Well-posedness of some initial-boundary-value problems for
dynamo-generated poloidal magnetic fields
[corrected version]

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Abstract

Given a bounded domain \(G \subset \mathbb{R}^d, d \geq 3\), we study smooth solutions of a linear
parabolic equation with non-constant coefficients in \(G\), which at the boundary have
to \(C^1\)-match with some harmonic function in \(\mathbb{R}^d \setminus \overline{G}\) vanishing at spatial infinity.

This problem arises in the framework of magnetohydrodynamics if certain
dynamo-generated magnetic fields are considered: For example, in the case of
axisymmetry or for non-radial flow fields, the poloidal scalar of the magnetic field
solves the above problem.

We first investigate the Poisson problem in \(G\) with the above described bound-
dary condition as well as the associated eigenvalue problem and prove the existence
of smooth solutions. As a by-product we obtain the completeness of the well-
known poloidal “free decay modes” in \(\mathbb{R}^3\) if \(G\) is a ball. Smooth solutions of the
evolution problem are then obtained by Galerkin approximation based on these
eigenfunctions.

Key words: magnetohydrodynamics, dynamo theory, poloidal field, harmonic field.

1 Introduction

We are concerned in this paper with the following initial-boundary-value problem:

\[
\begin{align*}
\partial_t u - a \Delta u &= b \cdot \nabla u + cu \quad \text{in } G \times \mathbb{R}_+, \quad (1.1a) \\
\Delta u &= 0 \quad \text{in } \hat{G} \times \mathbb{R}_+, \quad (1.1b) \\
u \text{ and } \nabla u \text{ continuous} &\quad \text{in } \mathbb{R}^d \times \mathbb{R}_+, \quad (1.1c) \\
u(x, t) &\to u_\infty(t) \quad \text{for } |x| \to \infty, t \in \mathbb{R}_+, \quad (1.1d) \\
u(\cdot, 0) &= u_0 \quad \text{on } G \times \{t = 0\}. \quad (1.1e)
\end{align*}
\]

Here, \(G \subset \mathbb{R}^d, d \geq 3\) is a bounded domain with (sufficiently) smooth boundary \(\partial G\), and
\(\hat{G} := \mathbb{R}^d \setminus \overline{G}\). The scalar-valued coefficients \(a\) and \(c\), and the vector-valued coefficient \(b\)
are sufficiently smooth functions of \(x \in G\) and \(t \in \mathbb{R}_+\); \(a\) is, moreover, bounded from
below by $a_0 > 0$. The asymptotic behaviour of solutions at spatial infinity is described by the (given) function $u_\infty : \mathbb{R}_+ \to \mathbb{R}$, and the initial-value $u_0$ is prescribed on $G$ only.

Problem (1.1) arises in the context of magnetohydrodynamic dynamo theory: the generation of a magnetic field $B$ by motion of a liquid conductor (of magnetic diffusivity $\eta > 0$) according to some prescribed flow field $v$ is described by the induction equation (cf. [13])

$$\partial_t B = \nabla \times (v \times B) - \nabla \times (\eta \nabla \times B), \quad \nabla \cdot B = 0. \tag{1.2}$$

Equation (1.2) constitutes a system of parabolic equations for the magnetic field components. In general, the flow field couples these components in a nontrivial way which makes the question for “dynamo solutions”, i.e. solutions which do not decay in time, difficult to answer. Only in special situations does a field component or a related scalar quantity decouple, and a general decay result, a so-called antidynamo theorem, may be obtained. For instance, if the conductor fills a ball $B_R \subset \mathbb{R}^3$ with radius $R$, if the conductivity is radially symmetric, and if the flow field has no radial component the quantity $P := B \cdot x$ satisfies the scalar problem (cf. appendix A or [11]):

$$\partial_t P - \eta \Delta P = -\nabla \cdot (vP) \quad \text{in } B_R \times \mathbb{R}_+, \tag{1.3a}$$

$$\Delta P = 0 \quad \text{in } \mathbb{R}^3 \times \mathbb{R}_+, \tag{1.3b}$$

$$P \text{ and } \nabla P \text{ continuous} \quad \text{in } \mathbb{R}^3 \times \mathbb{R}_+, \tag{1.3c}$$

$$P(x, t) = O(|x|^{-2}) \quad \text{for } |x| \to \infty, \ t \in \mathbb{R}_+, \tag{1.3d}$$

$$P(\cdot, 0) = P_0, \quad \langle P_0 \rangle = 0 \quad \text{on } B_R \times \{t = 0\}. \tag{1.3e}$$

Note that conditions (1.1b) and (1.1d) with $u_\infty \equiv 0$ imply the spatial decay condition $P(x, t) = O(|x|^{-1})$ (see Appendix C). The stronger condition (1.3d) is a consequence of the additional zero-spherical-mean condition

$$\langle P_0 \rangle := \frac{1}{4\pi r^2} \int_{S_r} P_0 \, ds = 0, \quad r \in (0, R),$$

on the initial value. This condition is preserved by eqs. (1.3a,b) and holds, consequently, for $P$ on $\mathbb{R}^3 \times \mathbb{R}_+$. Equation (1.3b) describes a vacuum field outside the conductor and condition (1.3c) guarantees a continuous magnetic field throughout space.

Another instance is the axisymmetric dynamo problem in ordinary space $\mathbb{R}^3$. Again, a scalar quantity describing the poloidal part of the magnetic field decouples and, if reformulated in $\mathbb{R}^5$, is precisely a solution of problem (1.1) (see appendix B). Note that the conductor is here assumed to be axisymmetric but need not be a ball. It is this application which motivates the investigation of problem (1.1) in more than 3 dimensions and in domains more general than balls.

The focus of dynamo theory is less on existence theorems than on decay results for the magnetic field under certain restrictions on the magnetic field and/or the flow field, thus excluding dynamo action under these restrictions. However, proving decay results requires sometimes the solution of an auxiliary problem. For instance, in proving a “non-radial velocity theorem” for solutions of problem (1.3) one needs positive solutions
of an auxiliary problem of type (1.1); and it is this application which requires a non-zero asymptotic condition like (1.1d) (cf. [11]). Similarly in the axisymmetric problem, Backus makes use of solutions of an auxiliary problem (cf. [3]). He made the existence of such solutions plausible by physical arguments but could not establish them rigorously. It is the aim of the present paper to prove rigorously the existence of smooth solutions of problem (1.1).

A problem closely related to (1.1) has been treated in [16]. It is inspired by the dynamo problem with plane symmetry (which means \(d = 2\)) and differs from (1.1) in that \(c = 0\) and by a different asymptotic condition at spatial infinity. The authors treat this problem and two related ones in arbitrary dimensions and prove existence of solutions and, moreover, some decay results. However, this problem does not precisely meet the requirements of the non-radial problem (1.3) nor those of the axisymmetric problem. Moreover, only weak solutions are established, whose behaviour at the boundary remains open.

The basic idea of our treatment is to consider (1.1) as parabolic problem in a bounded domain with non-local boundary condition, and to carry over the well-established methods for linear parabolic equations with standard boundary conditions such as Dirichlet’s or Neumann’s boundary condition to our situation (cf. e.g. [7]). In section 2 we solve weakly a Poisson-type problem related to (1.1) in all space. The regularity of the weak solution follows with standard arguments in \(G\) and \(\hat{G}\), only at \(\partial G\) we need special considerations. In section 3 we treat the corresponding eigenvalue problem, introduce the associated Fourier series, and characterize the elements of various function spaces which are useful in the remainder of the paper by the behaviour of their Fourier coefficients. In section 4 problem (1.1) is solved by a Galerkin procedure and, for sufficiently smooth data and suitable compatibility conditions, the smoothness of the obtained solution is established. In two appendices the relation between problem (1.1) and the non-radial-flow as well as the axisymmetric problem is elucidated. A third appendix collects some facts about exterior harmonic functions, and clarifies the relation between different kinds of spatial decay conditions – a topic about which there has been some debate in the literature. Finally, in a fourth appendix the completeness of the so-called poloidal free decay modes is proved: a commonly believed property which to our knowledge, however, has never been proved.

### 2 A Poisson problem

We establish in this section smooth solutions of the following Poisson problem in all space with suitable right-hand side \(f\):

\[-\Delta u = f \quad \text{in } G, \quad (2.1a)\]

\[\Delta u = 0 \quad \text{in } \hat{G}, \quad (2.1b)\]

\[u \text{ and } \nabla u \text{ continuous} \quad \text{in } \mathbb{R}^d, \quad (2.1c)\]

\[u(x) \to 0 \quad \text{for } |x| \to \infty. \quad (2.1d)\]
To obtain a weak formulation of problem (2.1) let us multiply (2.1a) and (2.1b) with a test function \( v \in C^\infty_0(\mathbb{R}^d) \) and integrate over \( G \) and \( B_R \setminus \overline{G}, \overline{G} \subset B_R \), respectively. Assuming \( \partial G \in C^1 \) one obtains after integration by parts:

\[
\int_G (\nabla u \cdot \nabla v - fv) \, dx - \int_{\partial G} n \cdot \nabla u v \, ds = 0, \tag{2.2a}
\]

\[
\int_{B_R \setminus \overline{G}} \nabla u \cdot \nabla v \, dx + \int_{\partial G} n \cdot \nabla u v \, ds - \int_{S_R} \frac{x}{|x|} \cdot \nabla u v \, ds = 0. \tag{2.2b}
\]

Here \( S_R \) denotes a sphere with radius \( R \) and \( n \) is the exterior unit normal at \( \partial G \).

Adding up (2.2a) and (2.2b), one finds in the limit \( R \to \infty \) that

\[
\int_{\mathbb{R}^d} \nabla u \cdot \nabla v \, dx = \int_G f v \, dx. \tag{2.3}
\]

In view of (2.3) it is reasonable to consider functions satisfying the “finite energy condition” (cf. remark A.1)

\[
\int_{\mathbb{R}^d} |\nabla v|^2 \, dx =: \|v\|_{H}^2 < \infty. \tag{2.4}
\]

Condition (2.4) together with condition (2.1d) motivate the definition of the real Hilbert space

\[
\mathcal{H}_0 := \text{clos}\{C^\infty_0(\mathbb{R}^d), \| \cdot \|_{\mathcal{H}}\}
\]

with scalar product \((v, w)_{\mathcal{H}} := \int_{\mathbb{R}^d} \nabla v \cdot \nabla w \, dx\). Observe that \( v \in \mathcal{H}_0 \) is locally square-integrable. In fact, the Gagliardo-Nirenberg-Sobolev-inequality (cf. [7, p263]) implies

\[
\|v\|_{L^p(\mathbb{R}^d)} \leq C\|v\|_{\mathcal{H}}, \quad p = \frac{2d}{d-2} \tag{2.5}
\]

for any \( v \in \mathcal{H}_0 \) and a constant \( C \) depending only on \( d \). Combining (2.5) with Hölder’s inequality yields for any \( K \in \mathbb{R}^d \):

\[
\|v\|_{L^2(K)} \leq |K|^{1/d}\|v\|_{L^p(K)} \leq C|K|^{1/d}\|v\|_{\mathcal{H}}. \tag{2.6}
\]

So, defining (for later use)

\[
\mathcal{H} := \{ v \in H^1_{loc}(\mathbb{R}^d) : \|v\|_{\mathcal{H}} < \infty \},
\]

there holds clearly \( \mathcal{H}_0 \subset \mathcal{H} \). Inequality (2.6) implies, in particular,

\[
\|v\|_{L^2(G)} \leq C_G\|v\|_{\mathcal{H}}, \quad v \in \mathcal{H}_0 \tag{2.7}
\]

with \( C_G := C|G|^{1/d} \); thus, \( v \in \mathcal{H}_0 \) yields in eq. (2.3) with \( f \in L^2(G) \) a finite right-hand side as well.

A function \( u \in \mathcal{H}_0 \) is now called a weak solution of problem (2.1) with \( f \in L^2(G) \) iff (2.3) holds for all \( v \in \mathcal{H}_0 \). Rewriting (2.3) in the form

\[
(u, v)_{\mathcal{H}} = (f, v)_{L^2(G)} \quad \text{for any} \ v \in \mathcal{H}_0 \tag{2.8}
\]
and noting that $(f, \cdot)_{L^2(G)}$ defines a bounded linear functional on $H_0$ (due to (2.7)) the existence of a unique weak solution follows immediately from the Riesz representation theorem.

Concerning regularity of the weak solution let us define $\mathcal{F} : \mathbb{R}^d \to \mathbb{R}$ by $\mathcal{F} = f$ on $G$ and $\mathcal{F} = 0$ on $\hat{G}$. It is a standard result about interior regularity (cf. e.g. [7, p309f]) that $\mathcal{F} \in H^k(\mathbb{R}^d)$ implies $u \in H^k_{loc}(\mathbb{R}^d)$, $k \in \mathbb{N}_0$. This result means in particular:

$$
u \in H^2_{loc}(\mathbb{R}^d), \quad (2.9a)$$
$$f \in H^k(G) \Rightarrow \nu \in H^k_{loc}(G), \quad k \in \mathbb{N}_0, \quad (2.9b)$$
$$\nu \in C^\infty(\hat{G}). \quad (2.9c)$$

With the usual Sobolev embeddings (2.9b) implies $u \in C^2(G)$ if $f \in H^k(G)$, $k > d/2$. So, choosing suitable test functions in (2.3) we find $u$ satisfying (2.1b) in $\hat{G}$ and, for sufficiently regular $f$, (2.1a) in $G$. At $\partial G$, however, we need some finer considerations.

As usual we flatten $\partial G$ locally by means of a diffeomorphism $\Phi : U \to W$, $x \mapsto y$ with $\Phi(x_0) = 0$, $x_0 \in U \cap \partial G$, $\det D\Phi = 1$ (cf. e.g. [7, p626f]). Setting $v(y) := u(\Phi^{-1}(y))$ and $g(y) := f(\Phi^{-1}(y))$, $\Phi$ transforms eqs. (2.1) localized on $U$ into

$$
-Lv = g \quad \text{in } W^-, \quad (2.10a)
$$
$$Lv = 0 \quad \text{in } W^+, \quad (2.10b)
$$
$$v \text{ and } \nabla v \text{ continuous} \quad \text{in } W. \quad (2.10c)
$$

Here,

$$L(\cdot) = \sum_{i,j=1}^d \partial_y(a_{ij} \partial_y \cdot)$$

is a uniformly elliptic operator with coefficients determined by $\Phi$, $W^- := W \cap \{ y_d < 0 \} = \Phi(U \cap G)$, $W^+ := W \cap \{ y_d > 0 \} = \Phi(U \cap \hat{G})$, and $W^0 := W \cap \{ y_d = 0 \} = \Phi(U \cap \partial G)$. A function $v \in H^1(W)$ is called a weak solution of problem (2.10) with $g \in L^2(W^-)$ iff

$$
\sum_{i,j=1}^d \int_W a_{ij} \partial_y v \partial_y w \, dy = \int_{W^-} g w \, dy
$$

for any $w \in H^1_0(W)$. Defining “non-isotropic” Sobolev spaces $H^k_{tan}(W)$, $k \in \mathbb{N}_0$ by

$$H^k_{tan}(W) := \{ v : W \to \mathbb{R} \mid D^\alpha v \in L^2(W) \text{ for any multiindex } \alpha \text{ with } |\alpha| \leq k, \alpha_d = 0 \}$$

with norm

$$\|v\|_{H^k_{tan}(W)} := \sum_{|\alpha| \leq k} \|D^\alpha v\|_{L^2(W)}^2,$$

the regularity of $v$ over $W^0$ is characterized by

**Lemma 2.1** Let $g \in H^k_{tan}(W^-)$ with $k > (d + 1)/2$, let $v$ be a weak solution of (2.10), and let $r$ be so small that the cylinder $V := \{ y = (y', y_d) \in \mathbb{R}^d : |y'| < r, |y_d| < 2r \} \subseteq W$. Then $v \in C^1(V_{1/2})$ with $V_{1/2} := V \cap \{ |y_d| < r \}$ and we have the estimate

$$
\|v\|_{C^1(V_{1/2})} \leq C(\|g\|_{H^k_{tan}(W^-)} + \|v\|_{L^2(W)}) \quad (2.11)
$$

5
with $C$ depending on $W$, $V$, $k$, and $d$.

**Proof:** We prove first the estimate

$$
\|v\|_{C^1(V_{1/2})}^2 \leq C \sum_{i=0,1,2} \sum_{|\alpha| \leq k} \|\partial^i_{y_d} D^\alpha v\|_{L^2(V)}^2
$$

(2.12)

for functions $v \in C^{k+2}(\overline{V})$ by applying Sobolev embeddings separately for $y'$ and $y_d$. Let us start with a 1-dimensional estimate on the interval $(-r, r)$. For functions $f \in C^1((-2r, 2r))$ we have

$$
|f(y)| \leq \frac{1}{2r} \int_{y-r}^{y+r} |f(y) - f(x)| \, dx + \frac{1}{2r} \int_{y-r}^{y+r} |f(x)| \, dx
$$

\[ \leq \int_{-2r}^{2r} |f'(x)| \, dx + \frac{1}{2r} \int_{-2r}^{2r} |f(x)| \, dx. \]

This implies that

$$
\max_{|y| \leq r} |f|^2 \leq C \left( \int_{-2r}^{2r} |f'(x)|^2 \, dx + \int_{-2r}^{2r} |f(x)|^2 \, dx \right)
$$

(2.13)

for some constant $C = C(r)$. On the other hand, Sobolev embeddings imply for $k > (d + 1)/2$

$$
\|f\|_{C^1(\overline{B}_r')}^2 \leq C \|f\|_{H^k(B_r')}^2 = C \sum_{|\alpha| \leq k} \int_{B_r^d} |D^\alpha f(y')|^2 \, dy'
$$

(2.14)

for some constant $C = C(k, d, r)$ and $B_r' := \{ y' \in \mathbb{R}^{d-1} : |y'| < r \}$. Combining (2.13) and (2.14) we obtain

$$
\max_{|y_d| \leq r} \|v(\cdot, y_d)\|_{C^1(B_r')}^2 \leq C \sum_{i=0,1} \sum_{|\alpha| \leq k} \int_{B_r^d} \int_{B_r^d} |\partial^i_{y_d} D^\alpha v(y', y_d)|^2 \, dy' \, dy_d
$$

\[ = C \sum_{i=0,1} \sum_{|\alpha| \leq k} \|\partial^i_{y_d} D^\alpha v\|_{L^2(V)}^2, \]

(2.15)

and analogously

$$
\max_{|y_d| \leq r} \|\partial^i_{y_d} v(\cdot, y_d)\|_{C^0(\overline{B}_r')}^2 = C \sum_{i=1,2} \sum_{|\alpha| \leq k} \|\partial^i_{y_d} D^\alpha v\|_{L^2(V)}^2.
$$

This is (2.12). Now let $v \in L^2(V)$ with bounded right-hand side in (2.12). It follows with standard arguments that $v$ has a representative in $C^1(\overline{V}_{1/2})$.

Next, let $\hat{g}$ be the trivial extension of $g$ on $W$. $g \in H^k_{tan}(W^-)$ implies then $\hat{g} \in H^k_{tan}(W)$, i.e. $D^\alpha \hat{g} \in L^2(W)$ for $|\alpha| \leq k$, $\alpha_d = 0$. From interior regularity for weak
solutions of (2.10) follows \( D^\beta v \in L^2(V) \) for \(|\beta| \leq k + 2, \beta_d \leq 2 \) and the estimate
\[
\sum_{|\beta| \leq k+2, \beta_d \leq 2} \int_V |D^\beta v|^2 \, dy \leq C \left( \sum_{|\alpha| \leq k} \int_W |D^\alpha \tilde{g}|^2 \, dy + \|v\|_{L^2(W)}^2 \right) \tag{2.16}
\]
with \( C \) depending on \( W, V, k, \) and \( d \). Together with (2.12) this proves (2.11). \( \square \)

**Remark 2.2** Observe for later use that lemma 2.1 applies in particular to the case \( g := \lambda v|_{W^-} \) with \( \lambda \in \mathbb{R} \). In fact, given any \( k \in \mathbb{N} \) we find \( v|_{W^-} \in R_{\tan}^{2k}(\tilde{W}^-) \) for some \( V \subset \tilde{W} \subset W \) just by iterating the interior regularity argument. So, lemma 2.1 applies to this eigenvalue problem as well and yields \( C^1 \)-smoothness of solutions over the boundary.

Concerning the original problem (2.1), lemma 2.1 implies for \( \partial G \in C^{k+2} \) (i.e. \( \Phi \in C^{k+2} \)) and \( f \in H^k(G), k > (d+1)/2 \) that \( u \in C^1 \) over \( \partial G \). Collecting the foregoing results we have

**Theorem 2.3 (Solution of the Poisson problem)** Let \( G \subset \mathbb{R}^d, d \geq 3 \) be a bounded domain with boundary \( \partial G \) and \( f \in L^2(G) \). The Poisson problem (2.1) has then a unique weak (in the sense of eq. (2.3)) solution \( u \in H^1_{\text{loc}}(\mathbb{R}^d) \). Moreover, if \( \partial G \in C^{k+2} \) and \( f \in H^k(G) \) with \( k > (d+1)/2 \), then \( u \) is a classical solution, i.e. \( u \in C^1(\mathbb{R}^d) \cap C^2(G \cup \tilde{G}) \), satisfying pointwise eqs. (2.1).

**Remark 2.4** The condition \( d \geq 3 \) is crucial for theorem 2.3. In lower dimensions problem (2.1) is overdetermined; to obtain nontrivial solutions one has to relax condition (2.1c) or (2.1d). As an example in \( \mathbb{R} \) consider the problem \(-u'' = 1 \) in \( G := (-1, 1), u'' = 0 \) in \( \tilde{G} \), and \( \lim_{|x| \to \infty} u(x) = 0 \). The only continuous solution is \( u(x) = \frac{1}{2} - \frac{1}{4}x^2 \) in \( G \) and \( u \equiv 0 \) in \( \tilde{G} \), which is not \( C^1 \) over \( \partial G \). Similarly, in \( \mathbb{R}^2 \), the only continuous solution of \(-\Delta u = 1 \) in \( G := \{ x \in \mathbb{R}^2 : |x| < 1 \}, \Delta u = 0 \) in \( \tilde{G} \), \( \lim_{|x| \to \infty} u(x) = 0 \) is \( u(x) = \frac{1}{2} - \frac{1}{4}|x|^2 \) in \( G \) and \( u \equiv 0 \) in \( \tilde{G} \).

On the other side, \( C^1 \)-solutions of these problems exhibit linear (in \( \mathbb{R} \)) or logarithmic (in \( \mathbb{R}^2 \)) growth at infinity. \( d = 3 \) is the lowest dimension, in which \( C^1 \)-solutions vanish at infinity. In fact, the analogous problem in \( \mathbb{R}^3 \) has the \( C^1 \)-solution \( u(x) = \frac{1}{2} - \frac{1}{6}|x|^2 \) in \( G \) and \( u(x) = \frac{1}{3}|x|^{-1} \) in \( \tilde{G} \).

**Remark 2.5** Concerning the asymptotic behaviour of \( u \) for large \( x \) we refer to well-known facts about exterior harmonic functions (cf. appendix C). In particular, condition (2.1d) implies the representation (C.1), which yields the asymptotics
\[
u = Y_0 |x|^{2-d} + O(|x|^{1-d}), \quad Y_0 \in \mathbb{R}, \quad \text{for } |x| \to \infty. \tag{2.17}
\]
The decay is faster (viz. \( O(|x|^{1-d}) \)), if \( f \) has vanishing mean over \( G \). In fact, one obtains from (2.1), (2.17), and Gauss’s theorem:
\[
\int_G \Delta u \, dx = \int_{\partial G} n \cdot \nabla u \, ds = \int_{S_R} \frac{x}{|x|} \cdot \nabla u \, ds = (d-2)|S_1|Y_0. \tag{2.18}
\]
Here, $\mathcal{J} := \frac{1}{|G|} \int_G f \, dx$ and $S_R \subset \hat{G}$; the last equality arises in the limit $R \to \infty$. So, $\mathcal{J} = 0$ implies $Y_0 = 0$.

Let us note as an aside that in general $\pi = 0$ can only be achieved at the expense of relaxing (2.1d) to $u(x) = O(1)$ for $|x| \to \infty$. In this case the function $u - \pi$ with $u$ being a solution of (2.1) is obviously a zero-mean solution (see also remark 2.6).

**Remark 2.6** If $G$ is a ball $B_R$ it makes sense to consider spherically symmetric solutions of problem (2.1), i.e. $\langle u \rangle (r) = 0$ for any $r > 0$. Obviously, $\langle u \rangle = 0$ implies $\langle f \rangle = 0$; on the other hand, any solution of (2.1) with spherically symmetric $f$ is spherically symmetric. This follows from the unique solvability of the sub-problem for the spherical mean arising from (2.1):\(^1\)

\[
-\left(\langle u \rangle'' + \frac{(d-1)}{r} \langle u \rangle'\right) = \langle f \rangle, \quad 0 \leq r \leq R,
\]
\[
\langle u \rangle(R) = Y_0/R^{d-2}, \quad \langle u \rangle'(R) = (2-d)Y_0/R^{d-1}.
\]

Observe here the representation (C.1), which implies $\langle u \rangle = Y_0 r^{2-d}$ for $r > R$, and (2.18), which implies $Y_0 = 0$ if $\langle f \rangle = 0$.

Note, finally, that spherically symmetric solutions decay at least like $|x|^{1-d}$ for large $x$.

### 3 The Eigenvalue problem

We treat in this section the eigenvalue problem corresponding to the Poisson problem (2.1) of the previous section:

\[
\begin{align*}
-\Delta u &= \lambda u \quad \text{in } G, \quad \text{(3.1a)} \\
\Delta u &= 0 \quad \text{in } \hat{G}, \quad \text{(3.1b)} \\
u \text{ and } \nabla u \text{ continuous} &\quad \text{in } \mathbb{R}^d, \quad \text{(3.1c)} \\
u(x) &\to 0 \quad \text{for } |x| \to \infty. \quad \text{(3.1d)}
\end{align*}
\]

According to theorem 2.3 we have for any $f \in L^2(G)$ a unique weak solution $u \in \mathcal{H}_0$ of (2.1), defining thus a Green operator

\[
\tilde{G} : L^2(G) \to \mathcal{H}_0, \quad f \mapsto u.
\]

Using (2.7), (2.8) we obtain \(^2\)

\[
\|\tilde{G} f\|_{\mathcal{H}}^2 = (u, u)_{\mathcal{H}} = (f, u)_{L^2} \leq \|f\|_{L^2} \|\tilde{G} f\|_{L^2} \leq C_G \|\tilde{G} f\|_{\mathcal{H}} \|f\|_{L^2} \quad \text{(3.2)}
\]

and

\[
(f, \tilde{G} f)_{L^2} = (f, u)_{L^2} = (u, u)_{\mathcal{H}} = \|\tilde{G} f\|_{\mathcal{H}}^2 \geq 0.
\]

\(^1\)Prime means differentiation with respect to $r$.

\(^2\)The symbols $L^2$ and $H^k$ without specified domain mean always $L^2(G)$ and $H^k(G)$, respectively.
Therefore, $\tilde{G}$ is a bounded linear operator between Hilbert spaces, which is, furthermore, positive and hence symmetric. Restricting $u$ on $G$ one obtains the operator

$$G : L^2(G) \to L^2(G), \quad f \mapsto u|_G,$$

which is likewise bounded and symmetric, and, moreover, compact due to the Rellich-Kondrachov theorem and the observation $\{u|_G : u \in H_0\} = \{u|_G : u \in H\} = H^1(G)$. The spectral theorem for symmetric compact operators in Hilbert spaces establishes now a complete (in $L^2(G)$) orthonormal system $\{v_n : n \in \mathbb{N}\}$ of eigenvectors of $G$:

$$G v_n = \lambda_n^{-1} v_n, \quad n \in \mathbb{N}, \quad (3.3)$$

with real, positive eigenvalues $\lambda_n^{-1}$ of finite multiplicity and $\lim_{n \to \infty} \lambda_n^{-1} = 0$. In order to solve the original problem let us define the “harmonic extension” $\tilde{v}_n := \lambda_n \tilde{G} v_n$ of $v_n$ on $\mathbb{R}^d$. By definition the $\tilde{v}_n$ are weak solutions of the Poisson problem (2.1) with $f := v_n$ and $u = \lambda^{-1} \tilde{v}_n$; thus, eq. (2.8) takes now the form

$$(\tilde{v}_n, v)_H = \lambda_n (v_n, v)_{L^2} \quad \text{for any } v \in H_0, \quad (3.4)$$

i.e. the pair $(\tilde{v}_n, \lambda_n)$ is a weak solution of the eigenvalue problem (3.1). Note, finally, the uniqueness of the harmonic extensions in $H_0$, which is implied by the uniqueness of weak solutions.

Concerning regularity we have $\tilde{v}_n \in C^\infty(G \cup \hat{G}) \cap C^1(\mathbb{R}^d)$, which follows from (2.9c), iterating (2.9b), and remark 2.2. So, $(\tilde{v}_n, \lambda_n)$ is also a classical solution of problem (3.1). We summarize these results in

**Theorem 3.1 (Solution of the eigenvalue problem)** Let $G \subset \mathbb{R}^d$, $d \geq 3$ be a bounded domain with $C^k$-boundary, $k > (d + 5)/2$. The eigenvalue problem (3.1) then has a countable set of eigensolutions $\{(\tilde{v}_n, \lambda_n) : n \in \mathbb{N}\}$ satisfying (3.4), and their restrictions $\{v_n : n \in \mathbb{N}\}$ constitute an orthonormal basis of $L^2(G)$.

Powers of the inverse Green operator and their domains of definition turn out to provide the right setting for the solution of the evolution problem in Section 4. The elements of these spaces can be characterized by the decay behaviour of their Fourier coefficients when expanded in the above eigenfunctions. This motivates the

**Definition 3.2** Let $\{v_n : n \in \mathbb{N}\}$ be the complete orthonormal system with associated eigenvalues $\lambda_n$ according to theorem 3.1, and $\alpha \in \mathbb{R}$. We define then the space of “formal series”

$$S := \left\{ \sum_{n=1}^{\infty} c_n v_n : c_n \in \mathbb{R} \right\},$$

with non-negative functional

$$\| \cdot \|_\alpha : S \to [0, \infty], \quad \sum_{n=1}^{\infty} c_n v_n \mapsto \left( \sum_{n=1}^{\infty} \lambda_n^{2\alpha} |c_n|^2 \right)^{1/2},$$

In this section a quantity with tilde always means the “harmonic extension” on $\mathbb{R}^d$ of a quantity defined on $G$. 

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linear mapping
\[ A^\alpha : S \to S, \quad \sum_{n=1}^{\infty} c_n v_n \mapsto \sum_{n=1}^{\infty} \lambda_n^\alpha c_n v_n, \]
and subspaces
\[ D^\alpha := D(A^\alpha) = \{ v \in S : \|v\|_\alpha < \infty \} \subset S. \]
Obviously, \( A^\alpha \) maps \( D^\alpha \) into \( L^2(G) \). Furthermore, there is \( D^\alpha \subset D^\beta \) if \( \alpha \geq \beta \) and \( D_0 = L^2(G) \). Thus, if \( \alpha \geq 0 \), there is \( D^\alpha \subset L^2(G) \) and \( v \in D^\alpha \) has the representation
\[ v = \sum_{n=1}^{\infty} (v_n, v)_{L^2} v_n. \] (3.5)

With the pairing
\[ \langle w, v \rangle := \sum_{n=1}^{\infty} d_n c_n \quad \text{for} \quad w = \sum_{n=1}^{\infty} d_n v_n \in D_{-\alpha}, \quad v = \sum_{n=1}^{\infty} c_n v_n \in D^\alpha, \] (3.6)
\( D_{-\alpha} \) is the dual space of \( D^\alpha \).

Applying \( A \) on (finite) linear combinations of \( v_n \) we find \( A = G^{-1} \), and \( D(A) \) turns out to be the maximal domain of definition of \( G^{-1} \). Similarly, \( D_{1/2} \) is related to \( H_0 \). More precisely, we have

**Theorem 3.3** Let \( G \subset \mathbb{R}^d, \ d \geq 3 \) be a bounded domain with \( C^\infty \)-boundary \( \partial G \) and let \( \{v_n : n \in \mathbb{N}\} \) be the complete orthonormal system defined by the eigenvalue problem (3.1). Let, furthermore, \( A \) and \( D^\alpha \) be as defined in definition 3.2, and \( G \) be the Green operator associated to the Poisson problem (2.1). Then,
\[ D_0 = H^0(G) = L^2(G), \]
\[ D_{1/2} = \{ v|_G : v \in H_0 \text{ and } v|_G \text{ is harmonic} \} = H^1(G), \] (3.7)
i.e., in particular, any \( v \in D_{1/2} \) has a unique harmonic extension \( \tilde{v} \in H_0 \), and
\[ D_1 = G(L^2(G)) = \{ v \in H^2(G) : \tilde{v} \in H^2_{\text{loc}}(\mathbb{R}^d) \}. \] (3.8)

Higher order spaces are characterized by
\[ D_{k/2} = \left\{ v \in H^k(G) : \Delta^{i-1} v \in H^2_{\text{loc}}(\mathbb{R}^d) \text{ for } i = 1, \ldots, \lfloor k/2 \rfloor \right\} \quad k \in \mathbb{N} \setminus \{1\}, \] (3.9)
where \( \tilde{w} \in H_0 \) again denotes the harmonic extension of a function \( w \in D_{1/2} = H^1(G) \) and \( \lfloor r \rfloor := \max\{ j \in \mathbb{N} : j \leq r \} \) is the integer part of \( r \).

On \( D_{k/2} \) we have the equivalence of norms:
\[ \| \cdot \|_{k/2} \sim \| \cdot \|_{H^k}, \quad k \in \mathbb{N}_0. \] (3.10)

To prove the theorem the following lemma is helpful. It improves the regularity result (2.9b) and provides the pertinent estimate.
Lemma 3.4 Let $G \subset \mathbb{R}^d$, $d \geq 3$ be a bounded domain with $C^{k+2}$-boundary $\partial G$ and $f \in H^k(G)$, $k \in \mathbb{N}_0$. Let, furthermore, $u \in \mathcal{H}_0$ be the weak solution of problem (2.1). Then $u \in H^{k+2}(G)$ and we have the bound

$$
\|u\|_{H^{k+2}(G)} \leq C\|f\|_{H^k(G)} = C\|\Delta u\|_{H^k(G)}
$$

(3.11)

with a constant $C$ depending on $G$, $k$, and $d$.

Proof: The case $k = 0$ is already implied by the interior regularity result (2.9a). In fact, $u \in H^2_{\text{loc}}(\mathbb{R}^d)$ means (see [7], p309)

$$
\|u\|_{H^2(G)} \leq C\left(\|\hat{f}\|_{L^2(K)} + \|u\|_{L^2(K)}\right),
$$

(3.12)

where $\hat{f}$ denotes the trivial extension of $f$ onto $\mathbb{R}^d$ and $K$ some bounded domain such that $G \subset K$. Combining (2.6) with (3.2) we obtain

$$
\|u\|_{L^2(K)} \leq C_K\|u\|_{\mathcal{H}} \leq C_KC_G\|f\|_{L^2(G)},
$$

(3.13)

and thus (3.12) takes the form

$$
\|u\|_{H^2(G)} \leq C\|f\|_{L^2(G)}.
$$

(3.14)

No boundary regularity is required for this result.

The case $k > 0$ needs separate considerations of tangential and normal derivatives at $\partial G$. We refer in the following to the situation, where $\partial G$ has already been flattened as explained in the paragraph before lemma 2.1 and we use the notation introduced there. So, given $g \in L^2(W^-)$ we assume $v \in H^1(W)$ to be a (weak) solution of

$$
\sum_{i,j=1}^d \int_W a_{ij}\partial_y v \partial_y w \,dy = \int_{W^-} g \,w \,dy
$$

(3.15)

for any $w \in H^1_0(W)$. Let $\hat{g}$ be again the trivial extension of $g$ onto $W$. Now we assume higher tangential regularity of $g$, i.e. $D^\alpha g \in L^2(W^-)$ for $|\alpha| \leq k$, $\alpha_d = 0$, which implies $D^\alpha \hat{g} \in L^2(W)$. From interior regularity for weak solutions it follows that $D^\beta v \in L^2(V)$ for $|\beta| \leq k + 2$, $\beta_d \leq 2$ and any $V \subset W$, together with the estimate

$$
\sum_{|\beta| \leq k+2} \int_{V^-} |D^\beta v|^2 \,dy \leq C\left(\sum_{|\alpha| \leq k} \int_{W^-} |D^\alpha g|^2 \,dy + \|v\|^2_{L^2(W)}\right).
$$

(3.16)

As to normal derivatives, note that higher interior regularity implies

$$
-D^\alpha \sum_{i,j=1}^d \partial_y(a_{ij}\partial_y v) = D^\alpha g
$$

(3.17)

to hold a.e. in $W^-$. Writing (3.17) with $\alpha = (0, \ldots, 0, 1)$ in the form

$$
a_{dd} \partial^2_{yy} v = - \sum_{i,j=1}^d \partial_{y\alpha}(a_{ij}\partial_y v) - 2 \partial_{y\alpha}a_{dd} \partial^2_{yy} v - \partial^2_{yy} a_{dd} \partial_y v - \partial_y g,
$$

(3.18)
we find, by uniform ellipticity, $\partial_{y^d}^2 v$ to be bounded in $W^-$ by the right-hand side of (3.18), which is at most of second order in $\partial_y v$. So, (3.16) may be applied and we arrive at
\[
\int_{W^-} |\partial_{y^d}^2 v|^2 \, dy \leq C \left( \sum_{|\alpha| \leq k, \alpha_d \leq 0} \int_{W^-} |D^\alpha g|^2 \, dy + \|v\|_{L^2(W)}^2 \right).
\] (3.19)

The case of arbitrary higher derivatives is now easily proved by induction. So, we find, finally, that (3.16) holds without restriction on $\alpha_d$ and $\beta_d$, respectively.

To complete the proof one has, as usual, to cancel the change of variables, to cover $G$ by local patches, to sum up the corresponding local estimates, and to use once more (3.13).

PROOF [OF THEOREM 3.3]: The case $k = 0$ is trivial. To prove (3.7) let $v \in D_{1/2}$ be decomposed as in (3.5), i.e., $v = \sum_{n=1}^{\infty} (v_n, v)_{L^2} v_n$ with $\sum_{n=1}^{\infty} \lambda_n |(v_n, v)_{L^2}|^2 < \infty$. We define
\[
\tilde{v} := \sum_{n=1}^{\infty} (v_n, v)_{L^2} \tilde{v}_n
\]
with $\tilde{v}_n \in \mathcal{H}_0$ being the unique harmonic extension of $v_n$. Computing
\[
\|\tilde{v}\|_{\mathcal{H}}^2 = \sum_{n,m=1}^{\infty} (v_n, v)_{L^2} (\tilde{v}_m, \tilde{v}_n)_{\mathcal{H}} = \sum_{n=1}^{\infty} \lambda_n |(v_n, v)_{L^2}|^2,
\] (3.20)
where we used (3.4), we find $\tilde{v} \in \mathcal{H}_0$. In order to prove that $\tilde{v}|_G$ is harmonic it suffices to show $(\tilde{v}, \tilde{v})_{\mathcal{H}} = 0$ for any $\tilde{v} \in C_0^\infty(G)$ with supp $\tilde{v} \subset \hat{G}$; this follows immediately with (3.4):
\[
(\tilde{v}, \tilde{v})_{\mathcal{H}} = \sum_{n=1}^{\infty} (v_n, v)_{L^2} (\tilde{v}_n, \tilde{v})_{\mathcal{H}} = \sum_{n=1}^{\infty} \lambda_n |(v_n, v)_{L^2}|^2 = 0,
\]
since $(v_n, v)_{L^2} = \int_G v_n \tilde{v} \, dx = 0$. Therefore, $v \in \{v|_G : v \in \mathcal{H}_0$ and $v|_G$ is harmonic\}. The opposite inclusion follows again with (3.20).

The inclusion $\{v|_G : v \in \mathcal{H}_0$ and $v|_G$ is harmonic\} $\subset H^1(G)$ is obvious; the opposite inclusion follows with theorem C.1: Let $v \in H^1(G)$, $v_0$ an $H^1$-extension of $v$ on $\mathbb{R}^d$ with bounded support, $w := v_0|_{\hat{G}} \in \mathcal{H}$ and $\tilde{u}$ the exterior harmonic solution of (C.6). The function $u$ defined by $v$ in $G$ and $\tilde{u}$ in $\hat{G}$ is then the sought-after function $\in \mathcal{H}_0$. This proves (3.7).

To estimate the $1/2$-norm we supply the above construction with bounds: Let supp $v_0 \subset K$, then
\[
\|v_0\|_{H^1(\mathbb{R}^d)} \leq C(G, K)\|v\|_{H^1(G)}.
\]
Thus, using the minimizing property of solutions of (C.6),
\[
\||\tilde{v}|\|_{\mathcal{H}} \leq \|w\|_{\mathcal{H}} \leq \|v_0\|_{\mathcal{H}} \leq C(G, K)\|v\|_{H^1(G)},
\]
and with (3.20):
\[
\|v\|_{1/2}^2 = \||\tilde{v}|\|_{\mathcal{H}}^2 = \|\nabla v\|_{L^2(G)}^2 + \||\tilde{u}|\|_{\mathcal{H}}^2 \leq C\|v\|_{H^1(G)}^2,
\]

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which is one half of (3.10) for $k = 1$. The other half follows by (2.7).

We show next that $D_1 = \mathcal{G}(L^2)$. Let $u \in \mathcal{G}(L^2)$ and $f = \sum_{n=1}^{\infty} (v_n, f)_{L^2} v_n \in L^2$ such that $u = \mathcal{G} f$. Computing the coefficients of $u$ we find with (3.3):

\[(v_m, u)_{L^2} = (v_m, \mathcal{G} f)_{L^2} = \sum_{n=1}^{\infty} (v_n, f)_{L^2} (v_m, \mathcal{G} v_n)_{L^2} = \lambda_m^{-1} (v_m, f)_{L^2}.
\]

Thus,

\[\sum_{m=1}^{\infty} \lambda_m^2 |(v_m, u)_{L^2}|^2 = \sum_{m=1}^{\infty} |(v_m, f)_{L^2}|^2 < \infty,
\]

hence $u \in D_1$. If, on the other hand,

\[u = \sum_{n=1}^{\infty} (v_n, u)_{L^2} v_n \quad \text{and} \quad \sum_{n=1}^{\infty} \lambda_n^2 |(v_n, u)_{L^2}|^2 < \infty,
\]

then

\[f := \sum_{n=1}^{\infty} \lambda_n (v_n, u)_{L^2} v_n \in L^2
\]

is well-defined and we find $\mathcal{G} f = u$.

The inclusion $\mathcal{G}(L^2(G)) \subset \{ v \in H^2(G) : \tilde{v} \in H^2_{\text{loc}}(\mathbb{R}^d) \}$ is an immediate consequence of the $H^2$-regularity of weak solutions. To prove the opposite inclusion let $w \in H^2(G)$ with harmonic extension $\tilde{w} \in \mathcal{H}_0 \cap H^2_{\text{loc}}(\mathbb{R}^d)$. Defining $f := -\Delta w \in L^2$ the Poisson problem (2.1) yields a solution $\tilde{u} \in \mathcal{H}_0 \cap H^2_{\text{loc}}(\mathbb{R}^d)$. So, we have pointwise a.e. $\Delta (\tilde{w} - \tilde{u}) = 0$ in $\mathbb{R}^d$ for $\tilde{w} - \tilde{u} \in \mathcal{H}_0 \cap H^2_{\text{loc}}(\mathbb{R}^d)$, which means $\tilde{w} - \tilde{u}$ is harmonic in $\mathbb{R}^d$ (by Weyl’s lemma) and, moreover, $\tilde{w} - \tilde{u} = 0$ (by Liouville’s theorem). Thus, $\tilde{w} = \tilde{u}$ and, in particular, $w = u = \mathcal{G}(f)$.

To estimate the 1-norm of $v \in D(A)$ observe that for its harmonic extension holds: $\tilde{v} \in \mathcal{H}_0 \cap H^2_{\text{loc}}(\mathbb{R}^d)$, and for the eigenfunctions $v_n$: $v_n \in C^1(\mathbb{R}^d)$. So, by (3.4) we can calculate

\[-(\lambda_n v_n, v)_{L^2(G)} = - \int_{\mathbb{R}^d} \nabla \tilde{v}_n \cdot \nabla \tilde{v} \, dx = \int_{\mathbb{R}^d} \tilde{v}_n \Delta \tilde{v} \, dx = (v_n, \Delta v)_{L^2(G)} \quad (3.21)
\]

and therefore obtain

\[\|v\|_1^2 = \|A v\|_2^2 = \sum_{n=1}^{\infty} \lambda_n^2 |(v_n, v)_{L^2}|^2 = \sum_{n=1}^{\infty} |(v_n, \Delta v)_{L^2}|^2 \leq \|\Delta v\|_2^2, \quad (3.22)
\]

which implies $\|v\|_1 \leq C \|v\|_{H^2(G)}$ with a constant $C$ depending only on $d$. To prove the opposite inequality we combine (3.22) with (3.11)$_{k=0}$:

\[\|v\|_1 = \|\Delta v\| \geq \frac{1}{C} \|v\|_{H^2(G)}.
\]

This proves (3.10)$_{k=2}$.

The case $k > 2$ is proved by induction. Let $v \in D_{k/2+1}$, $k \in \mathbb{N}$, i.e. $A v \in D_{k/2}$. By assumption we have $A v \in H^k(G)$ and

\[\Delta^{k-1} A v \in H^2_{\text{loc}}(\mathbb{R}^d) \quad \text{for} \quad i = 1, \ldots, [k/2]. \quad (3.23)
\]
(Note that for $v \in D_{3/2}$ condition (3.23) does not yet make sense and can be omitted.)

By (3.22) the condition $Av \in H^k(G)$ means $\Delta v \in H^k(G)$, and lemma 3.4 implies $v \in H^{k+2}(G)$. Moreover, we have $\tilde{v} \in H^2_{loc}(\mathbb{R}^d)$, which complements condition (3.23). So, we conclude

$$v \in \left\{ v \in H^{k+2}(G) : \Delta^{i-1} v \in H^2_{loc}(\mathbb{R}^d) \text{ for } i = 1, \ldots, [k/2] + 1 \right\}.$$  

To prove the opposite inclusion let $v$ be as in (3.24). We set $w := \Delta v$ and have by assumption

$$w \in \left\{ v \in H^k(G) : \Delta^{i-1} v \in H^2_{loc}(\mathbb{R}^d) \text{ for } i = 1, \ldots, [k/2] \right\} = D_{k/2}.$$ 

Computing the $k/2 + 1$-norm of $v$ we find with (3.21)

$$\|v\|_{k/2+1} = \sum_{n=1}^{\infty} \lambda_n^k \lambda_n (v, v)_{L^2} = \sum_{n=1}^{\infty} \lambda_n^k (v, w)_{L^2}^2 = \|w\|_{k/2}^2 < \infty,$$

and thus, $v \in D_{k/2+1}$. This completes the proof of (3.9).

As to the equivalence (3.10) we proceed likewise by induction. Assuming $v \in D_{k/2+1}, k \in \mathbb{N}$ we find by (3.25) and by assumption

$$\|v\|_{k/2+1} = \|\Delta v\|_{k/2} \leq C \|\Delta v\|_{H^k} \leq \tilde{C} \|v\|_{H^{k+2}},$$

whereas the opposite inequality follows by (3.11):

$$\|v\|_{H^{k+2}} \leq C \|\Delta v\|_{H^k} \leq \tilde{C} \|\Delta v\|_{k/2} = \tilde{C} \|v\|_{k/2+1}.$$ 

This completes the proof. \qed

**Remark 3.5** Iterating (3.21) one finds on $D_\alpha$ for integer values $\alpha$ the following alternative formulation of the $\alpha$-norm:

$$\|v\|_\alpha = \|\Delta^k v\|_{L^2(G)}, \quad v \in D_k, \quad k \in \mathbb{N},$$

and by (3.20) for half-integer values:

$$\|v\|_{k+1/2} = \|\nabla \Delta^k v\|_{L^2(\mathbb{R}^d)}, \quad v \in D_{k+1/2}, \quad k \in \mathbb{N}_0.$$

### 4 The evolution problem

We solve in this section the evolution problem (1.1) by means of the spaces $D_\alpha$ provided in the last section. According to eq. (3.7), $v \in D_{1/2}$ has a harmonic extension $\tilde{v}$ on $\mathbb{R}^d$; so, when working with these spaces it is sufficient to consider problem (1.1) on the simpler domain $G \times \mathbb{R}_+$. However, a nontrivial asymptotic function $u_\infty$ does not fit
into this framework. Therefore, in a first step, \( u_{\infty} \) is eliminated by the time-dependent shift \( u - u_{\infty} := u_s \). In terms of \( u_s \) problem (1.1) reads

\[
\begin{align*}
\partial_t u_s - a \Delta u_s &= b \cdot \nabla u_s + c u_s + f & \text{in } G \times \mathbb{R}_+, \quad (4.1a) \\
\Delta u_s &= 0 & \text{in } \hat{G} \times \mathbb{R}_+, \quad (4.1b) \\
u_s \text{ and } \nabla u_s \text{ continuous} & \quad \text{in } \mathbb{R}^d \times \mathbb{R}_+, \quad (4.1c) \\
u_s(x, t) &\to 0 & \text{for } |x| \to \infty, t \in \mathbb{R}_+, \quad (4.1d) \\
u_s(\cdot, 0) &= u_s(0) & \text{on } G \times \{t = 0\} \quad (4.1e)
\end{align*}
\]

with \( f := c u_{\infty} - \frac{d}{dt} u_{\infty} \).

The “simplified” problem takes then the form

\[
\begin{align*}
\dot{v} &= -a \mathcal{A} v + \mathcal{B} v + f, \quad (4.2a) \\
v(0) &= v_0, \quad (4.2b)
\end{align*}
\]

with the operator \( \mathcal{A} \) as defined in Definition 3.2 and the lower-order operator \( \mathcal{B} \) defined by \( \mathcal{B} v := b \cdot \nabla v + c v \). Here \( v \) is a mapping from \([0, T), T > 0 \) into some function space over \( G \). As explained above a reasonable such space is \( D_{k/2} = H^k(G) \) with at least \( k = 1 \) (cf. theorem 3.3). Moreover, for \( T > 0 \), when starting with \( v_0 \in D_{1/2} \) and taking into account parabolic smoothing we expect \( v \in L^2((0, T), D_1) \) which means in view of (4.2a) \( \dot{v} \in L^2((0, T), L^2(G)) \). This motivates the

**Definition 4.1** Let \( T > 0 \) and \( v_0 \in D_{1/2} \). A function \( v \in L^2((0, T), D_1) \) with weak time derivative \( \dot{v} \in L^2((0, T), L^2(G)) \) satisfying (4.2a) as equality in \( L^2((0, T), L^2(G)) \) and (4.2b) as equality in \( D_{1/2} \) is called weak solution of problem (4.2).

Condition (4.2b) makes sense for weak solutions due to the following interpolation result:

**Lemma 4.2** Let \( G \) be a bounded domain with smooth boundary, \( T > 0 \), and \( k \in \mathbb{N}_0 \). Let, furthermore, \( v \in L^2((0, T), H^{k+1}(G)) \) and \( \dot{v} \in L^2((0, T), H^{k-1}(G)) \). Then

\[
v \in C([0, T], H^k(G));
\]

moreover, the mapping \( t \mapsto \|v(t)\|_{L^2(G)}^2 \) is absolutely continuous with derivative

\[
\frac{1}{2} \frac{d}{dt} \|v(t)\|_{L^2(G)}^2 = \langle \dot{v}(t), v(t) \rangle \quad (4.3)
\]

for a.e. \( t \in [0, T] \).

For a proof we refer to [7, p287ff]. We note only in the case \( k = 0 \) that \( D_{-1/2} \) is the dual space of \( D_{1/2} = H^1(G) \), thus \( D_{-1/2} \subset H^{-1}(G) \). \( \langle \cdot, \cdot \rangle \) denotes the dual pairing as defined by (3.6).

**Theorem 4.3 (Weak solution of the evolution problem)** Let \( T > 0 \) and \( a, b, c \in C(\overline{G} \times [0, T]) \), \( a \geq a_0 > 0 \). Let, furthermore, \( v_0 \in D_{1/2} \) and \( f \in C([0, T], L^2(G)) \). Then problem (4.2) has a unique weak solution \( v \).
PROOF: We start with the construction of Galerkin approximations using the complete system \( \{ v_\nu : \nu \in \mathbb{N} \} \) from theorem 3.1. Let \( P_\nu \) be the orthogonal projection in \( L^2(G) \) onto \( \text{span} \{ v_\nu \} \). \( P^{(n)} := \bigoplus_{\nu=1}^n P_\nu \), and let \( v^{(n)}(t) \in P^{(n)}L^2(G) \) be the unique solution of the following finite-dimensional initial-value problem

\[
\begin{align*}
\frac{d}{dt} v^{(n)} &= P^{(n)}(-a \mathcal{A} v^{(n)} + \mathcal{B} v^{(n)} + f), \quad (4.4a) \\
v^{(n)}(0) &= P^{(n)}v_0. \quad (4.4b)
\end{align*}
\]

Note that \( P^{(n)} \) commutes with \( \mathcal{A} \) but not with \( a \) or \( \mathcal{B} \). From standard results about ordinary differential equations follows \( v^{(n)} \in C^1([0,T], D_1) \) for any \( n \in \mathbb{N} \).

Next we derive some a-priori estimates for \( v^{(n)} \), uniform in \( n \), which allow to extract a weakly convergent subsequence of the sequence \( (v^{(n)}) \) of Galerkin approximations. We first show that

\[
\max_{[0,T]} \| v^{(n)} \|_{1/2}^2 \leq C = C[v_0, f; T]. \quad (4.5)
\]

Taking the scalar product of (4.4a) with \( \mathcal{A} v^{(n)} \) we obtain

\[
\frac{1}{2} \frac{d}{dt} \| v^{(n)} \|_{1/2}^2 = \frac{1}{2} \frac{d}{dt} (\mathcal{A}^{1/2} v^{(n)}, \mathcal{A}^{1/2} v^{(n)})_{L^2} = (\mathcal{A} v^{(n)}, \frac{d}{dt} v^{(n)})_{L^2}
= (\mathcal{A} v^{(n)}, -a \mathcal{A} v^{(n)} + \mathcal{B} v^{(n)} + f)_{L^2}
\leq -a_0 \| v^{(n)} \|_1^2 + (\mathcal{A} v^{(n)}, \mathcal{B} v^{(n)})_{L^2} + (\mathcal{A} v^{(n)}, f)_{L^2}. \tag{4.6}
\]

Observing that \( \mathcal{B} \) is a bounded operator from \( C([0,T], D_{1/2}) \) into \( C([0,T], L^2(G)) \) there is a constant \( C_1 \) such that

\[
(\mathcal{A} v^{(n)}, \mathcal{B} v^{(n)})_{L^2} \leq (C_1 a_0)^{1/2} \| v^{(n)} \|_1 \| v^{(n)} \|_{1/2} \leq \frac{a_0}{2} \| v^{(n)} \|_1^2 + \frac{C_1}{2} \| v^{(n)} \|_{1/2}^2.
\]

Setting \( \max_{[0,T]} \| f \|_{L^2}^2 =: C_2 a_0 \) we thus obtain

\[
\frac{d}{dt} \| v^{(n)} \|_{1/2}^2 \leq C_1 \| v^{(n)} \|_{1/2}^2 + C_2,
\tag{4.7}
\]

and Gronwall’s inequality yields

\[
\| v^{(n)} \|_{1/2}^2 \leq e^{C_1 t} \| v^{(n)}(0) \|_{1/2}^2 + \frac{C_2}{C_1} (e^{C_1 t} - 1) \leq e^{C_1 T} \| v^{(n)}(0) \|_{1/2}^2 + \frac{C_2}{C_1} (e^{C_1 T} - 1)
\]

on \([0,T]\), and hence (4.5).

To obtain a bound on \( v^{(n)} \) in \( L^2((0,T), D_1) \) we estimate similarly to (4.6):

\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \| v^{(n)} \|_{1/2}^2 &\leq -a_0 \| v^{(n)} \|_1^2 + \frac{a_0}{4} \| v^{(n)} \|_1^2 + C_1 \| v^{(n)} \|_{1/2}^2 + \frac{a_0}{4} \| v^{(n)} \|_{1/2}^2 + C_2.
\end{align*}
\tag{4.8}
\]

Using (4.5) we rewrite (4.8) in the form

\[
a_0 \| v^{(n)} \|_1^2 \leq - \frac{d}{dt} \| v^{(n)} \|_{1/2}^2 + 2(C_1 C + C_2);
\]

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thus, integrating over \([0, T]\) and observing (4.5) once more yields the bound
\[
\int_0^T \|v^{(n)}\|_1^2 \, dt \leq \hat{C}.
\] (4.9)

With (4.9) the right-hand side in (4.4a) is obviously bounded in \(L^2((0, T), L^2(G))\), i.e. there is \(\hat{C}\) such that
\[
\int_0^T \left\| \frac{d}{dt} v^{(n)} \right\|_{L^2}^2 \, dt \leq \hat{C}
\] (4.10)
for the sequence of (classical) derivatives \(\left(\frac{d}{dt} v^{(n)}\right)\).

The bounds (4.9) and (4.10) imply that there is a subsequence \((n_l)\) and functions \(v \in L^2((0, T), D_1)\), \(\dot{v} \in L^2((0, T), L^2(G))\) such that \(v\) is the weak limit of \((v^{(n_l)})\) in \(L^2((0, T), D_1)\) and \(\dot{v}\) that of \(\left(\frac{d}{dt} v^{(n_l)}\right)\) in \(L^2((0, T), L^2(G))\), respectively, and moreover \(\dot{v}\) is the (weak) derivative of \(v\).

Testing (4.4a) with functions \(w\) of the form \(w(t) = \sum_{\nu=1}^m d_\nu(t) v_\nu \in C^1([0, T], L^2(G))\), where \(m \leq n\) and \(d_\nu : [0, T] \to \mathbb{R}\) are smooth functions, and integrating over \([0, T]\) yields
\[
\int_0^T \left( \frac{d}{dt} v^{(n)}, w \right)_{L^2} \, dt = \int_0^T \left( P^{(n)} (-a A v^{(n)} + B v^{(n)} + f), w \right)_{L^2} \, dt
\]
\[
= \int_0^T \left( -a A v^{(n)} + B v^{(n)} + f, w \right)_{L^2} \, dt. \tag{4.11}
\]

Setting \(n = n_l\) we find in the limit \(l \to \infty\) that
\[
\int_0^T (\dot{v}, w)_{L^2} \, dt = \int_0^T (-a A v + B v + f, w)_{L^2} \, dt. \tag{4.12}
\]

Since test functions of this type are dense in \(L^2((0, T), L^2(G))\), eq. (4.12) holds for any \(w \in L^2((0, T), L^2(G))\). This proves (4.2a) to be an equality in \(L^2((0, T), L^2(G))\).

Inserting \(w \in C^1([0, T], L^2(G))\) with \(w(T) = 0\) in (4.12), after integration by parts on the left-hand side we find
\[
- \int_0^T (v, \frac{d}{dt} w)_{L^2} \, dt + (v(0), w(0))_{L^2}.
\]

Doing the same in (4.11) yields in the limit \(n_l \to \infty\) on the left-hand side,
\[
- \int_0^T \left( \dot{v}, \frac{d}{dt} w \right)_{L^2} \, dt + (v(0), w(0))_{L^2}.
\]

Since \(w(0) \in L^2(G)\) is arbitrary we have \(v(0) = v_0\) in \(L^2(G)\). These results prove \(v\) to be a weak solution of problem (4.2).

Finally, to prove uniqueness of the weak solution consider \(v_1 - v_2 =: v_0\) satisfying
\[
\dot{v}_0 = -a A v_0 + B v_0, \quad v_0(0) = 0.
\]

Setting in eq. (4.12) \(v := v_0 \in L^2((0, T), D(A))\), \(w := A v_0\), and \(f = 0\) we obtain for a.e. \(t \in [0, T]\):
\[
(\dot{v}_0, A v_0)_{L^2} = - (a A v_0, A v_0)_{L^2} + (B v_0, A v_0)_{L^2}. \tag{4.13}
\]
The left-hand side of (4.13) combined with (4.3) takes the form
\[
(\dot{v}_0, A v_0)_{L^2} = (A^{1/2} \dot{v}_0, A^{1/2} v_0) = \frac{1}{2} \frac{d}{dt} \|A^{1/2} v_0\|_{L^2}^2 = \frac{1}{2} \frac{d}{dt} \|v_0\|_{L^2}^2,
\]
whereas estimates analogous to that leading to (4.7) show that the right-hand side of (4.13) can be bounded by \( \frac{1}{2} C_1 \|v_0\|_{L^2}^2 \). Since \( t \mapsto \|v_0(t)\|_{L^2} \) is absolutely continuous by lemma 4.2, applying Gronwall to the inequality
\[
\frac{d}{dt} \|v_0\|_{L^2}^2 \leq C_1 \|v_0\|_{L^2}^2 \quad \text{for a.e. } t \in [0, T]
\]
with \( \|v_0(0)\|_{L^2} = 0 \) yields the desired result \( v_0 \equiv 0 \). \( \square \)

**Remark 4.4** As to the original (shifted) problem (4.1) theorems 3.3 and 4.3 imply that \( u_s(x, t) := \tilde{v}(t)(x) \), where \( \tilde{v}(t) \) is the harmonic extension of \( v(t) \), satisfies (4.1a,b) for a.e. \( (x, t) \in \mathbb{R}^d \times [0, T] \).

Higher regularity of the weak solution depends on the smoothness of the coefficients and the initial-value, and suitable compatibility conditions among these data. There holds

**Theorem 4.5 (Higher regularity)** Let \( T > 0 \), \( k \in \mathbb{N} \setminus \{1\} \), and \( a, b, c \in C^k(\overline{G} \times [0, T]) \).

Let, furthermore, \( v_0 \in D_{(k+1)/2} \), \( -a(\cdot, 0)A v_0 + B_{t=0} v_0 + f(0) \in D_{(k-1)/2} \), and \( f \in C^1([0, T], H^k(G)) \). Then the weak solution \( v \) of problem (4.2) satisfies
\[
v \in L^2((0, T), D_{k/2+1}), \quad \dot{v} \in L^2((0, T), D_{k/2}), \quad \ddot{v} \in L^2((0, T), D_{k/2-1}).
\]

**Proof:** Higher spatial regularity is easily obtained via the operator \( A \): applying \( A^{k/2} \) on (4.2) and setting \( A^{k/2}v = : w \),
\[
A^{k/2} f := f^{(k)}, \quad A^{k/2} B A^{-k/2} := B^{(k)}, \quad (A - A^{k/2} a A^{-k/2}) A =: C^{(k)},
\]
and \( A^{k/2} v_0 := w_0 \), we obtain
\[
\dot{w} = -a A w + B^{(k)} w + C^{(k)} w + f^{(k)}, \quad w(0) = w_0.
\]

(4.14a) (4.14b)

\( f^{(k)} \) and \( w_0 \) fulfill the prerequisites of theorem 4.3,
\[
B^{(k)} : C([0, T], D_{1/2}) \to C([0, T], L^2(G))
\]
is again a bounded operator, and \( C^{(k)} \) is of the same type as \( B^{(k)} \). Thus theorem 4.3 applies to (4.14) with the result
\[
w \in L^2((0, T), D_1), \quad \dot{w} \in L^2((0, T), L^2(G)).
\]

---

\(^4\)In this notation the upper index at “\( C \)” refers to the order of spatial derivatives and the lower one (omitted if zero) to temporal derivatives; so, \( a \in C^k(\overline{G} \times [0, T]) \) means \( a, \partial_i a, \) and \( D^\alpha a, |\alpha| \leq k \) are all continuous functions on \( \overline{G} \times [0, T] \).
i.e.
\[ v \in L^2((0, T), D_{k/2}), \quad \dot{v} \in L^2((0, T), D_{k/2}). \]

To obtain higher temporal regularity we need some more a-priori estimates for the Galerkin approximations \( w^{(n)} \). Note that \( a, b, c \) are at least \( C^3([0, T]) \), \( v_0 \in D_{3/2} \), and \( f \in C^1([0, T], D(A)) \). So, inserting \( w^{(n)} := A w^{(n)} \) into (4.14a) and taking the scalar product with \( A w^{(n)} \) we obtain
\[ \frac{1}{2} \frac{d}{dt} \|w^{(n)}\|_{1/2}^2 = (A w^{(n)}, -a A w^{(n)} + B^{(2)} w^{(n)} + C^{(2)} w^{(n)} + f^{(2)}) \|_{L^2} . \]

This is analogous to (4.6) and the subsequent estimates leading to (4.5) yield now
\[ \max_{[0,T]} \| w^{(n)} \|_{1/2}^2 = \max_{[0,T]} \| u^{(n)} \|_{3/2}^2 \leq C_3 , \quad (4.15) \]
and, via (4.4a),
\[ \max_{[0,T]} \| \frac{d}{dt} v^{(n)} \|_{1/2}^2 \leq C_4 . \quad (4.16) \]

On the other hand, differentiating (4.4a) with respect to \( t \), setting
\[ \frac{d}{dt} v^{(n)} =: \dot{v}^{(n)}, \quad \frac{d}{dt} f =: \dot{f} \quad \text{and} \quad (\partial_t b \cdot \nabla v^{(n)} + \partial_t c v^{(n)}) =: \dot{B} v^{(n)}, \]
we obtain
\[ \frac{d}{dt} \dot{v}^{(n)} = P^{(\ast)} (\partial_t a \dot{v}^{(n)} + B \dot{v}^{(n)} + \dot{f} - \partial_t a A v^{(n)} + \dot{B} v^{(n)}) , \quad (4.17) \]
which is of type (4.4a). Complementing (4.17) by the initial-value \( \dot{v}^{(n)}(0) = P^{(\ast)} \dot{v}_0 \), where \( \dot{v}_0 := -a(\cdot, 0) A v_0 + B_{t=0} v_0 + f(0) \in D_{1/2} \), we obtain again a bound of type (4.5), now on \( \dot{v}^{(n)} \). Next, we modify for (4.17) the argument which leads from (4.5) to (4.9). Taking the scalar product of (4.17) with \( A \dot{v}^{(n)} \) and observing that \( \dot{B} \) is of the same type as \( B \), thus using the bounds
\[ \| B \dot{v}^{(n)} \|_{L^2} \leq (C_4 a_0)^{1/2} \| \dot{v}^{(n)} \|_{1/2} , \quad \| \dot{B} v^{(n)} \|_{L^2} \leq (C_6 a_0)^{1/2} \| v^{(n)} \|_{1/2} \]
as well as \( \max_{[0,T]} \| v^{(n)} \|_{3/2}^2 \leq C_5, \ (4.5), \) and \( a \geq a_0, \ |\partial_t a| \leq A, \ max_{[0,T]} \| f \|_{L^2}^2 \leq C_2 a_0, \)
we obtain
\[ \frac{1}{2} \frac{d}{dt} \| \dot{v}^{(n)} \|_{1/2}^2 \leq -a_0 \| \dot{v}^{(n)} \|_{1/2}^2 + \frac{a_0}{8} \| \dot{v}^{(n)} \|_{1/2}^2 + 2 C_1 \| \dot{v}^{(n)} \|_{1/2}^2 + \frac{a_0}{8} \| \dot{v}^{(n)} \|_{1/2}^2 + 2 C_2 + \frac{a_0}{8} \| \dot{v}^{(n)} \|_{1/2}^2 + 2 C_1 C . \]

This estimate is analogous to (4.8), so we have
\[ \int_0^T \| \dot{v}^{(n)} \|_1^2 dt \leq C_6 \]
and, using (4.17) once more,
\[ \int_0^T \frac{d}{dt} \| \dot{v}^{(n)} \|_{L^2}^2 dt \leq C_7 . \]
Recalling the reasoning after (4.10) there is thus a subsequence \((n_{im}) =: (m) \) of \((n)\) and a function \(\tilde{v} \in L^2((0, T), L^2(G))\) such that \(\tilde{v}\) is the weak limit of \(\left(\frac{d}{dt} u^{(m)}\right)\) and the weak time derivative of \(v\). Finally, inspecting again (4.17) we find that there is enough regularity of the data \(a, b, c, f,\) and \(v_0\) left to improve the spatial regularity of \(\tilde{v}\) by the order \(k - 2\), i.e. we have \(\tilde{v} \in L^2((0, T), D^{k/2 - 1})\).

In view of lemma 4.2 and Sobolev’s embedding theorems, theorem 4.5 implies the existence of smooth solutions. The following corollary formulates for the original evolution problem (1.1) sufficient (and not necessarily sharp) conditions in terms of classical derivatives for existence of classical solutions. In particular, to express the compatibility condition problem (1.1) sufficient (and not necessarily sharp) conditions in terms of classical theorems, from theorem 4.5 we find

\[
|\tilde{v}|_{H^{k/2}} \leq C |D_{\nu} v_0|_{H^{-k}(G)} \leq \tilde{C} \|S_{mn}\|_{C([0, T], D_{\nu/2})}.
\]

**Corollary 4.6 (Classical solution of the evolution problem)** Let \(G \subset \mathbb{R}^d, d \geq 3\) be a bounded domain with \(C^{k+3/2}\)-boundary, \(k > 1 + d/2, u_0 \in C^{k+1}(\overline{G}),\) and \(a, b, c \in C^1_b(\mathbb{R}^d)\) for any \(T > 0\). Let, furthermore, \(u_0 = u_{\infty}(0),\) \(\Delta^i u_0,\) and \(\Delta^{i-1}(a_0 \Delta u_0 + b \cdot \nabla u_0 + c_0 u_0 - \dot{u}_{\infty}(0)), i = 1, \ldots, [(k - 1)/2],\) where \(a_0 = a(\cdot, 0)\) etc., all \(C^1\)-match to their harmonic extensions. Then problem (1.1) has a unique classical solution \(u,\) i.e. \(u \in C^2(G \times \mathbb{R}_+) \cap C^2(\mathbb{G} \times \mathbb{R}_+)\) satisfies pointwise eqs. (1.1).

**Proof:** Fixing some \(T > 0,\) by theorem 3.3, lemma 4.2, and Sobolev’s embedding theorems, from theorem 4.5 we find

\[
v \in C([0, T], H^{k+1}(G)) \subset C([0, T], C^2(\mathbb{G})),
\]

\[
\dot{v} \in C([0, T], H^{k-1}(G)) \subset C([0, T], C(\mathbb{G})).
\]

Thus, setting \(u_s(x, t) := [\tilde{v}(t)](x)\) with \(\tilde{v}(t)\) being the harmonic extension of \(v(t) \in D_{1/2},\) we have \(u_s \in C^2(\mathbb{G} \times [0, T])\) satisfying (4.1a) and (4.1e). Since \(\tilde{v}(t) \in H_0\) is harmonic in \(\mathbb{G}\) conditions (4.1b) and (4.1d) hold for \(u_s\) as well. To prove (4.1c) note that this condition holds for \(\tilde{v}_n.\) Recalling the maximum principle for harmonic functions this implies

\[
\max_{x \in \mathbb{R}^d} |D^a_x \tilde{v}_n(x)| = \max_{x \in \mathbb{G}} |D^a_x \tilde{v}_n(x)|
\]

for any multiindex \(\alpha\) with \(|\alpha| \leq 1.\) Setting \(S_{mn}(x, t) := \sum_{\nu=mn}^n c_\nu(t) \tilde{v}_\nu(x)\) we have then with Sobolev and relation (3.10)

\[
\max_{t \in [0, T]} \sum_{|\alpha| \leq 1} \max_{x \in \mathbb{R}^d} |D^a_x S_{mn}(x, t)| = \max_{t \in [0, T]} \sum_{|\alpha| \leq 1} \max_{x \in \mathbb{G}} |D^a_x S_{mn}(x, t)| \leq C \max_{t \in [0, T]} \|S_{mn}(\cdot, t)\|_{H^{k/2}(G)} \leq \tilde{C} \|S_{mn}\|_{C([0, T], D_{\nu/2})}.
\]

(4.18)

So, convergence of \(v(t) = \sum_{n=1}^{\infty} c_n(t) v_n\) in \(C([0, T], D_{k/2})\) implies convergence of \(\tilde{v}(t) = \sum_{n=1}^{\infty} c_n(t) \tilde{v}_n\) in \(C([0, T], C^1(\mathbb{R}^d))\), i.e. \(u_s \in C^1(\mathbb{R}^d \times [0, T])\).

Similarly, fixing any \(K \Subset \mathbb{G}\) and using the interior derivative estimate (cf. [9, p23])

\[
\max_{x \in K} |D^a_x \tilde{v}_n(x)| \leq \tilde{C} \max_{x \in \mathbb{G}} |\tilde{v}_n(x)| \leq \tilde{C} \max_{x \in \mathbb{G}} |\tilde{v}_n(x)|
\]

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with $|\alpha| = 2$ we find $u_s \in C^2(K \times [0,T])$ and hence $u_s \in C^2(H \times [0,T])$.

Observing, finally, that $T > 0$ is arbitrary we have in conclusion that $u_s$ is a classical solution of problem (4.1) and hence $u := u_s + u_\infty$ is a classical solution of problem (1.1).

\[ \square \]

**Remark 4.7** In $d = 3$ we may choose $k = 3$ and the compatibility conditions amount to

\[ u_0 - u_\infty(0), \ \Delta u_0 \in C^1(\mathbb{R}^3) \]

and

\[ (a_0 \Delta u_0 + b_0 \nabla u_0 + c_0 u_0 - \dot{u}_\infty(0)) \sim \in C^1(\mathbb{R}^3). \]

So, in the case $u_\infty = 0$ admissible initial values $u_0$ are for instance $C^4(G)$-functions with $\partial_n u_0 \big|_{\partial G} = 0$, $i = 0, \ldots, 3$, where $\partial_n$ denotes the normal derivative at $\partial G$. In the case $u_0 = u_\infty = \text{const} > 0$, which was interesting in applications [11], condition (4.20) requires the coefficient $c_0$ to have a $C^1$-smooth harmonic extension.

**Appendices**

**A The non-radial-flow problem**

In the framework of magnetohydrodynamics the kinematic dynamo problem reads [4]:

\[
\begin{align*}
\partial_t B + \nabla \times (\eta \nabla \times B) &= \nabla \times (v \times B), \quad \nabla \cdot B = 0 \quad &\text{in } G \times \mathbb{R}_+, & \quad (A.1a) \\
\nabla \times B &= 0, \quad \nabla \cdot B = 0 \quad &\text{in } \hat{G} \times \mathbb{R}_+, & \quad (A.1b) \\
B &= \text{continuous} \quad &\text{in } \mathbb{R}^3 \times \mathbb{R}_+, & \quad (A.1c) \\
B(x,t) &= O(|x|^{-3}) \quad &\text{for } |x| \to \infty, t \in \mathbb{R}_+, & \quad (A.1d) \\
B(\cdot,0) &= B_0 \quad &\text{on } G \times \{t = 0\}. & \quad (A.1e)
\end{align*}
\]

Here, the induction equation (A.1a) describes the generation of the magnetic field $B$ by the motion (with prescribed flow field $v$) of a conducting fluid (with conductivity $\eta > 0$) in a bounded region $G \subset \mathbb{R}^3$. Outside the fluid region there are no further sources of magnetic field. Thus, $B$ continues in $\hat{G} = \mathbb{R}^3 \setminus \overline{G}$ as a vacuum field and vanishes at spatial infinity.

If $G$ is a ball $B_R$ (or a spherical shell) the so-called poloidal-toroidal decomposition of solenoidal fields is especially useful [4, 14]:

\[ B = B_P + B_T = -\nabla \times \Lambda S - \Lambda T, \quad \langle S \rangle = \langle T \rangle = 0. \]

$\Lambda$ denotes here the non-radial derivative operator $\Lambda := x \times \nabla$, $\Lambda \cdot \Lambda =: \mathcal{L}$ is the Laplace-Beltrami-operator on the unit sphere $S_1$, and $\langle \cdot \rangle$ denotes the spherical mean. The poloidal and toroidal scalars $S$ and $T$, resp., are uniquely determined (e.g. in $L^2(B_R)$) by $B$:

\[ x \cdot B = -\mathcal{L} S, \quad x \cdot \nabla \times B = -\mathcal{L} T. \]
In the following we refer to $P := x \cdot B$ instead of $S$ as the poloidal scalar.

In the case of a non-radial flow field, i.e. $v \cdot x \equiv 0$, and spherically symmetric conductivity, problem (A.1) implies the scalar sub-problem (1.3) for $P$: (1.3a) is just the radial component of the first part of (A.1a) and (1.3b) is obtained by applying $\Lambda$ on the first part of (A.1b). Condition (1.3c) is in fact enough to ensure a continuous magnetic field $B$, i.e. continuous second-order derivatives of $S = -L^{-1}P$, since $B$ involves not more than one radial derivative of $S$. The equivalence of (A.1d) with (1.3d) is clear for $B \cdot (x/|x|)$ and follows for the non-radial components with the divergence-constraint.

**Remark A.1** In the mathematical treatment of problem (1.1) it turns out to be useful to consider functions $v$ satisfying the integral condition $\int_{\mathbb{R}^d} |\nabla v|^2 \, dx < \infty$. In the context of problem (1.3) this condition can be interpreted as one guaranteeing finite total magnetic energy. In fact, the total energy of the poloidal magnetic field reads

$$E[B_P] = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla \times S|^2 \, dx = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla L^{1/2} S|^2 \, dx,$$

and with the variational estimate

$$\inf_{f \neq 0, (f) = 0} \frac{\|L f\|_{L^2(S_1)}}{\|f\|_{L^2(S_1)}} = 2,$$

one obtains the bound on $E[B_P]$:

$$E[B_P] \leq \frac{1}{4} \int_{\mathbb{R}^3} |\nabla L S|^2 \, dx = \frac{1}{4} \int_{\mathbb{R}^3} |\nabla P|^2 \, dx.$$

### B The axisymmetric problem

The central assumption is here an axisymmetric magnetic field with representation

$$B = \nabla P \times \nabla \phi + A \nabla \phi = -\frac{1}{\rho} \partial_\rho P e_\rho + \frac{1}{\rho} A e_\phi + \frac{1}{\rho} \partial_\phi P e_z \quad (B.1)$$

by two scalar quantities, the poloidal one $P$ and the toroidal or azimuthal one $A$, depending (besides on $t$) on $\rho$ and $z$ with $(\rho, \phi, z)$ being cylindrical coordinates in $\mathbb{R}^3 \setminus \{\rho = 0\}$. Inserting (B.1) into the dynamo equation (3.11) the following sub-problem for the poloidal scalar $P$ arises [3, 12, 10]:

$$\begin{align*}
\partial_t P - \eta \Delta_s P &= -v \cdot \nabla P & \text{in } G_2 \times \mathbb{R}_+, \\
\Delta_s P &= 0 & \text{in } \hat{G}_2 \times \mathbb{R}_+, \\
P(\rho, z, t) &\to 0 & \text{for } \rho \to 0, \ (z, t) \in \mathbb{R} \times \mathbb{R}_+, \\
|\nabla P(\rho, z, t)| &= O(\rho) & \text{for } \rho \to 0, \ (z, t) \in \mathbb{R} \times \mathbb{R}_+, \\
\left|\frac{1}{\rho} \nabla P(\rho, z, t)\right| &= O((\rho^2 + z^2)^{-3/2}) & \text{for } \rho^2 + z^2 \to \infty, \ t \in \mathbb{R}_+, \\
P(\cdot, \cdot, 0) &= P_0, \quad P_0 \text{ satisfying (B.2d,e)} & \text{on } G_2 \times \{t = 0\}. \quad (B.2g)
\end{align*}$$
\( \Delta_s \) is the elliptic operator \( \Delta_s := \frac{\partial^2}{\rho^2} - \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{\partial^2}{\partial z^2} \) on the half-plane \( H := \mathbb{R}_+ \times \mathbb{R} \). \( G_2 \subset H \) is the “cross-section” of some bounded region \( G_3 \subset \mathbb{R}^3 \); more precisely, \( \overline{G_3} \setminus \partial G_3 \) with \( G_3 := G_2 \times \{0 \leq \phi < 2\pi\} \) is a bounded domain in \( \mathbb{R}^3 \) with smooth boundary. Note that the axisymmetric flow field need not be solenoidal, the azimuthal component, however, w.l.o.g. can be assumed to be zero. Condition (B.2e) ensures a finite magnetic field on the symmetry axis \( \{\rho = 0\} \). It implies the limit \( \lim_{\rho \to 0} P(\rho, \cdot, t) = P_s(t) \), where \( P_s \) depends only on \( t \). As \( P_s \) does not affect the magnetic field it is set to zero for simplicity (condition (B.2d)). Note that in [12, 10] conditions (B.2d,e) are replaced by

\[
P(\rho, z, t) = O(\rho^3) \quad \text{for } \rho \to 0, \ (z, t) \in \mathbb{R} \times \mathbb{R}_+.
\]  

(B.3)

The cautious Backus [3] requires (B.2e) and (B.3). In fact, conditions (B.2d,e) imply (B.3), but not vice versa. In the view of the original problem (3.11), condition (B.2e) seems to be the natural one. Similarly, in these references the “natural” condition (B.2f) is replaced by

\[
P(\rho, z, t) = O((\rho^2 + z^2)^{-1/2}) \quad \text{for } \rho^2 + z^2 \to \infty, \ t \in \mathbb{R}_+.
\]  

(B.4)

These conditions are in fact equivalent for solutions of (B.2b) as becomes clear in the subsequent formulation of problem (B.2).

There is an elegant way to eliminate the “coordinate–singularity” at \( \rho = 0 \) in problem (B.2), namely by embedding (B.2) in \( \mathbb{R}^5 \). \( P \) is then considered as an axisymmetric function in \( \mathbb{R}^5 \) with symmetry axis in \( x_3 \)-direction. Identifying \( \rho^2 \) with \( \sum_{i=1}^4 x_i^2 \) and \( z \) with \( x_5 \), \( x \in \mathbb{R}^5 \), and introducing \( Q(x, t) := \tilde{Q}(\rho, z, t) := P(\rho, z, t)/\rho^2 \), the crucial observation is [5]

\[
\Delta_s P = (\partial^2/\rho^2 - \frac{1}{\rho} \partial_\rho + \partial_z^2)P = \rho^2(\partial^2/\rho^2 + \frac{3}{\rho} \partial_\rho + \partial_z^2)\tilde{Q} = \rho^2 \Delta_5 Q
\]

with \( \Delta_5 \) being the Laplacian in \( \mathbb{R}^5 \). With the further definitions

\[
b_i(x, t) := -v_\rho(\rho, z, t) x_i/\rho, \quad i = 1, \ldots, 4, \quad b_5(x, t) := -v_z(\rho, z, t).
\]

\[
c(x, t) := -2 v_\rho(\rho, z, t)/\rho, \quad a(x, t) := \eta(\rho, z, t)
\]

problem (B.2) takes in \( \mathbb{R}^5 \) the form

\[
\partial_t Q - a \Delta_5 Q = b \cdot \nabla Q + c Q \quad \text{in } G_5 \times \mathbb{R}_+, \quad \text{(B.5a)}
\]

\[
\Delta_5 Q = 0 \quad \text{in } G_5 \times \mathbb{R}_+, \quad \text{(B.5b)}
\]

\( Q \) and \( \nabla Q \) continuous \( \text{in } \mathbb{R}^5 \times \mathbb{R}_+, \quad \text{(B.5c)} \)

\[
Q(x, t) = O(|x|^{-3}) \quad \text{for } |x| \to \infty, \ t \in \mathbb{R}_+, \quad \text{(B.5d)}
\]

\[
Q(\cdot, 0) = Q_0, \quad Q_0 \text{ axisym.} \quad \text{on } G_5 \times \{t = 0\}. \quad \text{(B.5e)}
\]

\( G_5 \) is now an axisymmetric bounded region in \( \mathbb{R}^5 \). An axisymmetric initial field \( Q_0 \) implies axisymmetry of \( Q(\cdot, t) \) for all \( t > 0 \). A condition on the symmetry axis is no longer necessary; conditions (B.2d,e) (as well as (B.3)) are automatically satisfied by
$P := \rho^2 \tilde{Q}$. As to the behaviour for large $x$, $Q$ is in $\hat{G}$ an exterior harmonic function with representation (C.1); thus, (B.5d) implies $|\nabla Q(x, \cdot)| = O(|x|^{-4})$ for $|x| \to \infty$ and hence (B.2f) as well as (B.4). On the other side, in the view of (C.4) each of the conditions (B.2f) and (B.4) implies (B.5d).

**Remark B.1** Stredulinsky et al. [16] doubt the correctness of the boundary condition (B.4) in [10] and cite their own results (theorem 2) about solutions with non-vanishing asymptotic value $P_\infty(t)$ at spatial infinity. They mention the possibility of $\lim_{t \to \infty} P_\infty(t) \neq 0$, even when $\lim_{t \to 0} P_\infty(t) = 0$. In fact, condition (B.4) is correct as demonstrated above, which means $P_\infty \equiv 0$ in the axisymmetric problem. The discrepancy arises because in [16] the authors are especially interested in the two-dimensional case where their problem (1) makes physical sense (“dynamo problem with plane symmetry”). In $d = 2$, in fact, $P_\infty \neq 0$ cannot be avoided in general. In $d > 2$, however, problem (1) is underdetermined and the condition $P_\infty \equiv 0$ can be added. Observe in this context that in [16] the authors do not claim uniqueness for solutions of problem (1), uniqueness is claimed only for weak solutions which are “minimizers” (this is more than “harmonic”) in $\hat{G}$ (theorem 1).

### C Exterior harmonic functions

An exterior harmonic function $u$ is called “harmonic at infinity” iff $u(x) \to 0$ for $|x| \to \infty$. In $d \geq 3$ dimensions these functions have the series representation

$$u(x) = \sum_{n=0}^{\infty} |x|^{2-n-d} Y_n(x/|x|), \quad Y_n \in H_n,$$

absolutely and uniformly converging in the exterior of any ball $B_r \subset \mathbb{R}^d$ with $r > 1$ (cf. [8, p115]). $H_n$ is the space of all harmonic homogeneous polynomials of degree $n$ in $\mathbb{R}^d$ restricted to the unit sphere $S^{d-1}$ with dimension $D_n := \dim H_n = (2n + d - 2)(n + d - 3)!n!(d - 2)!^{-1}$ (cf. [8, p98f]). In particular, $\dim H_0 = 1$ and $Y_0 = \text{const}$; any other $Y_n$ has vanishing spherical mean, $\langle Y_n \rangle = 0$, $n \in \mathbb{N}$. The total of spaces $H_n$ spans $L^2(S^{d-1})$: $L^2(S^{d-1}) = \bigoplus_0^\infty H_n$. So, choosing orthonormal bases $\{Y_{nm} \mid 1 \leq m \leq D_n\}$ in $H_n$, any $f \in L^2(S^{d-1})$ allows a unique representation

$$f = \sum_{n=0}^{\infty} \sum_{m=1}^{D_n} c_{nm} Y_{nm},$$

with coefficients $c_{nm} := (f, Y_{nm})_{L^2} \in \mathbb{R}$. Obviously, (C.2) is the higher-dimensional analogue of the well-known representation

$$f = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} c_{nm} Y_{nm},$$

by spherical harmonics $\{Y_{nm} \mid n \in \mathbb{N}_0, |m| \leq n\}$ in $d = 3$. Note that the $Y_{nm}$ are in our setting real quantities; by taking suitable linear combinations this is true for spherical harmonics as well.
Exterior harmonic functions \( u \) satisfying the condition \( \int_{B_R} |\nabla u|^2 \, dx < \infty \) for some \( R > 0 \) are harmonic at infinity up to a constant \( c_0 \) (cf. [15, p41]). With (C.1) this implies the representation

\[
    u(x) = c_0 + \sum_{n=0}^{\infty} |x|^{2-n-d} Y_n(x/|x|), \quad c_0 \in \mathbb{R}, \quad Y_n \in H_n. \tag{C.4}
\]

(C.4) holds also in the case of exterior harmonic functions with asymptotic conditions \( u(x) = O(1) \) for \( |x| \to \infty \) (cf. [2, p64]) or \( |\nabla u| = O(|x|^{1-d}) \) for \( |x| \to \infty \). The latter statement follows from the former and the estimate

\[
    |u(x)| \leq |u(R x/|x|)| + \int_R^{|x|} |\nabla u(r x/|x|)| \, dr, \quad |x| \geq R > 0.
\]

Let us, finally, consider the exterior boundary-value problem

\[
    \begin{align*}
    \Delta u &= 0 \quad \text{in } \mathring{G}, \tag{C.5a} \\
    u &= \phi \quad \text{on } \partial \mathring{G}, \tag{C.5b} \\
    u(x) &\to c \quad \text{for } |x| \to \infty. \tag{C.5c}
    \end{align*}
\]

Here, \( \mathring{G} \subset \mathbb{R}^d, d \geq 3 \) is an exterior region, i.e. \( \mathring{G} = \mathbb{R}^d \setminus \overline{G} \) for some bounded domain \( G \subset \mathbb{R}^d \), with \( C^1 \)-boundary \( \partial \mathring{G} \). For \( \phi \in C(\partial G) \) and \( c \in \mathbb{R} \) we call \( u \in C(\mathring{G} \cup \partial G) \cap C^2(\mathring{G}) \) satisfying (C.5) a classical solution. A weak formulation is based on the spaces \( H_0 := \{ v \in H^1_{loc}(\mathring{G}) \mid \| v \|_R < \infty \} \) and \( \mathring{H}_0 := \text{clos}\{ C_0^\infty(\mathring{G}) \cap \mathbb{R} \} \) with \( \| v \|_R^2 := \int_{\mathring{G}} |\nabla v|^2 \, dx \).

Describing the boundary and asymptotic conditions by a function \( w \in \mathring{H} \), a weak version of (C.5) reads:

\[
    \int_{\mathring{G}} \nabla u \cdot \nabla v \, dx = 0 \quad \text{for any } v \in \mathring{H}_0, \tag{C.6a}
\]

\[
    u - w \in \mathring{H}_0. \tag{C.6b}
\]

For problems (C.5), (C.6) holds:

**Theorem C.1 (Solution of the exterior Dirichlet problem)** The exterior boundary-value problem (C.6) with given \( w \in \mathring{H} \) has a unique solution \( u \in \mathring{H} \). Moreover, \( u \in C^\infty(\mathring{G}) \) and \( \Delta u = 0 \) in \( \mathring{G} \). If \( w|_{\partial \mathring{G}} = \phi \in C(\partial \mathring{G}) \) and \( w \to c \) for \( |x| \to \infty \), then \( u \) is a classical solution of (C.5). Furthermore, \( u \) is the unique minimizer of the functional

\[
    \| \cdot \|_R \text{ on the set } \{ v + w \mid v \in \mathring{H}_0 \}.
\]

For a proof we refer to [6, p543]. We note only that \( \| \cdot \|_R \)-convergence already implies \( \| \cdot \|_{L^2(\mathring{G})} \)-convergence, thus \( \mathring{H}_0 \subset \mathring{H} \). In fact, the Gagliardo-Nirenberg-Sobolev-inequality (cf. [7, p263]) implies the estimate

\[
    \| v \|_{L^p(\mathring{G})} \leq C\| v \|_R, \quad p = \frac{2d}{d-2}
\]

for any \( v \in \mathring{H}_0 \) and a constant \( C \) depending only on \( d \). So, fixing some \( K \in \mathring{G} \) we obtain:

\[
    \| v \|_{L^2(K)} \leq C(K)\| v \|_{L^p(K)} \leq C(K)\| v \|_{L^p(\mathring{G})} \leq C(K)\| v \|_R.
\]

In particular, our \( \mathring{H}_0 \) coincides with the corresponding space \( B_0(\mathring{G}) \) in [6].
D  Poloidal free decay modes

The poloidal free decay modes are a countable set of explicit solutions of the eigenvalue problem (3.1) if $G$ is a ball $B_R$ in $\mathbb{R}^3$. In terms of spherical Bessel functions $j_n$ and spherical harmonics (cf. appendic C) they take the form

$$\tilde{p}_{lnm}(x) := \sqrt{\frac{2}{R^3}} \begin{cases} j_n(l^{n-1}|x|/R) Y_{nm}(x/|x|) & \text{in } B_R, \\ j_n(l^{n-1}) (|x|/R)^{-n-1} Y_{nm}(x/|x|) & \text{in } \hat{B}_R \end{cases}, \quad l \in \mathbb{N}, n \in \mathbb{N}_0, |m| \leq n$$

with eigenvalues $\lambda_{lnm} := \gamma_{ln} := (i_l^{n-1}/R)^2$; $i_l^n$ is the $l$-th positive zero of $j_n$. For their restrictions $p_{lnm} := \tilde{p}_{lnm}|_{B_R}$ holds

**Theorem D.1** The set of functions $\{p_{lnm} : B_R \to \mathbb{R} \mid l \in \mathbb{N}, n \in \mathbb{N}_0, |m| \leq n\}$ constitutes a complete orthonormal system in $L^2(B_R)$.

**Proof:** The orthonormality of the $p_{lnm}$ can be checked by explicit calculation using the orthonormality of the $Y_{nm}$ and the corresponding relation for the $j_n$ (cf. [1, p485, eq. 11.4.5]). To prove the completeness we show any solution of problem (3.1) being a linear combination of the $\tilde{p}_{lnm}$. Theorem 3.1 yields then the completeness of the $p_{lnm}$.

So, let $u$ be a solution of (3.1) with eigenvalue $\lambda > 0$. The representation (C.3) for $L^2$-functions on the unit sphere implies the representation

$$u(x) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} u_{nm}(|x|) Y_{nm}(x/|x|)$$

(D.1)

for $u$ with coefficients $u_{nm}$ depending only on $|x|$. For $|x| > R$, (C.1) yields

$$u_{nm}(|x|) = c_{nm} |x|^{-n-1}$$

with $c_{nm} \in \mathbb{R}$, whereas inserting (D.1) into (3.1a) yields the differential equation for $u_{nm}$ on $(0, R)$:

$$-\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d}{dr} u_{nm} \right) + \frac{1}{r^2} n(n+1)u_{nm} = \lambda u_{nm}$$

(D.2)

with $r := |x|$. After rescaling $s := \sqrt{\lambda} r$, (D.2) takes the spherical form of Bessel’s differential equation for $v_{nm}(s) := u_{nm}(s/\sqrt{\lambda})$:

$$\left( s^2 \frac{d^2}{ds^2} + 2s \frac{d}{ds} + s^2 - n(n+1) \right) v_{nm} = 0$$

with (in $s = 0$) regular solutions $j_n$, $n \in \mathbb{N}_0$. Matching inner and outer solutions according to (3.1c) yields the condition

$$\left( s \frac{d}{ds} j_n(s) + (n+1) j_n(s) \right)|_{s=\sqrt{\lambda} R} = (s j_n(s)-1)|_{s=\sqrt{\lambda} R} = 0$$

(D.3)

fixing the eigenvalue at

$$\lambda = \lambda_{ln} = (i_l^{n-1}/R)^2.$$

(D.4)
Note that (D.3) holds also in the case $n = 0$ with $j_{-1} = \cos s/s$.

According to theorem 3.1, the eigenvalue $\lambda$ is of finite multiplicity. Thus, only finitely many pairs $(l, n)$ satisfy (D.4), and (D.1) is in fact a (finite) linear combination of the $\tilde{p}_{lnm}$.

\begin{remark}
The condition of vanishing spherical mean eliminates all $n = 0$ modes; thus, $\{p_{lnm} | l \in \mathbb{N}, n \in \mathbb{N}, |m| \leq n\}$ is a complete orthonormal system in $\{v \in L^2(B_R) | \langle v \rangle = 0\}$.
\end{remark}

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References


