APPROXIMATELY AXISYMMETRIC ANTIDYNAMO THEOREMS
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Abstract. The axisymmetric antidynamo theorem rules out dynamo action by the motion of a conducting fluid in a bounded domain surrounded by vacuum, provided that magnetic field, flow field, magnetic diffusivity distribution, and the shape of the domain are axisymmetric. We present in this paper three versions of a generalized axisymmetric antidynamo theorem, which establishes decay of the magnetic field even in the presence of small amounts of nonaxisymmetry in the magnetic field, the flow field, and the diffusivity distribution. The first two versions hold only in the case of weak variations of compressibility and diffusivity of the fluid, whereas the third version is not subject to such a restriction. By proper choice of the diffusivity distribution modelling the conducting domain the third version allows even small deviations from axisymmetry of this domain. However the smallness-requirements of the third version are not as explicit as in the other versions and they are generally more severe. The first version refers only to the meridional part of the axisymmetric magnetic field and proves monotonic decay to zero of the corresponding scalar in the energy norm, whereas the other two versions demonstrate decay of functionals that involve both the meridional as well as the azimuthal scalars and the magnetic field itself. The smallness of the nonaxisymmetric part of the magnetic field is controlled by the ratio of energies of the nonaxisymmetric over the axisymmetric part; whereas the nonaxisymmetric parts of flow field and diffusivity distribution are controlled by their maximum values.

Key words. magnetohydrodynamics, dynamo theory, axisymmetric theorem

AMS subject classifications. 76W05, 85A30, 86A25

1. Introduction. The axisymmetric antidynamo theorem has been the subject of improvement and generalization ever since Cowling’s seminal paper was published more than 80 years ago (Cowling 1934). He started by considering the rather restricted situation of a steady purely meridional magnetic field sustained by a steady purely meridional flow in an incompressible fluid of constant diffusivity and he demonstrated its impossibility by his now famous (but somewhat informal) “neutral point argument”. Subsequently a number of researchers contributed to this field by more rigorous arguments and, in particular, by lifting almost all the restrictions mentioned above (see, for instance, Ivers and James 1984, which contains a detailed review section, or the reviews of Fearn et al. 1988 or of Núñez 1996, and references therein). The state of the art has recently been summarized (Kaiser and Tilgner 2014)1; any axisymmetric magnetic field solving the dynamo equation in an axisymmetric bounded domain surrounded by vacuum is pointwise bounded by a non-negative function that decays (in time) to zero. The flow field is assumed to be bounded and needs some regularity but does not need to be steady or divergence-free; likewise, the diffusivity is assumed to be bounded from above and below (away from zero) but is otherwise variable in space and time.

There is, however, one restriction that has never been questioned: axisymmetry itself. Of course, its complete removal is not possible, but small deviations from axisymmetry need not necessarily invalidate the axisymmetric theorem. In fact, the other prototypical antidynamo theorem, the toroidal velocity theorem, has recently been proved to be robust in the sense that a small amount of poloidal flow, slight compressibility of the fluid or weak variations of its diffusivity do not invalidate the theorem (Kaiser and Busse 2017). It is the aim of the present paper to derive a

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1Henceforth abbreviated by (KT14).
similar result for the axisymmetric theorem. Robustness is in fact relevant for applications: whenever the axisymmetric theorem is applied to a real-world system, exact axisymmetry cannot be expected and robustness of the theorem is (at least tacitly) presupposed. Here we make this property of the axisymmetric theorem explicit and provide, moreover, quantitative bounds.

As is well-known Cowling’s theorem does not hold in mean-field electrodynamics. In fact, early on researchers investigated the weakly nonaxisymmetric case by separating axisymmetric mean-fields from nonaxisymmetric “fluctuations” about the mean-field. Inserting this distinction (for magnetic field, flow field, and diffusivity) into the dynamo equation results typically in mean-field equations for the mean magnetic field components with additional fluctuation terms and equations for the magnetic fluctuations depending on mean and fluctuating quantities. The essence of mean-field theory is to exploit the latter equations such that the fluctuation terms in the mean-field equations can themselves be expressed by mean fields. Braginskii (1965a,b) did this successfully: based on some assumptions about the dominant parts of the mean-fields (with constant diffusivity) and using an expansion for small inverse magnetic Reynolds numbers, he was able to derive an $\alpha$-type term closing in this way the dynamo cycle for the (somewhat modified) mean magnetic field. Later on Soward (1972) confirmed these findings by a more systematic “pseudo-Lagrangian” approach. In this mean-field context our results can be viewed as providing necessary conditions for the validity of the mean-field approximation.

The paper is organized as follows: section 2 summarizes our results, viz. three types of approximately axisymmetric antidynamo theorems together with a number of comments. For easier access the results in this section are not presented in its most detailed form; more detailed versions are provided in the sections 4 – 6. Section 3 contains the basic equations for the subsequent calculations: the dynamo equation for the magnetic field $B$ and derived equations for suitable scalar quantities representing the axisymmetric part of $B$: basic inequalities and variational estimates needed later on are also provided in this section. Where possible we rely here on material already presented in (KT14). Sections 4, 5, and 6 are then devoted to the three types of approximately axisymmetric theorems considered in this paper. The proof of the last theorem depends on solutions of two auxiliary problems. One of them has already been considered in (KT14), the other is discussed in an appendix.

2. Results and comments. The following notation is used throughout the paper and is, moreover, compatible with that used in (KT14). Let $G$ be a bounded and (not necessarily simply) connected axisymmetric domain in $\mathbb{R}^3$ with smooth boundary $\partial G$ and exterior space $\hat{G} = \mathbb{R}^3 \setminus \overline{G}$. $R$ is the radius of the smallest ball $B_R$ enclosing $G$ and $n$ denotes the exterior normal at $\partial G$. In the following we will freely move between Cartesian coordinates $(x, y, z)$ and cylindrical ones $(\rho, \phi, z)$, where $e_z$ is pointing in the direction of the symmetry axis $S$ and $(\rho, \phi)$ are polar coordinates in the planes perpendicular to $S$ (see Figure 2.1). Representing the magnetic field $B$ and the flow field $v$ in cylindrical unit vectors $e_\rho$, $e_z$, and $e_\phi$ yields the meridional/azimuthal decompositions:

$$\begin{align*}
B &= (B_\rho e_\rho + B_z e_z) + B_\phi e_\phi =: B_{mc} + B_{az}, \\
v &= (v_\rho e_\rho + v_z e_z) + v_\phi e_\phi =: v_{mc} + v_{az}.
\end{align*}$$

$$\begin{align*}
(2.1)
\end{align*}$$
Another decomposition is obtained by azimuthal averaging:

\[ \frac{1}{2\pi} \int_{0}^{2\pi} f(\rho, \phi, z) \, d\phi =: \langle f \rangle, \]

which allows the representation of any function \( f \) defined on \( G, \hat{G} \), or \( \mathbb{R}^3 \) by its axisymmetric and nonaxisymmetric parts:

\[ f = \langle f \rangle + (f - \langle f \rangle) =: f^a + f^n. \]

This decomposition is applied to \( B, v \), and the diffusivity distribution \( \eta \):

\[ B = B^a + B^n, \quad v = v^a + v^n, \quad \eta = \eta^a + \eta^n. \]

We note yet for later use

\[ \langle f^n \rangle = 0, \quad \langle \partial_\phi f \rangle = 0, \quad \langle f^n g^n \rangle = 0, \]

whereas in general \( \langle f^n g^n \rangle \neq 0 \).

For \( B^a \) we make use of the well-known axisymmetric representation

\[ B^a = \nabla M \times \nabla \phi + A\nabla \phi = \left( -\frac{1}{\rho} \partial_z M \mathbf{e}_\rho + \frac{1}{\rho} \partial_\rho M \mathbf{e}_z \right) + \frac{1}{\rho} A \mathbf{e}_\phi = B^{a}_{me} + B^a_{az} \]

by the (axisymmetric) meridional scalar \( M \) and the azimuthal one \( A \). Later on, besides \( M \), the modified scalars \( N \) and \( A \) (instead of \( A \)) will play a role:

\[ N := M/\rho, \quad A := A/\rho^2. \]
Magnetic field and related scalars are measured by the energy norm on their respective domains:

\[
\int_G |\cdot|^2 \, dx =: \| \cdot \|_G^2, \quad \int_{B_R} |\cdot|^2 \, dx =: \| \cdot \|_{B_R}^2, \quad \int_{\mathbb{R}^3} |\cdot|^2 \, dx =: \| \cdot \|_\infty^2.
\]

The magnetic energy ratios

\[
\max_{[0,T]} \|B_{\text{me}}^n\|_\infty =: r_{\text{me}}^n, \quad \max_{[0,T]} \|B_{\text{mic}}^n\|_G =: r_{\text{mic}}^n, \quad \max \{r_{\text{me}}^n, r_{\text{mic}}^n\} =: r_{\text{mag}}.
\]

will turn out to be of particular importance. Control of the diffusivity \(\eta\) and its spatial variation is provided by the space-time-minimum

\[
\min_{G \times [0,T]} |\eta| =: \eta_0
\]

and various space-time maxima:

\[
\max_{G \times [0,T]} |\nabla \eta^n | R/\eta_0 =: \delta_1 \eta^n, \quad \max_{G \times [0,T]} |\partial_\rho \eta^n / \rho | R^2 / \eta_0 =: \delta_2 \eta^n,
\]

\[
\max_{G \times [0,T]} \eta^n / \eta_0 =: E^n, \quad \max_{G \times [0,T]} \eta^n / \rho R / \eta_0 =: E^n^*,
\]

\[
\max_{G \times [0,T]} |\nabla_{me} \eta^n | R / \eta_0 =: \delta \eta^n.
\]

Note that for a diffusivity distribution \(\eta\) that is smooth at the symmetry axis \(S\), \(\partial_\rho \eta^n\) and \(\eta^n\) vanish at \(S\) at least as \(\rho\) in the first power; \(\eta_0\) and powers of \(R\) have been inserted to obtain dimensionless measures; \(\nabla_{me}\) means the meridional part of \(\nabla\), viz. \(e_\rho \partial_\rho + e_z \partial_z\).

Concerning the flow field the following dimensionless space-time maxima will play a role:

\[
\max_{G \times [0,T]} |v_{me/az} | R / \eta_0 =: V_{me/az}, \quad \max_{G \times [0,T]} v_{me/az}^n | R / \eta_0 =: V_{me/az}^n,
\]

\[
\max_{G \times [0,T]} |v_{\rho}^n / \rho | R^2 / \eta_0 =: V_{\rho}^n, \quad \max_{G \times [0,T]} v_{\rho}^n / \rho | R^2 / \eta_0 =: V_{\rho}^n^*,
\]

\[
\max_{G \times [0,T]} v_{me/az}^n | R / \eta_0 =: V_{me/az}^n, \quad \max_{G \times [0,T]} v_{me/az}^n / \rho | R^2 / \eta_0 =: V_{me/az}^n^*,
\]

\[
\max \{V_{az}^*, V_{az}^{n*}\} =: V_{az}^*, \quad \max \{V_{me}, V_{az}\} =: V.
\]

Finally, violation of the divergence-constraint is measured by

\[
\max_{G \times [0,T]} |\nabla \cdot v^n | R^2 / \eta_0 =: DV^n.
\]

The following functionals of the magnetic field and/or its representing scalars are essential for the subsequent theorems:

\[
\mathcal{F}_3[M, N] := \frac{1}{2\eta_0} \left\{ \|M\|_G^2 + \nu^2 R^2 \|N\|_G^2 \right\}.
\]
\[ (2.15) \quad \mathcal{F}_2[M, N, A, B] := \frac{1}{2\eta_0} \left\{ \frac{1}{R^4} \| M \|_G^2 + \frac{\nu^2}{R^2} \| N \|_G^2 + \alpha^2 R^2 \| A \|_G^2 + \beta^2 \| B \|_\infty^2 \right\}, \]

\[ (2.16) \quad \mathcal{F}_3[N, A, B] := \frac{1}{2\eta_0} \left\{ \frac{1}{R^2} \| N \|_G^2 + \alpha^2 R^2 \| A \|_G^2 + \beta^2 \| B \|_\infty^2 \right\}, \]

\[ (2.17) \quad \tilde{\mathcal{F}}_3[N, A, B; Q, P] := \frac{1}{2\eta_0} \left\{ \frac{1}{R^2 \bar{Q}} \| NQ^{1/2} \|_G^2 + \frac{\alpha^2 R^2}{\bar{P}} \left\| \frac{A}{P^{1/2}} \right\|_G^2 + \beta^2 \| B \|_\infty^2 \right\}. \]

Here, \( \nu, \alpha, \) and \( \beta \) are constants that can explicitly be expressed by the bounds (2.9)–(2.12), whereas the functions \( Q \) and \( P \) with upper bounds \( \bar{Q} \) and \( \bar{P} \) appearing in \( \tilde{\mathcal{F}}_3 \) are solutions of certain auxiliary problems discussed in section 6. We are now in the position to state our results:

**Theorem 1.** Let \( M \) and \( N \) be the meridional scalars of the axisymmetric part of a magnetic field solving the dynamo problem (3.1). Then, \( F_1(t) = F_1[M(\cdot, t), N(\cdot, t)] \) with \( \nu^{-1} = V_\rho^a \geq 2 \) decays according to

\[ (2.18) \quad F_1(t) \leq F_1(0) e^{-((\nu_0/R^2)(1-D_1))t} \quad \text{on} \quad [0, T], \]

provided that

\[ (2.19) \quad D_1 := DK^a + 4(\delta_1 \eta^a + \delta_2 \eta^a) + 4 r_{me}^{mag} V_\rho^a(V_{me}^n + E^n + \delta \eta^n) < 1. \]

A more detailed expression for \( D_1 \) can be found in section 4 (eqs. (4.14)–(4.16)).

**Theorem 2.** Let \( M, N \) be the meridional scalars and \( A \) be the modified azimuthal scalar of the axisymmetric part of a magnetic field \( B \) solving the dynamo problem (3.1). Then, \( F_2(t) = F_2[M(\cdot, t), N(\cdot, t), A(\cdot, t), B(\cdot, t)] \) decays according to

\[ (2.20) \quad F_2(t) \leq F_2(0) e^{-((\nu_0/R^2)(1-D_2))t} \quad \text{on} \quad [0, T], \]

provided that

\[ (2.21) \quad D_2 := 2DK^a + 6 \delta_1 \eta^a + 8 \delta_2 \eta^a + 2 r_{me}^{mag}(V_\rho^a + 2) V_{me}^n + (1 + r_{me}^{mag})(3(1 + r_{me}^{mag})V_{az}^* + 8)(V_\rho^a + 1)VE^n < 1. \]

More detailed expressions for \( D_2 \) (eqs. (5.9) or (5.10)) and an even simpler one (eq. (5.11)), as well as explicit expressions for the constants \( \nu, \alpha, \) and \( \beta \) appearing in \( F_2 \) can be found in section 5.

**Theorem 3.** Let \( N \) be the modified meridional scalar and \( A \) be the modified azimuthal scalar of the axisymmetric part of a magnetic field \( B \) solving the dynamo problem (3.1). Then, \( F_3(t) = F_3[N(\cdot, t), A(\cdot, t), B(\cdot, t)] \) decays according to

\[ (2.22) \quad F_3(t) \leq C F_3(0) e^{-((\nu_0/C R^2)(1-D_3))t} \quad \text{on} \quad [0, T], \]

provided that

\[ (2.23) \quad D_3 := 2Cr_{me}^{mag} V_{me}^n + 2 C^2 (1 + r_{me}^{mag})^2 (V_{me}^n + V_{az}^*)^2 E^n < 1. \]

The constants \( \alpha \) and \( \beta \) appearing in \( F_3 \) are given explicitly in section 6. The constant \( C \) is related to the auxiliary functions appearing in \( \tilde{F}_3 \), which is a crucial ingredient.
in the proof of the theorem. $C$ depends on the axisymmetric bounds $V^a$, $V^{a*}$, $\delta_1 \eta^a$, and the domain (see section 6).

Some comments on the theorems are in order:

1. There is a basic difference between the axisymmetric theorem and approximately axisymmetric theorems. In the former case exact symmetry of the magnetic field results in unconditional decay to zero of the magnetic field $B^a = B^a$ for all time. In the latter case, a however small nonaxisymmetric part $B^n$ of the magnetic field can well be amplified by a (even axisymmetric) flow field of suitable form and sufficient strength. The interaction of $B^n$ with $v^n$ or $J^n$ with $\eta^n$ then provide source terms for the growth of $B^a$. Therefore, in the presence of (initially however small) nonaxisymmetric flow or diffusivity and of nonaxisymmetric magnetic field components, decay of $B^a$ for all time can no longer be expected. In other words, even in the kinematic setting, where $v^n$ and $\eta^n$ can be controlled for all time, $B^n$ and hence the magnetic energy ratios $r_{me/az}^{mag}$ cannot. The statements of theorems 1–3 refer, therefore, only to those time intervals $[0, T]$, where the respective conditions (2.19), (2.21), or (2.23) are satisfied.

2. In the case of strict axisymmetry, i.e. $B^n = 0$, $v^n = 0$, and $\eta^n = 0$, theorems 1–3 reproduce in principle known features of the axisymmetric theorem by partly novel methods (cf. KT14): Setting $r_{me}^{mag} = V_{me}^{n*} = E^{n*} = 0$, condition (2.19) boils down to

$$DV^a + 4(\delta_1 \eta^a + \delta_2 \eta^a) < 1$$

and similarly for (2.21), whereas (2.23) vanishes altogether. So, theorems 1 and 2 apply to the case of a weakly compressible fluid of weakly variable diffusivity, whereas theorem 3 applies to the fully compressible/variable case. Theorem 1 then yields monotonic decay to zero of the meridional scalar, whereas theorems 2 and 3 provide (not necessarily monotonic) decay of both scalars and of the magnetic field itself to zero on a possibly very large time scale. These results hold for all time.

3. The energy balances of the magnetic field and of the representing scalars $M$ and $A$ of its axisymmetric part are decisive for the choice of the functionals (2.14)–(2.16). To obtain a decay result the interaction terms in the functional balance must be dominated by the dissipative terms. This is impossible for the $M$-balance alone and makes the inclusion of the “higher order” variable $N$ in $F_1$ necessary. Note that indeed $N$ approximates $\partial \rho M$ close to the symmetry axis. On the other hand the energy balance of $N$ alone contains axisymmetric coupling terms that are not small and that can only be compensated by the inclusion of $M$ (see section 4).

Similarly, in the $A$-balance a derivative-counting argument shows that the interaction terms cannot be dominated by the dissipative terms in this balance alone; inclusion of the higher-order field $B$, however, can remedy this problem as demonstrated for $F_2$ in section 5.

$F_3$ is dominated by $F_3$ and it is the latter functional that adds a surplus value to the result based on $F_2$: $F_3$ differs from $F_2$ by the inclusion of two more dynamic variables solving certain auxiliary axisymmetric problems with the effect that the axisymmetric parts of flow field and diffusivity variation do not appear in the $F_3$-balance. So, conditions on these quantities are no longer necessary and also the scalar $M$ turns out to be dispensable (see section 6).

4. Note that in all three conditions (2.19), (2.21), and (2.23) the bounds $V_{me}^{n}$ and $V_{me}^{n*}$ on the nonaxisymmetric part of the meridional flow are multiplied by the bounds $r_{me/az}^{mag}$ on the nonaxisymmetric part of the meridional/azimuthal magnetic field. This
mirrors the structure of the nonaxisymmetric source terms in the axisymmetric scalar evolution equations and means, in particular, that in order to specify these conditions one of the bounds can still be large if only the other is sufficiently small. This effect applies also to \(r_{me}^{mag}\) and the nonaxisymmetric variation \(E^n + \delta \eta^n\) of the magnetic diffusivity in condition (2.19), but does not apply to the conditions (2.21) and (2.23), where \(E^{n*}\) needs to be small.

5. Comparing theorems 1 and 2, which both apply to weakly compressible fluids of weakly variable diffusivity, one obtains monotonic decay of the meridional scalar in the former case and monotonic decay of the bounding functional \(F_2\) that contains both scalars and the magnetic field itself in the latter case. Note, however, that decay of \(F_2\) does not imply decay of every variable contained in \(F_2\) individually. Especially the nonaxisymmetric part \(B^a\) of \(B\) may initially grow whereas the axisymmetric variables decay. Ultimately, \(B^a\) must also decay – provided that the magnetic energy ratio \(r_{me}^{mag}\) remains in a range that leaves condition (2.21) fulfilled; otherwise theorem 2 is no longer applicable, which means that even \(B^a\) may then grow (see first comment).

6. Theorem 3 is our most general decay result: a restriction on \(\nabla \cdot \mathbf{v}\) does not appear in condition (2.23) and axisymmetric variations of the magnetic diffusivity may be arbitrarily large. On the other hand the constant \(C\) in (2.23), resp. \(C_{me/az}\) in the more detailed condition (6.10), still depends on axisymmetric flow and diffusivity bounds, which we do not make fully explicit. We have a closer look merely on the dependence on \(V_{me}^a\) in section 6, where we find the exponential dependence \(C \sim a^{V_{me}^a}\) for some \(a > 0\) and \(V_{me}^a \gtrsim 10\). For quite moderate values of \(V_{me}^a\), condition (2.23) is thus much more severe than conditions (2.19) or (2.21), which exhibit merely a (low order) polynomial dependence on \(V\).

7. Theorems 1–3 are formulated for axisymmetric conducting domains, mainly because boundary and transition conditions for the axisymmetric scalars are simple only at axisymmetric boundaries. However, theorem 3 admits the possibility to model weakly nonaxisymmetric conducting domains by suitably chosen “diffusivity wells”. For example, consider in cylindrical coordinates \((\rho, \phi, z)\) the diffusivity distribution \(\eta : B_R \to \mathbb{R}_+\) with

\[
\eta(\rho, \phi, z) := \eta^a(\rho, z) + \frac{1}{\epsilon} \left(1 + \tanh \left(\frac{g(\rho, \phi, z)}{\epsilon}\right)\right), \quad g(\rho, \phi, z) := g^a(\rho, z) + \epsilon^2 \delta g^a(\rho, \phi, z),
\]

where \(\epsilon\) and \(\delta\) are (small) parameters and \(\eta^a\), \(g^a\), and \(g^a\) are smooth functions, which together with their derivatives are of order one. Let, furthermore, \(g^a\) such that \(g^a = 0\) describes the boundary \(\Gamma^a\) of a smooth axisymmetric domain \(G^a \subset B_R\) and \(\mathbf{n} \cdot \nabla g^a|_{\Gamma^a} \geq 1\), where \(\mathbf{n}\) means the exterior normal at \(\Gamma^a\). For \(\delta \leq 1\) and small \(\epsilon\), \(g = 0\) then describes the boundary \(\Gamma\) of a weakly nonaxisymmetric domain \(G\) and \(\eta\) exhibits a sharp increase at \(\Gamma\). Note that \(\nabla_{me} \eta|_{\Gamma} \sim 1/\epsilon^2\) becomes necessarily large, but \(\partial_{\mathbf{n}} \eta|_{\Gamma} \sim \delta\) can be kept small. As to condition (2.23), large values of \(\delta_1 \eta^a\) and hence (possibly) of the constant \(C\) can thus be compensated by small values of \(E^{n*}\), i.e. for sufficiently small \(\delta\).

8. Concerning applications one should distinguish between stellar and planetary dynamos with typically very large magnetic Reynolds numbers, numerical simulations of these objects using lower “effective” Reynolds numbers, and laboratory experiments with Reynolds numbers of order one. In the first case numbers \(V_{me}^a \gg 1\), \(V_{me} \gg 1\), or \(V_{az} \gg 1\) admit only small values of the nonaxisymmetric bounds in conditions (2.19), (2.21), or (2.23) in order that the respective theorems be applicable. In other words, according to these theorems, already a small amount of nonaxisymmetry allows a
celestial body to escape the restrictions of the axisymmetric theorem. This may be different in numerical simulations, where conditions of type (2.19) could be valuable for the design and interpretation of computational models.

One should keep in mind, however, that theorems 1–3 formulate only sufficient conditions for the axisymmetric theorem to be applicable. More detailed energy balances, finer norms, or sharper estimates could well weaken these conditions and thus extend the range of applicability of the axisymmetric theorem. Especially within the third class of applications, experimental dynamos, it could well be worth taking into account the form of the fluid container, boundary conditions, or specific features of the flow field, in order to obtain improved conditions. These could be helpful to keep the design of the experiment simple, i.e. close but not too close to axisymmetry.

3. Basic equations, representations, and estimates. The kinematic dynamo problem in a bounded domain $G \subset \mathbb{R}^3$ surrounded by vacuum in $\hat{G} := \mathbb{R}^3 \setminus G$ can be summarized by the following set of equations (Backus 1958):

$$
\begin{aligned}
\partial_t \mathbf{B} &= -\nabla \times (\eta \nabla \times \mathbf{B}) + \nabla \times (\mathbf{v} \times \mathbf{B}), \quad \nabla \cdot \mathbf{B} = 0 \quad \text{in } G \times \mathbb{R}^+, \\
\nabla \times \mathbf{B} &= 0, \quad \nabla \cdot \mathbf{B} = 0 \quad \text{in } \hat{G} \times \mathbb{R}^+, \\
\mathbf{B} &= \text{continuous} \quad \text{in } \mathbb{R}^3 \times \mathbb{R}^+, \\
\mathbf{B}(x, \cdot) &= O(|x|^{-3}) \quad \text{for } |x| \to \infty, \\
\mathbf{B}(\cdot, 0) &= \mathbf{B}_0, \quad \nabla \cdot \mathbf{B}_0 = 0 \quad \text{on } G \times \{t = 0\}.
\end{aligned}
$$

(3.1)

The following analysis presupposes axisymmetry only for the domain $G$, whereas all other quantities, the magnetic field $\mathbf{B}$, the flow field $\mathbf{v}$, and the diffusivity distribution $\eta$ are decomposed according to (2.4) into axisymmetric and nonaxisymmetric parts. The first step consists in deducing from (3.1) the meridional and azimuthal subproblems for the (modified) scalars in the fundamental decomposition (2.6) of $\mathbf{B}^a$. Only the three-dimensional formulation matters here (with one exception, see the appendix). Issues that have already been discussed in detail in (KT14) and that need not to be modified are just reported.

Starting point is an azimuthally averaged version of the dynamo equation (3.1)$_{1a}$. One obtains by (2.2)–(2.5) for the axisymmetric part:

$$
\partial_t \mathbf{B}^a + \nabla \times (\eta^a \nabla \times \mathbf{B}^a - \mathbf{v}^a \times \mathbf{B}^a) = -\{\nabla \times (\eta^a \nabla \times \mathbf{B}^a - \mathbf{v}^a \times \mathbf{B}^a\}.
$$

(3.2)

Inserting for $\mathbf{B}^a$ the decomposition (2.6) and using the abbreviations $\mathbf{E}^a$ and $\mathbf{E}^n$ for the parentheses on the left-hand and the right-hand side of (3.2), respectively, one obtains componentwise

$$
\begin{aligned}
\partial_z (\partial_t M + \rho \mathbf{E}^a \cdot \mathbf{e}_\phi) &= -\partial_z \langle \rho \mathbf{E}^n \cdot \mathbf{e}_\phi \rangle, \\
\partial_t A + \rho \nabla \times \mathbf{E}^a \cdot \mathbf{e}_\phi &= -\langle \rho \nabla \times \mathbf{E}^n \cdot \mathbf{e}_\phi \rangle, \\
\partial_\rho (\partial_t M + \rho \mathbf{E}^a \cdot \mathbf{e}_\phi) &= -\partial_\rho \langle \rho \mathbf{E}^n \cdot \mathbf{e}_\phi \rangle.
\end{aligned}
$$

(3.3)

From (3.3)$_{1,3}$ one concludes

$$
\partial_t M + \rho^2 (\eta^a \nabla \times (\nabla M \times \nabla \phi + A \nabla \phi) - \mathbf{v}^a \times (\nabla M \times \nabla \phi + A \nabla \phi)) \cdot \nabla \phi
= -\langle \rho (\eta^a \nabla \times \mathbf{B}^a - \mathbf{v}^a \times \mathbf{B}^n) \cdot \mathbf{e}_\phi \rangle + c(t),
$$

(3.4)

\footnote{We take the opportunity to correct a mistake in eq. (2.3)$_2$ in (KT14), where the factor $\rho$ in front of $\nabla \times \mathbf{E}^a \cdot \mathbf{e}_\phi$ is missing. Subsequent calculations made use of the correct expression.}
where $c(t)$ is an integration “constant”. Evaluating the left-hand side of (3.4), re-defining $M$ to eliminate $c(t)$, and introducing the electric current $\mathbf{J} = \nabla \times \mathbf{B}$ yields the first equation of the following system representing the meridional subproblem:

\[
\begin{align*}
\partial_t M - \eta^2 (\Delta M - 2 \nabla \rho \cdot \nabla M / \rho) + \mathbf{v} \cdot \nabla M &= -\rho \langle \eta^2 \mathbf{J} \cdot \mathbf{e}_\phi - \mathbf{v} \times \mathbf{B} \cdot \mathbf{e}_\phi \rangle \quad \text{in } G \times \mathbb{R}_+,
\Delta M - 2 \nabla \rho \cdot \nabla M / \rho &= 0 \quad \text{in } \hat{G} \times \mathbb{R}_+,
M(x, y, \cdot, \cdot) = O(\rho^2), \nabla M(x, y, \cdot, \cdot) = O(\rho) \quad \text{for } \rho \to 0,
M(x, \cdot) = O(|x|^{-1}), \nabla M(x, \cdot) = O(|x|^{-2}) \quad \text{for } |x| \to \infty,
M(\cdot, 0) = M_0 \quad \text{on } G \times \{t = 0\}.
\end{align*}
\]

The modified meridional scalar $N = M / \rho$ cannot be controlled by $M$ and appears additionally to $M$ in $F_1$ and $F_2$, and instead of $M$ in $F_3$. The governing system derives directly from (3.5) and reads:

\[
\begin{align*}
\partial_t N - \eta^2 (\Delta N - N / \rho^2) + \mathbf{v} \cdot \nabla N + \nabla \rho \cdot \mathbf{v}^a N / \rho &= -\rho \langle \eta^2 \mathbf{J} \cdot \mathbf{e}_\phi - \mathbf{v} \times \mathbf{B} \cdot \mathbf{e}_\phi \rangle \quad \text{in } G \times \mathbb{R}_+,
\Delta N - N / \rho^2 &= 0 \quad \text{in } \hat{G} \times \mathbb{R}_+,
N(x, y, \cdot, \cdot) = O(\rho), \nabla N(x, y, \cdot, \cdot) = O(1) \quad \text{for } \rho \to 0,
N(x, \cdot) = O(|x|^{-2}), \nabla N(x, \cdot) = O(|x|^{-3}) \quad \text{for } |x| \to \infty,
N(\cdot, 0) = N_0 \quad \text{on } G \times \{t = 0\}.
\end{align*}
\]

On the other hand, evaluating (3.3)$_2$ yields the following evolution equation for the azimuthal scalar $A$:

\[
\begin{align*}
\partial_t A - \rho^2 \nabla \cdot \left( \frac{\eta^2 / \rho^2}{\nabla A} \right) + \rho^2 \nabla \cdot \left( \frac{\mathbf{v}^a A / \rho^2}{\rho} \right) - \rho^2 \nabla \cdot \left( \mathbf{B}^a_{\text{me}} \frac{v^a_\phi}{\rho} \right) &= -\langle \rho \nabla \times (\eta^2 \mathbf{J}) \cdot \mathbf{e}_\phi + (\rho \nabla \times (\mathbf{v}^n \times \mathbf{B}^n) \cdot \mathbf{e}_\phi),
\end{align*}
\]

which suggests to introduce the modified scalar $A := A / \rho^2$. In this variable the azimuthal subproblem takes then the form

\[
\begin{align*}
\partial_t A - \nabla \cdot \left( \frac{\eta^2 / \rho^2}{\nabla (\rho^2 A)} \right) + \nabla \cdot (\mathbf{v}^a A) - \nabla \cdot \left( \mathbf{B}^a_{\text{me}} \frac{v^a_\phi}{\rho} \right) &= -\nabla \cdot (\eta^2 \mathbf{J} \times \mathbf{e}_\phi / \rho) + \nabla \cdot ((\mathbf{v}^n \times \mathbf{B}^n) \times \mathbf{e}_\phi / \rho) \quad \text{in } G \times \mathbb{R}_+,
A(x, y, \cdot, \cdot) = O(1) \quad \text{for } \rho \to 0,
A(\cdot, 0) = A_0 \quad \text{on } \partial G \times \mathbb{R}_+,
A(\cdot, 0) = A_0 \quad \text{on } G \times \{t = 0\}.
\end{align*}
\]

In subsequent calculations frequent use is made of the following variational inequalities
for $M$ and $A$, respectively (see KT14):

\begin{align}
\lambda_1 \int_G M^2 dx & \leq \int_{\mathbb{R}^3} \left| \nabla M \right|^2 dx, \\
\mu_1 \int_G A^2 dx & \leq \int_{\mathbb{G}^3} \left| \nabla A \right|^2 dx + 2\pi \int_L A^2 dx,
\end{align}

where the line integral in (3.9) extends over that part of the symmetry axis that is covered by $G$. The numbers $\lambda_1 = \lambda_1(G)$ and $\mu_1 = \mu_1(G)$ depend still on $G$ and especially on the size of $G$; suitable nondimensional quantities based on the radius $R$ of the smallest ball $B_R$ enclosing $G$ are

\begin{align}
\lambda_0 = \lambda_0(G) := \sqrt{\lambda_1(G)} R, & \quad \mu_0 = \mu_0(G) := \sqrt{\mu_1(G)} R
\end{align}

with (absolute) lower bounds

\begin{align}
\lambda_0(G) & \geq \pi/2, \quad \mu_0(G) \geq \pi.
\end{align}

In (3.8) $G$ may be replaced by any ball $B_R$ enclosing $G$; in that case we define $\lambda_0(B_R) := \sqrt{\lambda_1(B_R)} R$. Other frequently used inequalities are that of Cauchy–Schwarz,

\begin{align}
\int u \cdot v \, dx & \leq \left( \int |u|^2 dx \right)^{1/2} \left( \int |v|^2 dx \right)^{1/2} = \|u\| \|v\|,
\end{align}

where the range of integration is $G, \hat{G}, \mathbb{R}^3$, or $B_R$, and that of Young in the form

\begin{align}
ab \leq \frac{\epsilon}{2} a^2 + \frac{1}{2\epsilon} b^2, \quad \epsilon > 0.
\end{align}

4. **Meridional decay.** The result of this section is based on a suitable combination of the energy balances of the meridional scalars $M$ and $N$. To avoid ambiguities due to the singular coefficients in these balances integration is first performed over regularized domains $G_\epsilon$ and $\tilde{G}_\epsilon$, where a solid cylinder around the symmetry axis is removed, and then the cylinder is shrunk to the axis. So, with $\epsilon > 0$ and the cylindrical distance $\rho = \rho(x)$ we define the domains $G_\epsilon := \{ x \in G : \rho(x) > \epsilon \}$ and $G_\epsilon := \{ x \in G : \rho(x) > \epsilon \}$, the surfaces $C_\epsilon := \{ x \in G : \rho(x) = \epsilon \}$, $\tilde{C}_\epsilon := \{ x \in \hat{G} : \rho(x) = \epsilon \}$ and $S_\epsilon := \{ x \in \partial G : \rho(x) \geq \epsilon \}$, and the line $L := \{ x \in G : \rho(x) = 0 \}$ (see Figure 4.1). We have then $\partial G_\epsilon = S_\epsilon \cup C_\epsilon$ and $\partial \tilde{G}_\epsilon = S_\epsilon \cup \tilde{C}_\epsilon$; note that in the limit $\epsilon \to 0$, $C_\epsilon$ shrinks to $L$.

The $M$-balance is now obtained by multiplying (3.5) by $M$ and integrating over $G_\epsilon$; integrating by parts using thereby $J^n = \nabla \times B^n$, $\nabla \cdot (\nabla \rho / \rho) = 0$, and the boundary condition $\mathbf{n} \cdot \mathbf{v} = 0$ at $S_\epsilon$ yields:

\begin{align}
\frac{1}{2} \frac{d}{dt} \int_{G_\epsilon} M^2 dx & = - \int_{G_\epsilon} \eta^n |\nabla M|^2 dx - \int_{G_\epsilon} \nabla \eta^n \cdot \left( \nabla MM - \frac{\nabla \rho}{\rho} M^2 \right) dx \\
& + \frac{1}{2} \int_{G_\epsilon} \nabla \cdot v^n M^2 dx + \int_{G_\epsilon} \nabla (\rho \eta^n M) \cdot B^n \times \mathbf{e}_\phi \, dx + \int_{G_\epsilon} \rho v^n \cdot B^n \times \mathbf{e}_\phi \, M \, dx \\
& + \int_{\partial G_\epsilon} \eta^n \left( \nabla MM - \frac{\nabla \rho}{\rho} M^2 \right) \cdot \mathbf{n} \, ds + \frac{1}{2} \int_{C_\epsilon} M^2 \nabla \rho \cdot v^n \, ds \\
& - \int_{\partial G_\epsilon} \rho \eta^n M B^n \times \mathbf{e}_\phi \cdot \mathbf{n} \, ds.
\end{align}
To cope with the boundary terms at $S_\epsilon$ we introduce continuous extensions $\tilde{\eta}^a$ and $\tilde{\eta}^n$ of $\eta^a$ and $\eta^n$, respectively, onto $\mathbb{R}^3$ with the properties: $\eta^0 \leq \tilde{\eta}^a \in C^1((\hat{G} \cup \hat{\hat{G}}) \times [0,T])$, $\nabla \tilde{\eta}^a = 0$ outside some ball $B_{\tilde{R}}$, $\eta^0 \leq \tilde{\eta}^n \in C^1((G \cup \hat{G}) \times [0,T])$, $\tilde{\eta}^n = 0$ outside some ball $B_{\tilde{\tilde{R}}}$, and, most importantly, $\tilde{\eta}^a$ and $\tilde{\eta}^n$ do not increase the bounds (2.11). In the case $\eta^a|_{\partial G} = \text{const}$ we can choose $\tilde{R} = R$, the radius of the smallest ball enclosing $G$ and analogously in the case $\eta^n|_{\partial G} = 0$, otherwise we have $\tilde{R}, \tilde{\tilde{R}} > R$ depending on the variation of $\eta^{a,n}$ at $\partial G$; by and large

\begin{equation}
\tilde{R} \leq 2R, \quad \tilde{\tilde{R}} \leq 2R
\end{equation}

seems to be enough to guarantee extensions of the above kind.

Using $\bar{\eta}^a$ and (3.5) a calculation analogous to (4.1) yields in $\hat{G} \cap \hat{B}_{\tilde{R}}$:

\begin{align}
0 &= \int_{\hat{G} \cap \hat{B}_{\tilde{R}}} \bar{\eta}^a \left( \Delta M - 2 \frac{\nabla \rho \cdot \nabla M}{\rho} \right) M \, dx \\
&= -\int_{\hat{G} \cap \hat{B}_{\tilde{R}}} \bar{\eta}^a |\nabla M|^2 \, dx - \int_{\hat{G} \cap \hat{B}_{\tilde{R}}} \nabla \bar{\eta}^a \cdot \left( \nabla MM - \frac{\nabla \rho M^2}{\rho} \right) \, dx \\
&\quad - \int_{S_\epsilon} \bar{\eta}^a \left( \nabla MM - \frac{\nabla \rho M^2}{\rho} \right) \cdot n \, ds - \int_{\hat{\hat{G}} \cap \hat{B}_{\tilde{\tilde{R}}} \cap \{ \rho > \epsilon \}} \bar{\eta}^a \left( \nabla \rho \cdot \nabla MM - \frac{M^2}{\rho} \right) \, ds \\
&\quad + \int_{\partial \hat{B}_{\tilde{R}} \cap \{ \rho > \epsilon \}} \bar{\eta}^a \left( \nabla MM - \frac{\nabla \rho M^2}{\rho} \right) \cdot \frac{x}{\tilde{R}} \, ds,
\end{align}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure41}
\caption{Notation for the regularized domain $G_\epsilon$.}
\end{figure}
and with $\bar{\eta}^n$ and (3.1)$_{2e}$ follows:

$$0 = -\int_{G \cap B_{\bar{R}}} \rho \bar{\eta}^n M \nabla \times \mathbf{B}^n \cdot \mathbf{e}_{\phi} \, dx$$

(4.4)

$$= \int_{G \cap B_{\bar{R}}} \nabla (\rho \bar{\eta}^n M) \cdot \mathbf{B}^n \times \mathbf{e}_{\phi} \, dx + \int_{S_\epsilon} \rho \bar{\eta}^n M \mathbf{B}^n \times \mathbf{e}_{\phi} \cdot \mathbf{n} \, ds$$

$$+ \int_{C_\epsilon \cap B_{\bar{R}}} \rho \bar{\eta}^n M \mathbf{B}^n \cdot \mathbf{e}_z \, ds - \int_{\partial B_{\bar{R}} \cap (\rho > \epsilon)} \rho \bar{\eta}^n M \mathbf{B} \times \mathbf{e}_{\phi} \cdot \frac{x}{\bar{R}} \, ds.$$

Summing up (4.1), (4.3), and (4.4), the surface integrals over $S_\epsilon$ cancel each other, those over $\partial B_{\bar{R}}$ vanish in the limit $\bar{R} \to \infty$ when observing (3.1)$_4$ and (3.5)$_5$, and those along $C_\epsilon$ and $\tilde{C}_\epsilon$ vanish in the limit $\epsilon \to 0$ observing (3.5)$_4$; it remains

$$\frac{1}{2} \frac{d}{dt} \int_G M^2 \, dx = -\int_{\mathbb{R}^3} \bar{\eta}^n |\nabla M|^2 \, dx - \int_{B_{\bar{R}}} \left( \nabla \bar{\eta}^n \cdot \nabla M M - \frac{\nabla \rho}{\rho} \cdot \nabla \bar{\eta}^n M^2 \right) \, dx$$

(4.5)

$$+ \frac{1}{2} \int_G \nabla \cdot \mathbf{v}^n M^2 \, dx + \int_{B_{\bar{R}}} \left( (\rho \nabla_{me} \bar{\eta}^n + \bar{\eta}^n \nabla \rho) M + \rho \bar{\eta}^n \nabla M \right) \cdot \mathbf{B}^n \times \mathbf{e}_{\phi} \, dx$$

$$+ \int_G \rho \mathbf{v}^n \cdot \mathbf{B}^n \times \mathbf{e}_{\phi} \, M \, dx,$$

where, as already noted, $\nabla_{me}$ means the meridional part of $\nabla$. Estimating (4.5) by (3.12) and using the notation (2.8) and the bounds (2.10)–(2.12) one obtains

(4.6)

$$\frac{1}{2} \frac{d}{dt} \|M\|^2_G$$

$$\leq \eta_0 \left\{ -\|\nabla M\|_\infty^2 + \delta_1 \eta^n \|\nabla M\|_\infty \|M\|_{\bar{R}/R} + \delta_2 \eta^n \|M\|_{\bar{R}/R}^2/R^2 + D\mathbf{n} \|M\|^2_G/(2R^2)$$

$$+ \left( (\delta \eta^n \bar{\eta} / R + E^n) \|M\|_{\bar{R}/R} + E^n \|\nabla M\|_\infty \bar{\eta} \|\mathbf{B}^n_{me}\|_\infty + V^n \|M\|_G \|\mathbf{B}^n_{me}\|_\infty \right) \right\}$$

$$\leq \eta_0 \left\{ [\delta_1 \eta^n \lambda_0^{-1} \bar{R}/R + \delta_2 \eta^n \lambda_0^{-2} (\bar{R}/R)^2 + D\mathbf{n} \lambda_0^{-2}/2 - 1] \|\nabla M\|_\infty^2$$

$$+ [V^n \lambda_0^{-1} \bar{R}/R + E^n (1 + \lambda_0^{-1}) \bar{R}/R + \delta \eta^n \lambda_0^{-1} (\bar{R}/R)^2] \|\nabla M\|_\infty \|\mathbf{B}^n_{me}\|_\infty \right\}.$$

In the last estimate we made moreover use of (2.9), (3.8), and (3.10)$_{1e}$. It is the term $\|\mathbf{B}^n_{me}\|_\infty = \|\frac{1}{\rho} \nabla M\|_\infty$ in (4.6), which cannot be controlled by $\|\nabla M\|_\infty$, that prevents a meridional decay result based on $M$ alone and that makes the inclusion of the "higher order" meridional scalar $N$ necessary. Starting from (3.6) a calculation
analogous to that for $M$ yields now instead of (4.1):

$$\frac{1}{2} \frac{d}{dt} \int_{G_e} N^2 \, dx = -\int_{G_e} \eta^a \left( |\nabla N|^2 + \left( \frac{N}{\rho} \right)^2 \right) \, dx - \int_{G_e} \nabla \eta^a \cdot \nabla N \, dx$$

$$+ \int_{G_e} \left( \frac{1}{2} \nabla \cdot v^a - \frac{\nabla \rho}{\rho} \cdot v^a \right) N^2 \, dx + \int_{G_e} \nabla (\eta^aN) \cdot B^n \times e_\phi \, dx$$

$$+ \int_{\partial G_e} \eta^a N \nabla N \cdot n \, ds + \frac{1}{2} \int_{G_e} N^2 \nabla \rho \cdot v^a \, ds - \int_{\partial G_e} \eta^a N B^n \times e_\phi \cdot n \, ds. \tag{4.7}$$

By (3.6)$_{2-5}$ and in the limit $\epsilon \to 0$ the surface terms in (4.7) vanish as for $M$ with the result (analogous to (4.5)):

$$\frac{1}{2} \frac{d}{dt} \int_{G} N^2 \, dx = -\int_{\mathbb{R}^3} \eta^a \left( |\nabla N|^2 + \left( \frac{N}{\rho} \right)^2 \right) \, dx - \int_{B_R} \nabla \eta^a \cdot \nabla N \, dx$$

$$+ \int_{G} \left( \frac{1}{2} \nabla \cdot v^a - \frac{\nabla \rho}{\rho} \cdot v^a \right) N^2 \, dx + \int_{B_R} (\nabla \phi \nabla N + \eta^a \nabla N) \cdot B^n \times e_\phi \, dx$$

$$+ \int_{G} v^a \cdot B^n \times e_\phi \, N \, dx. \tag{4.8}$$

Integrating by parts and using once more (3.6)$_{2-5}$ reveals for the dissipation term in (4.8) without $\eta^a$ the identity:

$$\int_{\mathbb{R}^3} \left( |\nabla N|^2 + \left( \frac{N}{\rho} \right)^2 \right) \, dx = \int_{\mathbb{R}^3} \frac{1}{\rho^2} |\nabla (\rho N)|^2 \, dx = \int_{\mathbb{R}^3} |B_{me}^a|^2 \, dx. \tag{4.9}$$

So, the $N$-balance may provide the missing term in the $M$-balance; on the other hand, a pure $N$-balance is not appropriate either, since the $v^a_n$-related term in (4.8) need not be small and can only be controlled in a suitable combination of both balances.

Using again the notation (2.8) and estimating (4.8) as in (4.6) one obtains

$$\frac{1}{2} \frac{d}{dt} \|N\|_G^2 \leq \eta_0 \left\{ -\|B_{me}^a\|_\infty^2 + \delta_1 \eta_0 \|\nabla N\|_\infty \|N\|_\bar{R}/R + DV^a/\|N\|_\infty \|G\| \|B_{me}^a\|_\infty / R \right\}$$

$$+ V^a, \|\rho N\|_G / \|G\| / \rho / R^2 + V_{me} \|N\|_G \|B_{me}^a\|_\infty / R$$

The last estimate we made repeated use of

$$\|N\|_G \leq R \|N/\rho\|_G \leq R \|N/\rho\|_\infty \leq R \|B_{me}^a\|_\infty$$

and analogous estimates for $\|N\|_\bar{R}$ and $\|N\|_\infty$; note also that we introduced $\|\nabla M\|_\infty$ by

$$\|\rho N\|_G = \|M\|_G \leq \lambda_0^{-1} R \|\nabla M\|_\infty.$$
We are now in the position to derive the time-derivative $\frac{d}{dt} F_1$ of the functional (2.14). By (4.6) and (4.10) one obtains:

\begin{equation}
\frac{d}{dt} F_1 \leq C_1 \|\nabla M\|_\infty^2 + C_2 \nu^2 R^2 \|B^a\|_\infty^2
\end{equation}

with

\[
C_1 := \frac{1}{2} DV^a \lambda_0^{-2} + \delta_1 \eta^a \lambda_0^{-1} \bar{R}/R + \delta_2 \eta^a \lambda_0^{-2} (\bar{R}/R)^2 + \frac{c_1}{2} \nu^2 V^a_\rho \lambda_0^{-1} + \frac{c_2}{2} [\nu_{me} \lambda_0^{-1} + E^n (1 + \lambda_0^{-1}) \bar{R}/R + \delta \eta^n \lambda_0^{-1} (\bar{R}/R)^2] r_{me}^{mag} - 1
\]

and

\[
C_2 := \frac{1}{2} DV^a + \frac{1}{2} \delta_1 \eta^a \bar{R}/R + \frac{1}{2} V^a_\rho \lambda_0^{-1} + [\nu_{me} + E^n + \delta \eta^n \bar{R}/R] r_{me}^{mag} + \frac{1}{2} \nu^2 [\nu_{me} \lambda_0^{-1} + E^n (1 + \lambda_0^{-1}) \bar{R}/R + \delta \eta^n \lambda_0^{-1} (\bar{R}/R)^2] r_{me}^{mag} - 1.
\]

Here, the mixed terms in (4.6) and (4.10) have been split by (3.13) introducing thereby the further parameters $\epsilon_1$ and $\epsilon_2$. These parameters together with $\nu$ are now chosen such that $C_1$ and $C_2$ have a common (as low as possible) upper bound; especially $V^a_\rho$, which need not be small, must be controlled. This is achieved by the choice

\begin{equation}
\epsilon_1 = \epsilon_2 = \frac{1}{\nu} \lambda_0^{-1},
\end{equation}

so that $C_1$ and $C_2$ may be estimated by

\begin{equation}
\frac{1}{2} \nu^2 \delta_1 \eta^a \bar{R}/R + \frac{1}{2} V^a_\rho \lambda_0^{-1} + \frac{1}{2} \nu_{me} \lambda_0^{-1} + \frac{1}{2} \nu_{me} \lambda_0^{-1} (\bar{R}/R)^2] r_{me}^{mag} - 1 =: \frac{1}{2} (D_1 - 1)
\end{equation}

with

\begin{equation}
\delta \eta^a := \max \{\delta_1 \eta^a \bar{R}/R, \ 2(\delta_1 \eta^a \lambda_0^{-1} \bar{R}/R + \delta_2 \eta^a \lambda_0^{-2} (\bar{R}/R)^2)\}
\end{equation}

and

\[
\left\{
\begin{array}{l}
c_1 = \lambda_0^{-2} + 2/V^a_\rho, \\
c_2 = (\lambda_0^{-1} + \lambda_0^{-2})(\bar{R}/R) + 2/V^a_\rho, \\
c_3 = \lambda_0^{-2} (\bar{R}/R)^2 + (2/V^a_\rho)(\bar{R}/R).
\end{array}
\right.
\]

Finally, by (3.8), (3.10) a, (3.11) a, and (4.11) one obtains

\[
\|\nabla M\|_\infty^2 + \nu^2 R^2 \|B^a_{me}\|_\infty^2 \geq \lambda_0^2 \|M\|_G^2/R^2 + \nu^2 \|N\|_G^2
\]

\[
\geq \frac{1}{R^2} (\|M\|_G^2 + \nu^2 R^2 \|N\|_G^2) = 2 \frac{\eta_0}{R^2} F_1 [M,N]
\]

and thus by (4.12) and (4.14) the differential inequality

\[
\frac{d}{dt} F_1 \leq -\frac{\eta_0}{R^2} (1 - D_1) F_1 \quad \text{on} \ [0,T]
\]

with solution (2.18), provided that $D_1 < 1$. This condition may be simplified (and hence strengthened) by estimating the domain-dependent coefficients in (4.14). By (3.11) a and (4.2) one obtains for (4.15) and (4.16)

\[
\delta \eta^a \leq 4(\delta_1 \eta^a + \delta_2 \eta^a), \quad c_i \leq 4, \quad i = 1, 2, 3.
\]

For the $c_i$-estimates we assumed, additionally, $V^a_\rho \geq 2$. This proves theorem 1.
5. Full decay in the weakly compressible/variable case. We include in this section, additionally to the meridional balances of the last section, the $A$-balance. With the notation of the last section the $A$-balance is obtained by multiplying (3.7) by $A$ and integrating over $G_c$; integrating by parts using thereby the zero boundary conditions for $v$, section the introduction second order derivatives of $A$ variable (which will be $B_J$ and $J$), which, by (3.7) this section, additionally to the meridional balances of the last section, the $A$-balance may thus be estimated by

\begin{align}
\frac{1}{2} \frac{d}{dt} \int_{G_c} A^2 \, dx &= - \int_{G_c} \eta^s \| \nabla A \|^2 \, dx + \int_{G_c} \frac{\nabla \rho}{\rho} \cdot \nabla \eta^s A^2 \, dx - \frac{1}{2} \int_{G_c} \nabla \cdot v^a A^2 \, dx \\
& \quad - \int_{G_c} \left[ \eta^s \frac{v^a}{\rho} B^a_{me} \cdot \nabla A \, dx + \int_{G_c} \left[ \frac{\eta^s}{\rho} J^n - \left( \frac{v^n}{\rho} \times B^n \right) \right] \times e_\phi \cdot \nabla A \, dx \\
& \quad - \int_{G_c} \left( \frac{\eta^s}{\rho} A^2 + \eta^s \nabla \rho \cdot \nabla A A \right) \, ds + \frac{1}{2} \int_{G_c} \nabla \rho \cdot v^a A^2 \, ds \\
& \quad - \int_{G_c} \nabla \rho \cdot B^a_{me} \frac{v^a}{\rho} A \, ds + \int_{G_c} \nabla \rho \cdot \left[ \frac{\eta^s}{\rho} J^n - \left( \frac{v^n}{\rho} \times B^n \right) \right] \times e_\phi A \, ds.
\end{align}

(5.1)

In the limit $\epsilon \to 0$ all surface integrals vanish with one exception:

$$\lim_{\epsilon \to 0} \int_{G_c} \frac{\eta^s}{\rho} A^2 \, ds = 2\pi \int_{L} A^2 \, dz := \|A\|^2_L,$$

which, by (3.7), need not be zero. Furthermore, note that, differently to the meridional balance, $J^n$ can here not be eliminated by integration by parts. This would introduce second order derivatives of $A$, which cannot be controlled by the dissipation term. $J^n$ remains therefore in the balance and makes the inclusion of a higher-order variable (which will be $B$) necessary. Using again the notation (2.8) and the bounds (2.10)–(2.12), (5.1) may thus be estimated by

\begin{align}
\frac{1}{2} \frac{d}{dt} \|A\|^2_G &\leq \eta_0 \left\{ - \| \nabla A \|^2_G + \delta_2 \eta^s \| A \|^2_G / R^2 + D v^n \| A \|^2_G/(2 R^2) \\
& \quad + V^a_{me} \| B^a_{me} \|_G \| \nabla A \|_G / R^2 + E^n \| J^n_{me} \|_G \| \nabla A \|_G / R \\
& \quad + (V^n_{az} \| B^a_{me} \|_G + V^n_{me} \| B^a_{me} \|_G) \| \nabla A \|_G / R^2 - \| A \|^2_L \right\} \\
& \leq \eta_0 \left\{ [\delta_2 \eta^s \mu_0^{-2} + D v^n \mu_0^{-2}/2 + V^a_{me} r_{az} \mu_0^{-1} - 1] (\| \nabla A \|^2_G + \| A \|^2_L) \\
& \quad + [(V^a_{az} + r_{me} V^n_{az}) \| B^a_{me} \|_\infty / R^2 + E^n \| J^n_{me} \|_G / R] \| \nabla A \|_G \right\}.
\end{align}

(5.2)

In the last estimate we made furthermore use of (2.9), (3.9), and (3.10).

Once $J^n$ is allowed to appear in the balance there is no need to eliminate it from the meridional balances as done in the last section. The following estimates of $M$ and $N$, which replace (4.6) and (4.10), retain $J^n$ and do without a condition of type $\delta \eta^s$:

\begin{align}
\frac{1}{2} \frac{d}{dt} \| M \|^2 \leq \eta_0 \left\{ [\delta_1 \eta^s \lambda_0^{-1} \tilde{R}/R + \delta_2 \eta^s \lambda_0^{-2} (\tilde{R}/R)^2 + D v^n \lambda_0^{-2}/2 - 1] \| \nabla M \|^2_\infty \\
& \quad + [V^n_{me} \lambda_0^{-1} \| B^a_{me} \|_\infty R + E^n \lambda_0^{-1} \| J^n_{az} \| G / R^2] \| \nabla M \|_\infty \right\},
\end{align}

(5.3)
By (5.4) one obtains:

\[
\frac{1}{2} \frac{d}{dt} \|N\|^2_G \leq \eta_0 \left\{ \delta_1 \eta^a \tilde{R}/(2R) + D V_n^a/2 + V_n^m r_{me}^{mag} - 1 \|B_{me}^a\|^2_G \\
+ \|V_{\rho}^{as} \lambda_0^{-1} \|\nabla M\|_\infty/R + E^n \|J_{az}\|_G R \|B_{me}^a\|_\infty \right\}.
\]

Finally, the B-balance is quite standard (see Backus 1958). Decomposing the right-hand side into meridional and azimuthal components and (only) the magnetic field into axisymmetric and nonaxisymmetric parts yields

\[
\frac{1}{2} \frac{d}{dt} \|B\|^2_G = \int_G v \times B \cdot \mathbf{J} \, dx - \int_G \eta J^2 \, dx
\]

\[
\leq \|v_{me} \times B_{me} \cdot e_\phi\|_G \|J_{az}\|_G + \|(v \times B) \times e_\phi\|_G \|J_{me}\|_G - \eta_0 \|J\|^2_G
\]

\[
\leq \eta_0 \left\{ V_{me}(\|B_{me}^a\|_G + \|B_{za}^a\|_G)) \|J_{az}\|_G/R \\
+ V_{me}(\|B_{az}^a\|_G + \|B_{za}^a\|_G)) \|J_{me}\|_G/R - \eta_0 \left\{ \|J_{me}\|^2_G + \|J_{az}\|^2_G \right\} \right\}
\]

\[
= \eta_0 \left\{ - \|J_{me}\|^2_G - \|J_{az}\|^2_G + \left[ V_{me}(1 + r_{me}^{mag}) \mu_0^{-1}(\|\nabla A\|_G + \|A\|^2_G) \right]^{1/2} R \\
+ V_{az}(1 + r_{me}^{mag}) \|B_{me}^a\|_\infty/R \|J_{me}\|_G \\
+ V_{me}(1 + r_{me}^{mag}) \|B_{me}^a\|_\infty/R \|J_{az}\|_G \right\}
\]

We are now in the position to estimate the time derivative \( \frac{d}{dt} F_2 \) of the functional (2.15). By (5.2)–(5.5) one obtains:

\[
\frac{d}{dt} F_2 \leq C_1 \|\nabla M\|_G^2/R^4 + C_2 \nu^2 \|B_{me}^a\|^2_G/R^2 \\
+ C_3 \alpha^2(\|\nabla A\|_G^2 + \|A\|^2_G) R^2 + C_4 \beta^2 \|J_{me}\|^2_G + C_5 \beta^2 \|J_{az}\|^2_G
\]

with

\[
C_1 := \frac{1}{2} D V^a \lambda_0^{-2} + \delta_1 \eta^a \lambda_0^{-1} \tilde{R}/R + \delta_2 \eta^a \lambda_0^{-2} (\tilde{R}/R)^2 + \frac{\epsilon_1}{2} V_{me}^n \lambda_0^{-1} r_{me}^{mag} \\
+ \frac{\epsilon_2}{2} E^n \lambda_0^{-1} + \frac{\nu^2}{2 \sigma_1} V_{\rho}^{as} \lambda_0^{-1} - 1,
\]

\[
C_2 := \frac{1}{2} D V^a + \frac{1}{2} \delta_1 \eta^a \tilde{R}/R + V_{me}^n r_{me}^{mag} + \sigma_1 \lambda_0^{-1} V_{\rho}^{as} + \frac{\sigma_2}{2} E^n \\
+ \frac{1}{2 \epsilon_1 \nu^2} V_{me}^n \lambda_0^{-1} r_{me}^{mag} + \frac{\omega_3^2}{2 \sigma_1 \nu^2} (V_{az}^{as} + r_{me}^{mag} V_{az}^{ns}) \\
+ \frac{\beta^2}{\nu^2} (\omega_3 / 2 V_{me} + \omega_2 / 2 V_{az})(1 + r_{me}^{mag}) - 1,
\]

\[
C_3 := \frac{1}{2} D V^a + \frac{1}{2} \delta_1 \eta^a \tilde{R}/R + V_{me}^n r_{me}^{mag} + \sigma_1 \lambda_0^{-1} V_{\rho}^{as} + \frac{\sigma_2}{2} E^n \\
+ \frac{1}{2 \epsilon_1 \nu^2} V_{me}^n \lambda_0^{-1} r_{me}^{mag} + \frac{\omega_3^2}{2 \sigma_1 \nu^2} (V_{az}^{as} + r_{me}^{mag} V_{az}^{ns}) \\
+ \frac{\beta^2}{\nu^2} (\omega_3 / 2 V_{me} + \omega_2 / 2 V_{az})(1 + r_{me}^{mag}) - 1,
\]

\[
C_4 := \frac{1}{2} D V^a + \frac{1}{2} \delta_1 \eta^a \tilde{R}/R + V_{me}^n r_{me}^{mag} + \sigma_1 \lambda_0^{-1} V_{\rho}^{as} + \frac{\sigma_2}{2} E^n \\
+ \frac{1}{2 \epsilon_1 \nu^2} V_{me}^n \lambda_0^{-1} r_{me}^{mag} + \frac{\omega_3^2}{2 \sigma_1 \nu^2} (V_{az}^{as} + r_{me}^{mag} V_{az}^{ns}) \\
+ \frac{\beta^2}{\nu^2} (\omega_3 / 2 V_{me} + \omega_2 / 2 V_{az})(1 + r_{me}^{mag}) - 1,
\]

\[
C_5 := \frac{1}{2} D V^a + \frac{1}{2} \delta_1 \eta^a \tilde{R}/R + V_{me}^n r_{me}^{mag} + \sigma_1 \lambda_0^{-1} V_{\rho}^{as} + \frac{\sigma_2}{2} E^n \\
+ \frac{1}{2 \epsilon_1 \nu^2} V_{me}^n \lambda_0^{-1} r_{me}^{mag} + \frac{\omega_3^2}{2 \sigma_1 \nu^2} (V_{az}^{as} + r_{me}^{mag} V_{az}^{ns}) \\
+ \frac{\beta^2}{\nu^2} (\omega_3 / 2 V_{me} + \omega_2 / 2 V_{az})(1 + r_{me}^{mag}) - 1,
\]
The mixed terms in (5.2)–(5.5) have again been split by (3.13) introducing thereby the further parameters \( \epsilon_{1/2}, \sigma_{1/2}, \tau_{1/2}, \) and \( \omega_{1/2/3}. \) As in the last section these parameters are chosen such that \( C_1 - C_5 \) have an optimal common bound, and \( \nu \) is used to control \( V_{\rho}^{a\ast} :\)

\[
\epsilon_1 = \frac{1}{\nu}, \quad \epsilon_2 = \frac{1}{\beta}, \quad \sigma_1 = \nu, \quad \sigma_2 = \frac{\nu}{\beta},
\[
\tau_1 = \frac{\alpha}{\nu}, \quad \tau_2 = \frac{\alpha}{\beta}, \quad \omega_1 = \frac{\alpha}{\beta}, \quad \omega_2 = \omega_3 = \frac{\nu}{\beta},
\[
\nu^{-1} = 2V_{\rho}^{a\ast}\lambda_0^{-1}.
\]

With the further abbreviations

\[
\begin{align*}
\delta \eta^a &:= \max \{ \delta_1 \eta^a R/R, 2(\delta_1 \eta^a \lambda_0^{-1} R/R + \delta_2 \eta^a \lambda_0^{-2} (R/R)^2), 2 \delta_2 \eta^a \mu_0^{-2} \}, \\
DV_{G}^{a} &:= \max \{ 1, \lambda_0^{-2}(G), \mu_0^{-2}(G) \} DV^a,
\end{align*}
\]

\( C_1 - C_5 \) can thus be estimated by

\[
C_1 \leq \frac{1}{2} DV_{G}^{a} + \frac{1}{2} \delta \eta^a + V_{\rho}^{a\ast} V_{me}^{n\ast} \mu_0^{-2} r_{me}^{mag} + \frac{1}{2\beta} E^n \lambda_0^{-1} + \frac{1}{4} - 1,
\]

\[
C_2 \leq \frac{1}{2} DV_{G}^{a} + \frac{1}{2} \delta \eta^a + V_{\rho}^{a\ast} V_{me}^{n\ast} \mu_0^{-2} r_{me}^{mag} + \frac{1}{2\beta} E^n \lambda_0^{-1} + \frac{1}{4} - 1,
\]

\[
+ \alpha V_{\rho}^{a\ast} \lambda_0^{-1}(V_{az}^{a\ast} + r_{me}^{mag} V_{az}^{n\ast}) + \beta V_{\rho}^{a\ast} \lambda_0^{-1}(V_{me} + V_{az})(1 + r_{me}^{mag}) + \frac{1}{4} - 1,
\]

\[
C_3 \leq \frac{1}{2} DV_{G}^{a} + \frac{1}{2} \delta \eta^a + V_{me}^{n\ast} \mu_0^{-1} r_{az}^{mag} + \alpha V_{\rho}^{a\ast} \lambda_0^{-1}(V_{az}^{a\ast} + r_{me}^{mag} V_{az}^{n\ast})
\]

\[
+ \frac{\beta}{2\alpha} V_{me} \mu_0^{-1}(1 + r_{az}^{mag}) + \frac{\alpha}{2\beta} E^n \lambda_0^{-1} - 1,
\]

\[
C_4 \leq \frac{\beta}{2\alpha} V_{me} \mu_0^{-1}(1 + r_{az}^{mag}) + \beta V_{\rho}^{a\ast} \lambda_0^{-1} V_{az}(1 + r_{me}^{mag}) + \frac{\alpha}{2\beta} E^n \lambda_0^{-1} - 1,
\]

\[
C_5 \leq \beta V_{\rho}^{a\ast} \lambda_0^{-1} V_{me}(1 + r_{me}^{mag}) + \frac{1}{2\beta} E^n \lambda_0^{-1} + \frac{1}{4\beta} E^n \lambda_0/V_{\rho}^{a\ast} - 1.
\]
A common bound on $C_1 - C_5$ then is

$$
\frac{1}{2} D k + \frac{1}{2} \delta \eta^a + \max \{ V_n r^{mag} + V_n L_0^{-1/2} \}
+ V^{ra} V_n \lambda_0^{-2} r^{mag} + \alpha V^a r_0^{-1} (V^{as} + r^{mag} V_n)
+ \frac{\beta}{2\alpha} V_m \mu_0^{-1} (1 + r^{mag}) + V^a r_0^{-1} (V + V_n) (1 + r^{mag})
+ \frac{1}{2\beta} \max \left\{ E_n \lambda_0^{-1} + \frac{1}{2} E_n \lambda_0 / r^{as}, \alpha E^{as} \right\} + \frac{3}{4} - 1.
$$

We make now use of the remaining parameters $\alpha$ and $\beta$ to control the remaining "large" terms in (5.8), i.e. we set

$$
\alpha^{-1} := 4 V^a \lambda_0^{-1} (V^{as} + r^{mag} V_n),
$$

$$
\beta^{-1} := 4 V^a \lambda_0^{-1} \left[ 2 V_n (V^{as} + r^{mag} V_n) \mu_0^{-1} (1 + r^{mag}) + (V + V_n) (1 + r^{mag}) \right]
$$
to obtain

$$
\frac{1}{2} D k + \frac{1}{2} \delta \eta^a + \max \{ V_n r^{mag} + V_n L_0^{-1/2} \}
+ V^{ra} V_n \lambda_0^{-2} r^{mag} + \frac{2}{2\alpha} \left[ 2 V_n (V^{as} + r^{mag} V_n) \mu_0^{-1} (1 + r^{mag}) + (V + V_n) (1 + r^{mag}) \right]
\times \max \left\{ E_n V^a \lambda_0^{-1} + \frac{1}{2} E_n, E^{as}, \left[ 4 (V^{as} + r^{mag} V_n) \right]^{-1} \right\} + \frac{3}{4} = \frac{1}{4} (D_2 - 1),
$$

which defines $D_2$. Finally, by (3.8)–(3.11), (4.11), and the estimate $\|J\|_G \geq (\pi / R)^2 \|B\|_{L^\infty}$ (cf. Backus 1958) one obtains

$$
\|\nabla M\|_\infty^2 / R^4 + \nu^2 \|B_{me}\|_\infty^2 / R^2 + \alpha^2 (\|\nabla A\|_G^2 + \|A\|_L^2)^2 R^2 + \beta^2 \|J\|_G^2
\geq \frac{1}{R^2} \left\{ (\pi / 2)^2 \|M\|_G^2 / R^4 + \nu^2 \|N\|_G^2 / R^2 + \pi^2 \alpha^2 R^2 \|A\|_G^2 + \pi^2 \beta^2 \|B\|_G^2 \right\}
\geq \frac{2\eta_0}{R^2} F_2 [M, N, A, B]
$$

and thus by (5.6) and (5.9) the differential inequality

$$
\frac{d}{dt} F_2 \leq -\frac{\eta_0}{2R^2} (1 - D_2) F_2 \quad \text{on} \quad [0, T]
$$

with solution (2.20), provided that $D_2 < 1$. Eliminating the domain-dependent quantities $\lambda_0$ and $\mu_0$ by (3.11), using (5.7) and (4.2), and estimating $V_n$ by $V^{ns}$ and $E^{as}$, respectively, the clumsy expression (5.9) may be simplified:

$$
D_2 \leq 2 D k + 6 \delta \eta^a + 8 \delta \eta^2 + 4 \max \{ V^{mag} r^{mag} / 3 \}
+ 2 V_n \mu_0^{-1} r^{mag} V^{as} + [3 (1 + r^{mag}) (V^{as} + r^{mag} V_n) ] V_n
+ 4 (1 + r^{mag}) (V + V_n) ] (V^{ns} + 1) E^{ns}.
$$
Here also $V^a_{az} \geq 1/2$ has been used. The even more simple (and rough) expression for $D_2$, given in (2.21), is obtained by use of (2.9)$_2$ and (2.12)$_4$. Finally, for large axisymmetric velocities, i.e. $V^a_{\rho} \gg 1$, $V^a_{az} \gg 1$, and $r_{mag} \leq 1$, (2.21) may be replaced by

$$2DV^a + 6 \delta_1 \eta^a + 8 \delta_2 \eta^a + 2V^a_{\rho}r_{mag}V^a_{me} + 12V^a_{az}V^a_{\rho}VE^a_{n} < 1.$$ 

This proves theorem 2.

6. Decay in the fully variable/compressible case. The result of this section is based on axisymmetric solutions of two auxiliary problems, which allow us to eliminate the flow field completely and, moreover, the variation of the diffusivity from the meridional as well as from the azimuthal balances.

The first, the azimuthal, problem reads

$$\begin{cases} 
\partial_t \rho - \nabla \cdot \left( \frac{\rho^a}{\rho^2} \nabla (\rho^2 P) \right) + \nabla \cdot (v^a P) = 0 & \text{in } G \times (0, T), \\
A(x, y, \cdot, \cdot) = O(1) & \text{for } \rho \to 0, \\
n \cdot \nabla P = 0 & \text{on } \partial G \times (0, T), \\
P(\cdot, 0) = 1 & \text{on } G \times \{t = 0\}. 
\end{cases}$$

(6.1)

This problem has already been discussed in (KT14). We formulate the main property (for our purposes) in the following lemma.

**Lemma 1.** There exist positive bounds $P$ and $\bar{P}$, which depend on $G$, $\eta_0$, and $K$, where $K$ is a simultaneous upper bound on $|v^a|$, $|v^a_{\rho}|$, and $|\partial_\rho \eta^a|$, but which do not depend on $T$, such that the unique solution $P$ of (6.1) satisfies

$$P \leq \bar{P} \leq P \text{ in } G \times [0, T].$$

(6.2)

The second, the meridional, auxiliary problem is not quite standard in that it involves a backward parabolic equation. Following (Stredulinsky et al. 1986) it can, however, be reduced to an ordinary parabolic problem by a “time inversion”. Let $\bar{\eta}^a$ be a $C^1$-extension of $\eta^a$ onto $\mathbb{R}^3$ of class $C^2((G \cup \tilde{G}) \times [0, T]) \cap C^1(\mathbb{R}^3 \times [0, T])$ with the property $\bar{\eta}^a = 1$ outside some ball $B_{\bar{R}}$. Note that this extension, other than that in section 4, falls possibly below $\eta_0$: the difference, however, can be made arbitrarily small (at the price of large gradients) and is henceforth ignored, so that $\eta_0$ will serve as lower bound for $\bar{\eta}^a$ as well. We define now the time-inverted coefficients

$$v^a_T(\cdot, t) := v^a(\cdot, T - t), \quad \bar{\eta}^a_T(\cdot, t) := \bar{\eta}^a(\cdot, T - t)$$

and denote by $Q_T$ the solution of the following ordinary evolution problem:

$$\begin{cases} 
\partial_t Q_T - \Delta(\bar{\eta}^a_T Q_T) - 2\nabla \rho \cdot \nabla (\bar{\eta}^a_T Q_T) / \rho \\
- \nabla \cdot (v^a_T Q_T) + 2 \nabla \rho \cdot v^a_T Q_T / \rho = 0 & \text{in } G \times (0, T), \\
\Delta(\bar{\eta}^a_T Q_T) + 2 \nabla \rho \cdot \nabla (\bar{\eta}^a_T Q_T) / \rho = 0 & \text{in } \tilde{G} \times (0, T), \\
Q_T(\cdot, 0) = Q_0 & \text{on } G \times \{t = 0\}. 
\end{cases}$$

(6.3)
where \( Q_0 \) is some initial-value with lower bound \( \eta_0^{-1} \) and satisfying some compatibility conditions. In the appendix this problem is reformulated such that the singular coefficients vanish and the problem becomes amenable to standard solution techniques for parabolic problems. In particular, we can then apply results from (Kaiser and Uecker 2009) and (Stredulinsky et al. 1986), which yield the following analogue of the preceding lemma:

**Lemma 2.** There exist positive bounds \( Q \) and \( \overline{Q} \), which depend on \( G, \eta_0, \) and \( L \), where \( L \) is a simultaneous upper bound on \( \|
abla\eta\|, |\eta_0^a/\rho|, |\tilde{\eta}|, \) and \( |\nabla\tilde{\eta}| \), but which do not depend on \( T \), such that the unique solution \( Q_T \) of (6.3) satisfies

\[
Q \leq Q_T \leq \overline{Q} \quad \text{in} \quad \mathbb{R}^3 \times [0, T].
\]

Defining \( Q(\cdot, t) := Q_T(\cdot, T-t) \), obviously \( Q \) solves the following “final-value” problem

\[
\begin{aligned}
\partial_t Q + \Delta(\tilde{\eta}^a Q) + 2 \nabla \rho \cdot \nabla(\tilde{\eta}^a Q)/\rho \\
+ \nabla \cdot (\mathbf{v}^a Q) - 2 \nabla \rho \cdot \mathbf{v}^a Q/\rho = 0 \quad \text{in} \quad G \times (0, T), \\
Q(\cdot, 0) = Q_0 \quad \text{on} \quad G \times \{t = T\}, \\
\end{aligned}
\]

(6.4)

and satisfies, moreover, the bounds

\[
Q \leq Q \leq \overline{Q} \quad \text{in} \quad \mathbb{R}^3 \times [0, T].
\]

Note that \( Q \) will depend on \( T \), but the bounds \( Q \) and \( \overline{Q} \) do not.

With these preparations we set up the meridional balance first: Multiplying (3.6) by \( \partial_t N Q \) and (6.4) by \( N^2/2 \), summing up, integrating over \( G_\epsilon \), and integrating by parts yields

\[
\begin{aligned}
\frac{1}{2} \frac{d}{dt} \int_{G_\epsilon} N^2 dx &= \int_{G_\epsilon} (\partial_t N N Q + \partial_t Q N^2/2) dx \\
&= - \int_{G_\epsilon} \tilde{\eta}^a Q \left( |\nabla N|^2 - 2 \frac{\nabla \rho}{\rho} \cdot \nabla N + \frac{N^2}{\rho^2} \right) dx \\
&\quad + \int_{\partial G_\epsilon} \tilde{\eta}^a Q \left( \nabla N N - \frac{\nabla \rho}{\rho} N^2 \right) \cdot \mathbf{n} ds - \frac{1}{2} \int_{\partial G_\epsilon} N^2 \nabla (\tilde{\eta}^a Q) \cdot \mathbf{n} ds \\
&\quad + \frac{1}{2} \int_{G_\epsilon} N^2 \nabla \rho \cdot \mathbf{v}^a ds.
\end{aligned}
\]

The same procedure applied to (3.6) and (6.4) and integrated over \( \tilde{G} \cap B_R \) yields

\[
\begin{aligned}
0 &= \int_{\tilde{G} \cap B_R} \tilde{\eta}^a \left( \Delta N - \frac{N^2}{\rho^2} \right) N Q dx - \int_{\tilde{G} \cap B_R} \left( \Delta (\tilde{\eta}^a Q) + 2 \frac{\nabla \rho}{\rho} \cdot \nabla (\tilde{\eta}^a Q) \right) \frac{N^2}{2} dx \\
&= - \int_{\tilde{G} \cap B_R} \tilde{\eta}^a Q \left( |\nabla N|^2 - 2 \frac{\nabla \rho}{\rho} \cdot \nabla N + \frac{N^2}{\rho^2} \right) dx \\
&\quad + \int_{\partial (\tilde{G} \cap B_R)} \tilde{\eta}^a Q \left( \nabla N N - \frac{\nabla \rho}{\rho} N^2 \right) \cdot \mathbf{n} ds - \frac{1}{2} \int_{\partial (\tilde{G} \cap B_R)} N^2 \nabla (\tilde{\eta}^a Q) \cdot \mathbf{n} ds,
\end{aligned}
\]
where \( \mathbf{n} \) denotes the exterior (with respect to \( \tilde{G} \cap \mathcal{B}_R \)) normal at \( \partial(\tilde{G} \cap \mathcal{B}_R) \). Summing up, letting \( \epsilon \to 0 \), \( \bar{R} \to \infty \), and rewriting the dissipation term leads to

\[
\frac{1}{2} \frac{d}{dt} \int_G Q N^2 \, dx = - \int_{\mathbb{R}^3} \eta^a Q | \nabla (N/\rho) |^2 \rho^2 \, dx - \int_G Q \eta^a \mathbf{J}^n \cdot \mathbf{e}_\phi \, N \, dx \\
+ \int_G Q \mathbf{v}^n \cdot \mathbf{B}^n \times \mathbf{e}_\phi \, N \, dx,
\]

which, by (6.5) and (2.10)–(2.12), may be estimated by

\[
\frac{1}{2} \frac{d}{dt} \| Q^{1/2} N \|_{G}^2 \leq \eta_0 \left\{ - Q \| \nabla (N/\rho) \|_F^2 + \bar{Q} E^a R \| J^n_{az} \|_G \| N/\rho \|_G \\
+ \bar{Q} V_{me} \| B^a_{me} \|_G \| N/\rho \|_G \right\} \\
\leq \eta_0 \bar{Q} \left\{ [V_{me} \mathbf{r}_{me}^\text{mag} - \bar{Q}/\bar{Q}] \| B^a_{me} \|_\infty^2 + E^a \| B^a_{me} \|_\infty \| J^n_{az} \|_G \right\} R.
\]

In the last line we made use of (2.9) and the identity, similar to (4.9),

\[
\int_{\mathbb{R}^3} \| \nabla (N/\rho) \|_F^2 \rho^2 \, dx = \int_{\mathbb{R}^3} (| \nabla N |^2 + (N/\rho)^2) \, dx = \int_{\mathbb{R}^3} \| B^a_{me} \|_F^2 \, dx.
\]

The azimuthal balance is obtained by multiplying (3.7) by \( 2A/P \) and (6.1) by \( (A/P)^2 \); it appeared already in (KT14) (without the nonaxisymmetric terms) and is just reported here:

\[
\frac{1}{2} \frac{d}{dt} \int_G A^2 \frac{P}{P} \, dx = - \int_G \eta^a P \left| \nabla \left( \frac{A}{P} \right) \right|^2 \, dx - 2\pi \int \eta^a \frac{A^2}{P} \, dz \\
- \int_G \frac{v^a}{\rho} B^a_{me} \cdot \nabla \left( \frac{A}{P} \right) \, dx + \int_G \left[ \frac{\eta^a}{\rho} \mathbf{J}^n - \left( \frac{\mathbf{v}^n}{\rho} \times \mathbf{B}^n \right) \right] \times \mathbf{e}_\phi \cdot \nabla \left( \frac{A}{P} \right) \, dx.
\]

By (2.9)–(2.12), (3.9), (3.10) by (6.2) we have then the estimate

\[
\frac{1}{2} \frac{d}{dt} \| A / P^{1/2} \|_G^2 \leq \eta_0 \left\{ \bar{P} \left[ V_{me}^n \mathbf{r}_{az}^{-1} r_{az}^\text{mag} - \bar{P} / \bar{P} \right] \| \nabla (A/P) \|_G^2 + \| A/P \|_L^2 \right\} \\
+ \left\{ \left( V_{az}^n + r_{az}^\text{mag} V_{az}^n \right) \| B^a_{me} \|_\infty / R^2 + E^n \| J^a_{me} \|_G / R \right\} \\
\times \left( \| \nabla (A/P) \|_G^2 + \| A/P \|_L^2 \right)^{1/2}
\]

Computation and estimate of \( \frac{d}{dt} F_3 \) follows now closely that of \( \frac{d}{dt} F_2 \) in section 5. By (6.6), (6.7), and (5.5) one obtains:

\[
\frac{d}{dt} F_3 \leq C_2 \left( \frac{Q}{\bar{Q}} \right) \| B^a_{me} \|_L^2 / R^2 + C_3 \left( \bar{P} / \bar{P} \right) \alpha^2 \left( \| \nabla (A/P) \|_G^2 + \| A/P \|_L^2 \right) R^2 \\
+ C_4 \beta^2 \| J^a_{me} \|_G^2 + C_5 \beta^2 \| J^a_{az} \|_G^2
\]
with

\[
C_2 := \frac{Q}{Q} V_{me}^{n_{mag}} + \frac{\epsilon}{2} E^n + \frac{\alpha^2}{2 \tau_1} (V_{az}^{a_{mag}} + r_{me}^{mag} V_{az}^{n_{mag}}) \frac{1}{P} + \beta^2 \left( \frac{\omega_1}{2} V_{me} + \frac{\omega_2}{2} V_{az} \right) (1 + r_{me}^{mag}) - 1,
\]

\[
C_3 := \frac{P}{P} V_{me}^{n_{mag}} + \frac{\tau_1}{2} (V_{az}^{a_{mag}} + r_{me}^{mag} V_{az}^{n_{mag}}) \frac{1}{P} + \frac{\tau_2}{2} E^n + \frac{\omega_1}{2 \alpha^2} V_{me} \mu_0^{-1} (1 + r_{az}^{mag}) P^2 - 1,
\]

\[
C_4 := \frac{1}{2 \omega_1} V_{me} \mu_0^{-1} (1 + r_{az}^{mag}) + \frac{1}{2 \omega_2} V_{az} (1 + r_{me}^{mag}) + \frac{\alpha^2}{2 \tau_2 \beta^2} E^n \frac{1}{P} - 1,
\]

\[
C_5 := \frac{1}{2 \omega_3} V_{me} (1 + r_{me}^{mag}) + \frac{1}{2 \beta^2} E^n - 1.
\]

Note that we made use of (5.5) in a slightly modified form in that the first term in the last estimate is replaced by \( V_{me} (1 + r_{az}^{mag}) \mu_0^{-1} P \langle \nabla (A/P) \rangle \| G \| + \| A/P \| G \rangle^{1/2} \). This replacement is justified by \( \| A \| G \leq \frac{1}{P} \| A/P \| G \) and subsequent application of (3.9).

By the choice of parameters

\[
\epsilon = \frac{1}{\beta} \left( \frac{Q}{Q} \right)^{-1/2}, \quad \tau_1 = \alpha \left( \frac{Q}{Q} \right)^{1/2} \left( \frac{P}{P} \right)^{-1/2}, \quad \tau_2 = \frac{\alpha}{\beta} \left( \frac{P}{P} \right)^{-1/2},
\]

\[
\omega_1 = \frac{1}{\beta} \left( \frac{P}{P} \right)^{-1/2}, \quad \omega_2 = \omega_3 = \frac{1}{\beta} \left( \frac{Q}{Q} \right)^{-1/2},
\]

\( C_2 - C_5 \) take the form

\[
C_2 := \frac{Q}{Q} V_{me}^{n_{mag}} + \frac{\alpha}{2} \left( \frac{Q}{Q} \right)^{1/2} (P P)^{-1/2} (V_{az}^{a_{mag}} + r_{me}^{mag} V_{az}^{n_{mag}})
\]

\[+ \frac{\beta}{2} \left( \frac{Q}{Q} \right)^{1/2} (Q^{1/2} (V_{me} + V_{az}) (1 + r_{me}^{mag}) + \frac{1}{2 \beta} \left( \frac{Q}{Q} \right)^{1/2} E^n - 1,
\]

\[
C_3 := \frac{P}{P} V_{me}^{n_{mag}} + \frac{\alpha}{2} \left( \frac{Q}{Q} \right)^{1/2} (P P)^{-1/2} (V_{az}^{a_{mag}} + r_{me}^{mag} V_{az}^{n_{mag}})
\]

\[+ \frac{\beta}{2 \alpha} \left( \frac{P}{P} \right)^{1/2} P V_{me} \mu_0^{-1} (1 + r_{az}^{mag}) + \frac{\alpha}{2 \beta} \left( \frac{P}{P} \right)^{1/2} E^n - 1,
\]

\[
C_4 := \frac{\beta}{2 \alpha} \left( \frac{P}{P} \right)^{1/2} P V_{me} \mu_0^{-1} (1 + r_{az}^{mag}) + \frac{\beta}{2} \left( \frac{Q}{Q} \right)^{1/2} V_{az} (1 + r_{me}^{mag})
\]

\[+ \frac{\alpha}{2 \beta} \left( \frac{P}{P} \right)^{-1/2} E^n - 1,
\]

\[
C_5 := \frac{\beta}{2} \left( \frac{Q}{Q} \right)^{1/2} V_{me} (1 + r_{me}^{mag}) + \frac{1}{2 \beta} \left( \frac{Q}{Q} \right)^{1/2} E^n - 1.
\]
with common bound
(6.9)
\[
\max \left\{ \frac{Q}{Q} V_{n_{\text{me}}} r_{n_{\text{me}}}^{\text{mag}}, \frac{P}{P} V_{n_{\text{az}}}^{r_{n_{\text{az}}}^{\text{mag}}} \mu_{0}^{-1} r_{n_{\text{az}}}^{\text{mag}} \right\} + \frac{\alpha}{2} \left( \frac{Q}{Q} \right)^{1/2} \left( P \right)^{-1/2} \left( V_{n_{\text{az}}}^{r_{n_{\text{az}}}^{\text{mag}}} + r_{n_{\text{az}}}^{\text{mag}} V_{n_{\text{az}}}^{r_{n_{\text{az}}}^{\text{mag}}} \right)
\]
\[+ \beta \frac{1}{2} \max \left\{ \left( \frac{Q}{Q} \right)^{1/2} (V_{n_{\text{me}}} + V_{n_{\text{az}}})(1 + r_{n_{\text{me}}}^{\text{mag}}), \frac{1}{\alpha} \left( \frac{P}{P} \right)^{1/2} P V_{n_{\text{me}}} \mu_{0}^{-1} (1 + r_{n_{\text{az}}}^{\text{mag}}) \right\}
\]
\[+ \frac{1}{2 \alpha} \max \left\{ \left( \frac{Q}{Q} \right)^{1/2} E^{n}, \alpha \left( P \right)^{-1/2} E^{n} \right\} - 1.
\]
Finally, setting
\[
\alpha^{-1} := 2 \left( \frac{Q}{Q} \right)^{1/2} \left( \frac{P}{P} \right)^{-1/2} (V_{n_{\text{az}}}^{r_{n_{\text{az}}}^{\text{mag}}} + r_{n_{\text{az}}}^{\text{mag}} V_{n_{\text{az}}}^{r_{n_{\text{az}}}^{\text{mag}}}),
\]
\[
\beta^{-1} := 2 \left( \frac{Q}{Q} \right)^{1/2} \max \left\{ (1 + r_{n_{\text{me}}}^{\text{mag}})(V_{n_{\text{me}}} + V_{n_{\text{az}}}), 2 \frac{P}{P} \mu_{0}^{-1} (1 + r_{n_{\text{az}}}^{\text{mag}}) \right\}
\times (V_{n_{\text{az}}}^{r_{n_{\text{az}}}^{\text{mag}}} + r_{n_{\text{az}}}^{\text{mag}} V_{n_{\text{az}}}^{r_{n_{\text{az}}}^{\text{mag}}}) \right\} \right\}
\]
we obtain for (6.9):
(6.10) \[
\max \left\{ C_{n_{\text{me}}} V_{n_{\text{me}}} r_{n_{\text{me}}}^{\text{mag}}, C_{n_{\text{az}}} V_{n_{\text{az}}}^{r_{n_{\text{az}}}^{\text{mag}}} \right\} + C_{n_{\text{me}}}
\]
\[
\times \max \left\{ (V_{n_{\text{me}}} + V_{n_{\text{az}}}) r_{n_{\text{me}}}^{\text{mag}}, 2 C_{n_{\text{me}}} V_{n_{\text{me}}}^{r_{n_{\text{az}}}^{\text{mag}}} V_{n_{\text{az}}}^{r_{n_{\text{az}}}^{\text{mag}}}(1 + r_{n_{\text{az}}}^{\text{mag}}) \right\}
\]
\[
\times \max \left\{ E^{n}, E^{n}, 2 C_{n_{\text{me}}} V_{n_{\text{me}}}^{r_{n_{\text{az}}}^{\text{mag}}} V_{n_{\text{az}}}^{r_{n_{\text{az}}}^{\text{mag}}}]^{-1} \right\} + \frac{1}{2} - 1 =: \frac{1}{2} (D_{3} - 1),
\]
which defines $D_{3}$. We have here introduced the abbreviations
\[
\frac{Q}{Q} := C_{n_{\text{me}}} = C_{n_{\text{me}}}(L, G), \quad \left( \frac{P}{P} \right) \mu_{0}^{-1} := C_{n_{\text{az}}} = C_{n_{\text{az}}}(K, G)
\]
with $K$ and $L$ denoting the bounds introduced in the lemmata 1 and 2, respectively.

Under the condition $D_{3} < 1$ the right-hand side in (6.8) can now be estimated similarly as in section 5 with the result
\[
\frac{d}{dt} \tilde{\mathcal{F}}_{3} \leq \frac{1}{2} (D_{3} - 1) \left( 1 - \frac{2 R}{c} \right) \max \left\{ C_{n_{\text{me}}}^{-1}, C_{n_{\text{az}}}^{-1} \right\} \left\{ \frac{1}{R^{2} Q} \| N \|_{G}^{2} + \beta^{2} \| B \|_{G}^{2} \right\}
\]
\[\leq - \frac{1}{2 R} (1 - D_{3}) \min \left\{ C_{n_{\text{me}}}^{-1}, C_{n_{\text{az}}}^{-1} \right\} \left\{ \frac{1}{R^{2} Q} \| N \|_{G}^{2} + \beta^{2} \| B \|_{G}^{2} \right\}
\]
\[\leq - \frac{\eta_{0}}{c R} (1 - D_{3}) \tilde{\mathcal{F}}_{3} \quad \text{on } [0, T].
\]
In the last estimate we introduced $C := \max \{ C_{n_{\text{me}}}, C_{n_{\text{az}}}, C_{n_{\text{az}}} \} \geq 1$. The inclusion of $
\tilde{C}_{n_{\text{az}}} := \tilde{P}^{2}$ is convenient for a final estimate that connects $\tilde{\mathcal{F}}_{3}$ with the functional $\mathcal{F}_{3}$:
\[
\tilde{\mathcal{F}}_{3} \geq \frac{1}{2 n_{0} c R} \left\{ \frac{1}{R^{2} Q} \| N \|_{G}^{2} + \beta^{2} \| B \|_{G}^{2} \right\} \geq \frac{1}{C} \mathcal{F}_{3}.$
Together with $\tilde{\mathcal{F}}_{3}(0) \leq \mathcal{F}_{3}(0)$ we obtain therefore
\[
\mathcal{F}_{3}(t) \leq C \tilde{\mathcal{F}}_{3} \leq C \tilde{\mathcal{F}}_{3}(0) e^{-(\eta_{0}/c R^{2})(1-D_{3}) t} \leq C \mathcal{F}_{3}(0) e^{-(\eta_{0}/c R^{2})(1-D_{3}) t}.
\]
which is (2.22). The simpler expression for $D_3$ given in (2.23) is again obtained by using common bounds for meridional and azimuthal as well as for starred and non-starred quantities. This proves theorem 3.

In (KT14) the possibility of very slow magnetic decay for special axisymmetric flow fields has been demonstrated by example. More precisely, for a meridional model flow it has been proved that the decay rate $d_{me}$ of the meridional magnetic scalar can be estimated from above by a bound that vanishes exponentially fast with respect to the flow amplitude $V_{me}^a$, viz.

\[(6.11)\quad d_{me} \sim (V_{me}^a)^2 e^{-V_{me}^a} \quad \text{for } V_{me}^a \to \infty\]

or, equivalently,

\[(6.12)\quad d_{me} \sim b^{-V_{me}^a} \quad \text{for } V_{me}^a \to \infty\]

and any $b < e$. A numerical examination of the model flow confirmed, moreover, (6.11) for quite moderate values of $V_{me}^a$. A comparison of (6.12) with the decay rate $\eta_0/(CR^2)$ in (2.22) then yields

\[C \sim a V_{me}^a,\]

valid for $V_{me}^a \gtrsim 10$.

**Appendix A.** This appendix gives a five-dimensional description of the auxiliary problem (6.3), which avoids singular coefficients, and it makes the connection to well-known results for problems of this type.

Let $x = (x_1, x_2, x_3, x_4, x_5)$ be Cartesian coordinates in $\mathbb{R}^5$, $\nabla$ the corresponding gradient, and $\Delta_5$ the 5-dimensional Laplacian. When identifying $\rho^2$ with $\sum_{i=1}^{4} x_i^2$ and $z$ with $x_5$ and when defining $G_5 := \{ x \in \mathbb{R}^5 : (\rho, z) \in G \}$, axisymmetric functions in $\mathbb{R}^3$ can be identified with axisymmetric functions in $\mathbb{R}^5$ with symmetry axes in $z$ and $x_5$ direction, respectively. With the axisymmetric identity

\[\Delta_5 f = \frac{1}{\rho^3} \partial_{\rho}(\rho^3 \partial_{\rho} f) + \partial_z^2 f = \frac{1}{\rho} \partial_{\rho}(\rho \partial_{\rho} f) + \partial_z^2 f + \frac{2}{\rho} \partial_{\rho} f = \Delta_5 f + \frac{2}{\rho} \partial_{\rho} f \cdot \nabla f\]

and the definition $\mathbf{v}_5^a := \sum_{i=1}^{4} v_i^a (x_i/\rho) \mathbf{e}_i + v_z \mathbf{e}_5$, problem (6.3) takes the form

\[
\begin{aligned}
\partial_t Q - \Delta_5 (\tilde{\eta}^a Q) - \nabla \cdot (\mathbf{v}_5^a Q) + 4 (v_z^a/\rho) Q &= 0 \quad \text{in } G_5 \times (0, T), \\
\Delta_5 (\tilde{\eta}^a Q) &= 0 \quad \text{in } \hat{G}_5 \times (0, T), \\
Q(x, \cdot) &\to 1 \quad \text{for } |x| \to \infty, \\
Q(\cdot, 0) &= Q_0 \quad \text{on } G_5 \times \{ t = 0 \},
\end{aligned}
\]

(A.1)

where we have omitted throughout the index $T$ indicating the time-inversion in the coefficients $\tilde{\eta}^a$ and $v_5^a$. The set $\{ \rho = 0 \}$ does no longer play a distinguished role in (A.1), which makes a condition of type (6.3)$_4$ superfluous. Introducing the variable $R := \tilde{\eta}^a Q$, (A.1) may be further reformulated:

\[
\begin{aligned}
\partial_t R - \eta^a \Delta_5 R &= \mathbf{v}_5^a \cdot \nabla R + c R \quad \text{in } G_5 \times (0, T), \\
\Delta_5 R &= 0 \quad \text{in } \hat{G}_5 \times (0, T), \\
R(x, \cdot) &\to 1 \quad \text{for } |x| \to \infty, \\
R(\cdot, 0) &= R_0 := \eta^a(\cdot, T) Q_0 \quad \text{on } G_5 \times \{ t = 0 \},
\end{aligned}
\]

(A.2)
where we have set $c := \nabla \cdot \mathbf{v}^a_5 - 4(v^a_5 / \rho) - \mathbf{v}^a_5 \cdot \nabla \eta^a / \eta^a + \partial_t \eta^a / \eta^a$. We recall that $\tilde{\eta}^a$ is a $C^1$-extension of $\eta^a$ with $\tilde{\eta}^a \geq \eta_0$ and $\tilde{\eta}^a = 1$ outside some ball $B_R$.

According to Kaiser and Uecker (2009) problem (A.2) is well-posed and unique classical solutions are guaranteed for sufficiently regular boundary, initial-value, and coefficients, and if, moreover, these quantities satisfy some compatibility conditions at $\partial G_5 \times \{ t = 0 \}$. $\partial G_5 \in C^6$, $R_0 \in C^4(G_5)$, and $\eta^a$, $\mathbf{v}^a_5$, $c \in C^1[\overline{G_5} \times [0, T)]$ are sufficient but supposedly not necessary. The high degree of regularity required for the data is due to the use of embedding results in $\mathbb{R}^5$ ignoring the axisymmetry of the problem.

The compatibility conditions require in our case that $R_0 - 1$, $\Delta_5 R_0$, and $(\eta^a \Delta_5 + \mathbf{v}^a_5 \cdot \nabla + c) R_0(\cdot, T)$ have harmonic extensions onto $\mathbb{R}^5$, which $C^1$-match at $\partial G_5$. Taking for example $R_0 \equiv 1$ in $G_5$ with trivial extension 1 onto $\mathbb{R}^5$ these conditions are all satisfied provided that $c(\cdot, T)$ has a $C^1$-harmonic extension, which implies restrictions on $\nabla c(\cdot, T)$ at $\partial G_5$. Conversely, for given coefficients, $R_0$ and its derivatives up to third order are restricted at $\partial G_5$ by these conditions.

The positivity of solutions of (A.2) is easy to see: introducing the variable $S := R e^{-d t}$ with $d > \max_{\overline{G_5} \times [0, T]} |c|$, (A.2) takes the form

\[(A.3) \quad \partial_t S = \eta^a \Delta_5 S + \mathbf{v}^a_5 \cdot \nabla S + (c - d) S\]

with a non-positive zeroth-order coefficient. Applying (strong) elliptic and parabolic maximum principles and their boundary versions (see, e.g., Protter and Weinberger, 1967, Theorems 6.7, p. 64f. and Theorems 5-7, p. 167f.) on $\Delta_5 S = 0$ in $\overline{G_5}$ and on (A.3) in $G_5 \times [0, T]$, respectively, a non-positive minimum can be excluded at $G_5 \times [0, T]$, $\overline{G_5} \times [0, T]$, as well as at $\partial G_5 \times [0, T]$. $S$ and hence $R$ are thus positive throughout $\mathbb{R}^5 \times [0, T]$.

Harnack-type inequalities, which provide positive bounds that do not depend on $T$, are much harder to establish. We rely here on (Stredulinsky et al. 1986), who prove (6.5) for weak solutions of problem (A.1) without zeroth-order coefficient (see eqs. (2) and (11) and Theorem 2 in this reference). As in the azimuthal problem inclusion of a zeroth-order term does not invalidate the boundedness, it merely affects the bounds, which depend here then additionally on $\max_{\overline{G_5} \times [0, T]} |v^a_5 / \rho|$. This has been demonstrated in some detail for the azimuthal case in (KT14, appendix A) and essentially the same modifications apply to the present case, which is not further detailed.

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\[\text{In this notation upper and lower indices at } \text{"}C\text{"} \text{ refer to the orders of spatial and temporal derivatives, respectively, which have to be continuous in } \overline{G_5} \times [0, T].\]


