty,
\]
and, thus, \( P \in \mathcal{P}_1 \).

To prove the opposite inclusion let \( P \in \mathcal{P}_1 \) with representation \( P = \sum_{m=1}^{\infty} C_m P_m \), \( \sum_{m=1}^{\infty} \mu^2_m |C_m|^2 < \infty \). Let \( S_M := \sum_{m=1}^{M} c_m \Lambda \tilde{v}_m \) with \( c_m := \mu^{-1/2}_m t_m^{-1/2} C_m \) and let \( \tilde{S}_M = \sum_{m=1}^{M} c_m \Lambda \tilde{v}_m \) its scalar harmonic extension. By (3.11) any component of \( S_M \) is element of \( V_{k/2} \) for any \( k \in \mathbb{N} \), so the equivalence (3.12) applies (component-wise) to \( S_M \). By (3.12) \( k=3 \), (3.4), and (3.5) we then can compute
\[
\|S_M\|_{H^3(B_R)} \leq C \|\nabla (\Delta \tilde{S}_M)\|_{L^2(\mathbb{R}^3)} \leq C \left\| \nabla \left( \sum_{m=1}^{M} \mu_m C_m \Lambda \tilde{v}_m \right) \right\|_{L^2(\mathbb{R}^3)}
\]
\[
= C \sum_{m=1}^{M} \mu_m |C_m|^2 \leq C \|P\|_1 < \infty.
\]
Therefore, \((S_M)_{M\in\mathbb{N}}\) converges in \( H^3(B_R) \) and hence \((-\nabla \times S_M)_{M\in\mathbb{N}} = (\sum_{m=1}^{M} C_m P_m)_{M\in\mathbb{N}}\) in \( H^2(B_R) \), and we obtain, finally, \( P = \sum_{m=1}^{\infty} C_m P_m \in H^2(B_R) \) together with
\[
\|P\|_{H^2(B_R)} \leq \tilde{C} \|P\|_1.
\]

The matching condition \( \tilde{P} \in H^1(\mathbb{R}^3) \) follows immediately by \( P \in \mathcal{P}_1(B_R) \subset \mathcal{P}_{1/2}(B_R) \) and part (i) of the proof. This proves (3.15) \( k=2 \). Finally, the norm-equivalence (3.16) \( k=2 \) follows by (3.26), (3.29), and (3.30).

(iii) In the case \( k > 2 \) the first inclusion is proved by induction. Let
\[
P \in \{ P \in \mathcal{P} \subset H^k(B_R) : \Delta^i \tilde{P} \in H^1(\mathbb{R}^3) \text{ for } i = 0, \ldots, [(k-1)/2] \}.
\]

Setting \( Q := \Delta P \) we have by assumption
\[
Q \in \{ P \in \mathcal{P} \subset H^{k-2}(B_R) : \Delta^i \tilde{P} \in H^1(\mathbb{R}^3) \text{ for } i = 0, \ldots, [(k-3)/2] \} = \mathcal{P}_{k/2-1}.
\]

Computing the \( k/2 \)-norm of \( P = \sum_{m=1}^{\infty} (P, P_m)_{L^2(\mathbb{R}^3)} P_m \) we find by (3.27)
\[
\|P\|_{k/2}^2 = \sum_{m=1}^{\infty} \mu_m^{k-2} |(\tilde{P}_m, P_m)_{L^2(\mathbb{R}^3)}|^2 = \sum_{m=1}^{\infty} \mu_m^{k-2} |(\Delta \tilde{P}, \tilde{P}_m)_{L^2(\mathbb{R}^3)}|^2 = \|Q\|_{k/2-1} < \infty,
\]
and, thus, \( P \in \mathcal{P}_{k/2} \).

To prove the opposite inclusion let \( P \in \mathcal{P}_{k/2} \) with representation \( P = \sum_{m=1}^{\infty} C_m P_m \), \( \sum_{m=1}^{\infty} \mu^k_m |C_m|^2 < \infty \). As in the \( k=2 \)-case let \( S_M := \sum_{m=1}^{M} c_m \Lambda \tilde{v}_m \) with harmonic extension
\[ \tilde{S}_M = \sum_{m=1}^{M} c_m \alpha_{m} \] and with \( c_m := \mu_m^{1/2} \tau_m^{-1/2} C_m \). We have again (component-wise) \( S_M \in V_{k/2} \) for any \( k \in \mathbb{N} \) and, therefore, (3.12) at our disposal. Using, furthermore, (3.5) we can compute for odd \( k \):

\[
\|S_M\|_{H^{k+1}(B_R)}^2 \leq C \left\| \Delta^{(k+1)/2} \left( \sum_{m=1}^{M} c_m \alpha_{m} \right) \right\|_{L^2(B_R)}^2 = C \sum_{m=1}^{M} \mu_m^{k+1} \tau_m |c_m|^2
\]

and, additionally with (3.4), for even \( k \):

\[
\|S_M\|_{H^{k+1}(B_R)}^2 \leq C \left\| \nabla \left[ \Delta^{k/2} \left( \sum_{m=1}^{M} c_m \alpha_{m} \right) \right] \right\|_{L^2(\mathbb{R}^3)}^2 = C \sum_{m=1}^{M} \mu_m^{k+1} \tau_m |c_m|^2 \leq C \|P\|_{k/2}^2 < \infty.
\]

This yields as in the \( k=2 \)-case \( P \in H^k(B_R) \) together with

\[
\|P\|_{H^k(B_R)} \leq \tilde{C} \|P\|_{k/2}.
\]

Finally, the matching conditions \( \tilde{\Delta} P \in H^1(\mathbb{R}^3) \) for \( i = 0, \ldots, [(k-1)/2] \) follow as in the \( k=1 \)-case by (3.24) applied on \( \Delta^i P \) and (3.33) below. This proves (3.15).

As to the equivalences (3.16) we find by iterating (3.27) for \( P \in \mathcal{P}_{k/2} \) with even \( k \):

\[
\|P\|_{k/2}^2 = \sum_{m=1}^{\infty} \left| \mu_m \left( \tilde{\Delta}^{k/2} P, \tilde{P}_m \right)_{L^2(\mathbb{R}^3)} \right|^2 = \sum_{m=1}^{\infty} \left\| \left( \tilde{\Delta}^{k/2} P, \tilde{P}_m \right)_{L^2(\mathbb{R}^3)} \right|^2 = \left\| \tilde{\Delta}^{k/2} P \right\|_{L^2(\mathbb{R}^3)}^2
\]

and, additionally with (3.21) and (3.23), for odd \( k \):

\[
\|P\|_{k/2}^2 = \sum_{m=1}^{\infty} \left| \mu_m \left( \tilde{\Delta}^{(k-1)/2} P, \tilde{P}_m \right)_{L^2(\mathbb{R}^3)} \right|^2
\]

and, additionally with (3.21) and (3.23), for odd \( k \):

\[
\|P\|_{k/2}^2 = \sum_{m=1}^{\infty} \left| \mu_m \left( \tilde{\Delta}^{(k-1)/2} P, \tilde{P}_m \right)_{L^2(\mathbb{R}^3)} \right|^2
\]

Together with (3.32) this proves (3.16).

A common description of the higher regularity classes of poloidal and toroidal fields can be given by means of the eigenfunction system \( \{ (\tilde{B}_l, \lambda_l) : l \in \mathbb{N} \} \), characterized in theorem 3.2. Defining spaces, norms, and operators for \( \alpha \in \mathbb{R} \) by

\[
\mathcal{B}_\alpha := \left\{ B = \sum_{l=1}^{\infty} E_l B_l : \|B\|_\alpha < \infty \right\},
\]

\[
\|B\|_\alpha := \left( \sum_{l=1}^{\infty} \lambda_l^{2\alpha} |E_l|^2 \right)^{1/2} \quad \text{for} \quad B = \sum_{l=1}^{\infty} E_l B_l,
\]

and

\[
\mathcal{A}\alpha B = \sum_{l=1}^{\infty} \lambda_l^\alpha E_l B_l \quad \text{for} \quad B = \sum_{l=1}^{\infty} E_l B_l,
\]

respectively, the following corollary is an immediate consequence of lemma 2.2 and theorem 3.5.

**Corollary 3.6** For \( k \in \mathbb{N}_0 \) holds

\[
\mathcal{B}_{k/2} = \mathcal{P}_{k/2} \oplus \mathcal{T}_{k/2}
\]

and on \( \mathcal{B}_{k/2} \) we have the equivalence of norms

\[
\|B\|_{k/2} \sim \|B\|_{H^k(B_R)}.
\]
4 The evolution problem

We solve in this section the evolution problem (1.1) by means of the spaces \( \mathcal{B}_k \) provided in the last section. Elements of \( \mathcal{B}_{k/2} \) exhibit for \( k \geq 1 \) the correct behaviour outside \( B_R \), viz. they match to a unique harmonic extension and this matching will turn out to be continuous if \( k \) is large enough. So, we put (1.1) into the functional framework

\[
\dot{H} = -\eta A H + C H + F, \tag{4.1a}
\]

\[
H(0) = H_0 \tag{4.1b}
\]

with \( H \) denoting a mapping \([0, T) \to \mathcal{B}_{1/2} (T > 0)\), \( A = A^k \) denoting the operator (3.34), and \( C \) the lower-order operator according to (1.2):

\[
C H := -(\nabla \eta \nabla)^T H + (\nabla v)^T H - (v - \nabla \eta) \cdot \nabla H - \nabla \cdot v H.
\]

The inhomogeneity \( F \) has no counterpart in (1.1), the solution procedure, however, requires the inclusion of such a term in (4.1a). It is one of the advantages of the abstract formulation (4.1) of the dynamo problem that for its solution by a Galerkin procedure the details of the equation, of the lower-order operator \( C \), or of the spaces \( \mathcal{B}_k \) do not matter. The solution of (4.1) as demonstrated in [8] is based on the existence of a \( L^2 \)-complete set of eigenfunctions, the regularity property (3.35) of the spaces \( \mathcal{B}_{k/2} \), and the boundedness of the operator \( C : C([0, T], \mathcal{B}_{1/2}) \to C([0, T], L^2(\mathcal{B}_R)) \), which depends on the regularity of \( \eta \) and \( v \). So, those results of [8, 9] which are formulated in terms of \( \mathcal{B}_{k/2} \) directly carry over to the present situation, whereas in formulating the classical results the difference between the scalar-valued problem considered in [8] and the vector-valued problem considered here lead to minor modifications.

The definition of a weak solution of (1.1) is as in [8]: With \( T > 0 \) and \( H_0 \in \mathcal{B}_{1/2} \) a function \( H \in L^2((0, T), \mathcal{B}_1) \) with weak time derivative \( \dot{H} \in L^2((0, T), L^2(\mathcal{B}_R)) \) satisfying (4.1a) as equality in \( L^2((0, T), L^2(\mathcal{B}_R)) \) and (4.1b) as equality in \( \mathcal{B}_{1/2} \) is called a weak solution of problem (4.1). In fact, a weak solution takes continuously its initial value since by interpolation \( H \in L^2((0, T), H^2(\mathcal{B}_R)) \) and \( \dot{H} \in L^2((0, T), L^2(\mathcal{B}_R)) \) imply \( H \in C([0, T], H^2(\mathcal{B}_R)) \). We summarize the results of theorem 4.3 in [8] and theorem 4 in [9] as follows:

**Theorem 4.1 (Weak solution and higher regularity)** Let \( T > 0 \) and \( \eta \geq \text{const} > 0 \).

(i) Let \( H_0 \in \mathcal{B}_{1/2} \), \( \eta \) and \( v \in C^1(\overline{\mathcal{B}_R} \times [0, T]) \), and \( F \in C([0, T], L^2(\mathcal{B}_R)) \). Then problem (4.1) has a unique weak solution \( H \).

(ii) If for some \( k > 1 \), \( H_0 \in \mathcal{B}_{(k+1)/2} \), \(-\eta \cdot \partial_x A H_0 + C_{\mid t=0} H_0 + F(0) \in \mathcal{B}_{(k-1)/2} \), \( \eta \) and \( v \in C^{k+1}(\overline{\mathcal{B}_R} \times [0, T]) \), and \( F \in C^1([0, T], H^k(\mathcal{B}_R)) \), then for the weak solution of (4.1) holds:

\[
\begin{align*}
H \in L^2((0, T), \mathcal{B}_{k+1/2}) \quad \dot{H} \in L^2((0, T), \mathcal{B}_{k/2}) \quad \ddot{H} \in L^2((0, T), \mathcal{B}_{k-1/2}).
\end{align*}
\]

Recall that we are interested in classical solutions of problem (1.1) with conditions formulated in terms of classical derivatives. Of course, theorem 4.1 does not work with the weakest possible assumptions but it is enough for our purposes. So, setting \( B(x, t) := [\tilde{H}(t)](x) \), where \( \tilde{H} \) denotes the harmonic extension of \( H(t) \in \mathcal{B}_{1/2} \), classical solutions of (1.1) are obtained from theorem 4.1 with \( k = 3 \) by interpolation and by Sobolev's embedding theorems; for the opposite direction note that \( H \in C^1(\mathcal{B}_R) \) and \( \tilde{H} \in C(\mathbb{R}^3) \) imply \( \dot{H} \in H^1(\mathbb{R}^3) \). The following theorem is analogous to corollary 5 and remark 6 in [9]: the proof follows closely that of corollary 4.6 in [8]; for proving the matching condition observe that any component of a harmonic vector field is a harmonic function to which the maximum principle applies.
Theorem 4.2 (Classical solution of the evolution problem) Let $B_0 \in C^4(B_R)$ with $\nabla \cdot B_0 = 0$ and $\eta, v \in C^4_1(B_R \times [0, T])$ for any $T > 0$. Let, furthermore, $B_0$, $\Delta B_0$, and $\eta(\cdot, 0)\Delta B_0 + C|_{t=0} B_0$ all match continuously to their harmonic extensions. Then problem (1.1) has a unique classical solution $B$, i.e. $B \in C^2_t(B_R \times \mathbb{R}_+) \cap C^1_\hat{\mathbb{R}}$ satisfies the equations (1.1) pointwise.

Remark 4.3 The smoothness conditions on the coefficients and on the initial value are supposedly not optimal. Likewise, the compatibility conditions are supposedly not required by the solution but by the method of proof. Admissible initial values which satisfy these conditions are for instance solenoidal fields $B_0 \in C^4(B_R)$ with $B_0 = \nabla B_0 = \Delta B_0 = 0$ at $S_R$.

References


